# Searching, Sorting

part 1

# Week 3 Objectives

- Searching: binary search
- Comparison-based search: running time bound
- Sorting: bubble, selection, insertion, merge
- Sorting: Heapsort
- Comparison-based sorting time bound

# Brute force/linear search

- Linear search: look through all values of the array until the desired value/event/condition found
- Running Time: linear in the number of elements, call it O(n)
- Advantage: in most situations, array does not have to be sorted

# Binary Search

- Array must be sorted
- Search array A from index b to index e for value V
- Look for value V in the middle index m = (b+e)/2
  - That is compare V with A[m]; if equal return index m
  - If V<A[m] search the first half of the array
  - If V>A[m] search the second half of the array

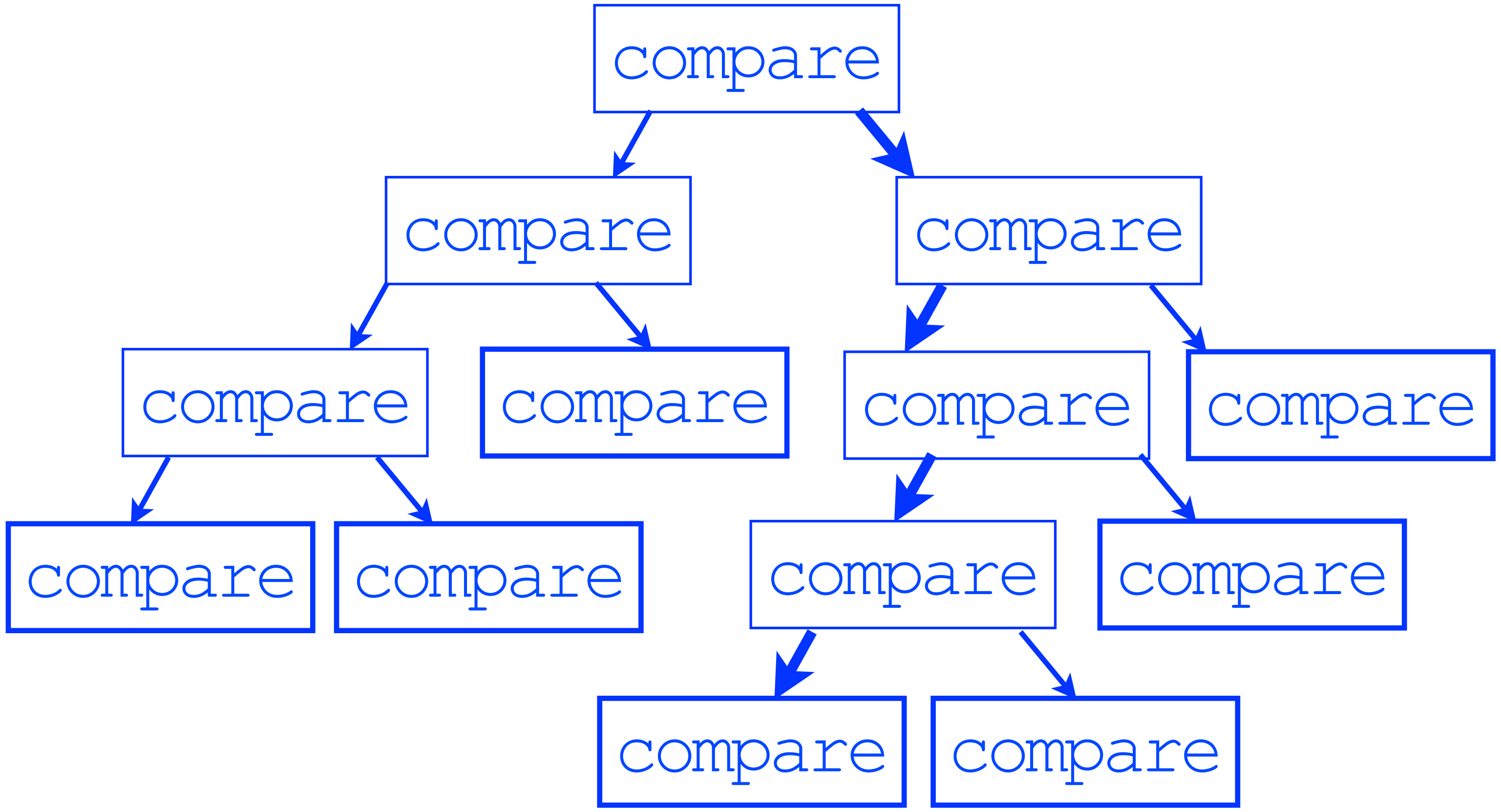
| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | b | | | | m | | | | | e |
| V=3 | -4 | -1 | 0 | 0 | 1 | 1 | 3 | 19 | 29 | 47 |

A[m]=1 < V=3 => search moves to the right half

# Binary Search Efficiency

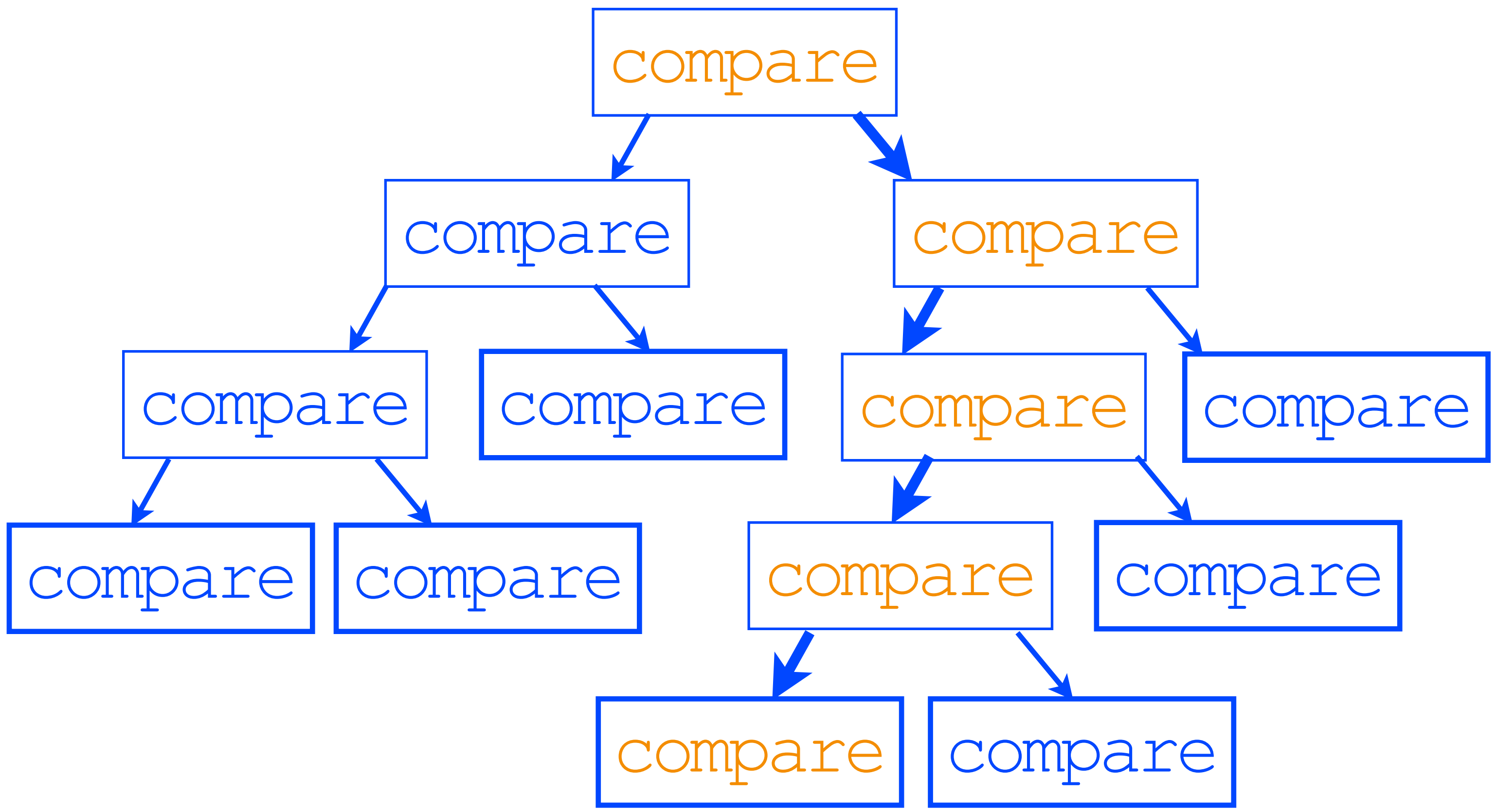
- every iteration/recursion
  - ends the procedure if value is found
  - if not, reduces the problem size (search space) by half
- worst case : value is not found until problem size=1
  - how many reductions have been done?
  - n / 2 / 2 / 2 / . . . . / 2 = 1. How many 2-s do I need ?
  - if k 2-s, then $n= 2^k$, so k is about log(n)
  - worst running time is O(log n)

# Search: tree of comparisons



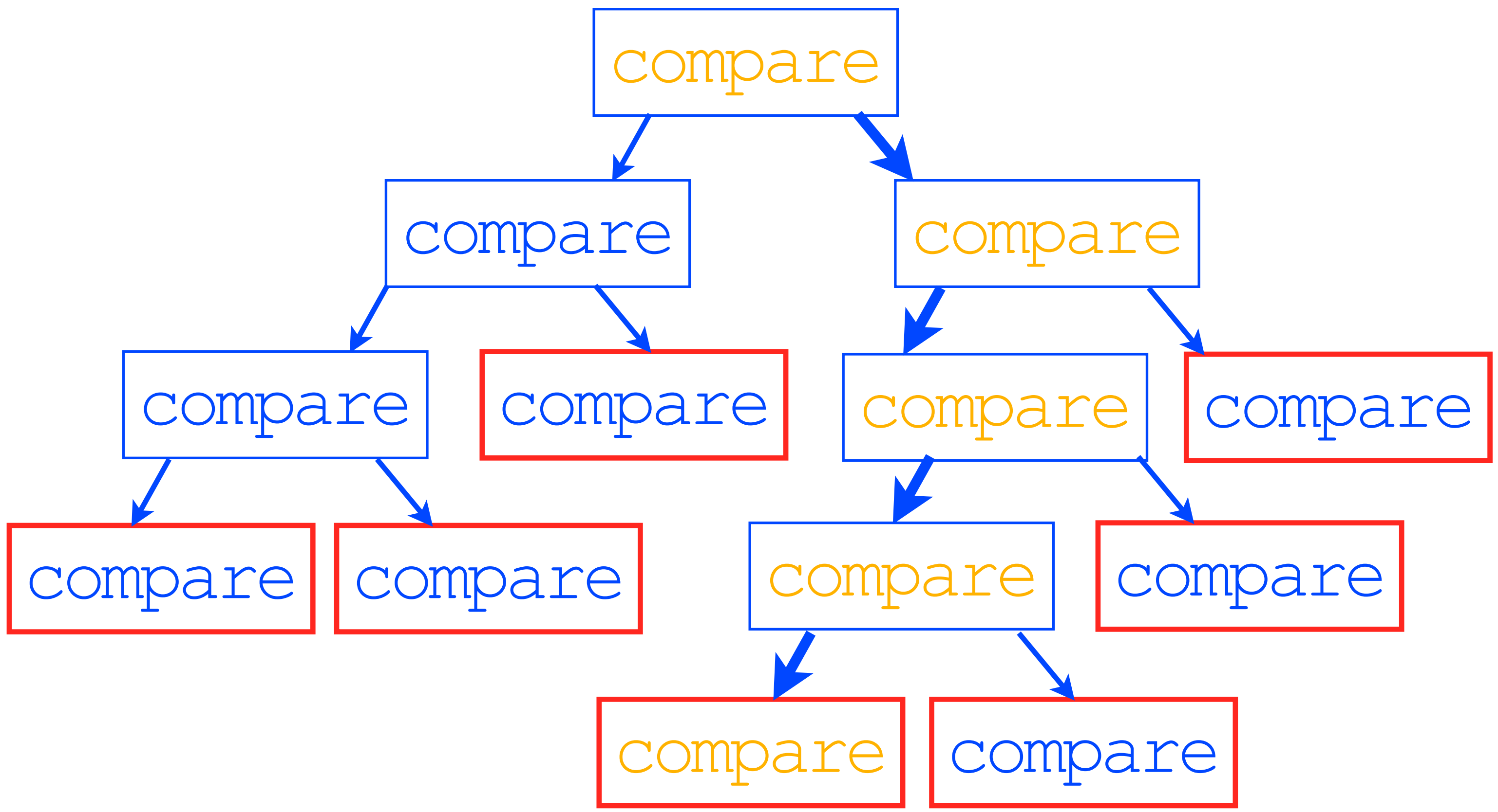

- tree of comparisons : essentially what the algorithm does

# Search: tree of comparisons




- tree of comparisons : essentially what the algorithm does
  - each program execution follows a certain path

# Search: tree of comparisons




- tree of comparisons : essentially what the algorithm does
  - each program execution follows a certain path
  - red nodes are terminal / output
  - the algorithm has to have at least n output nodes... why ?

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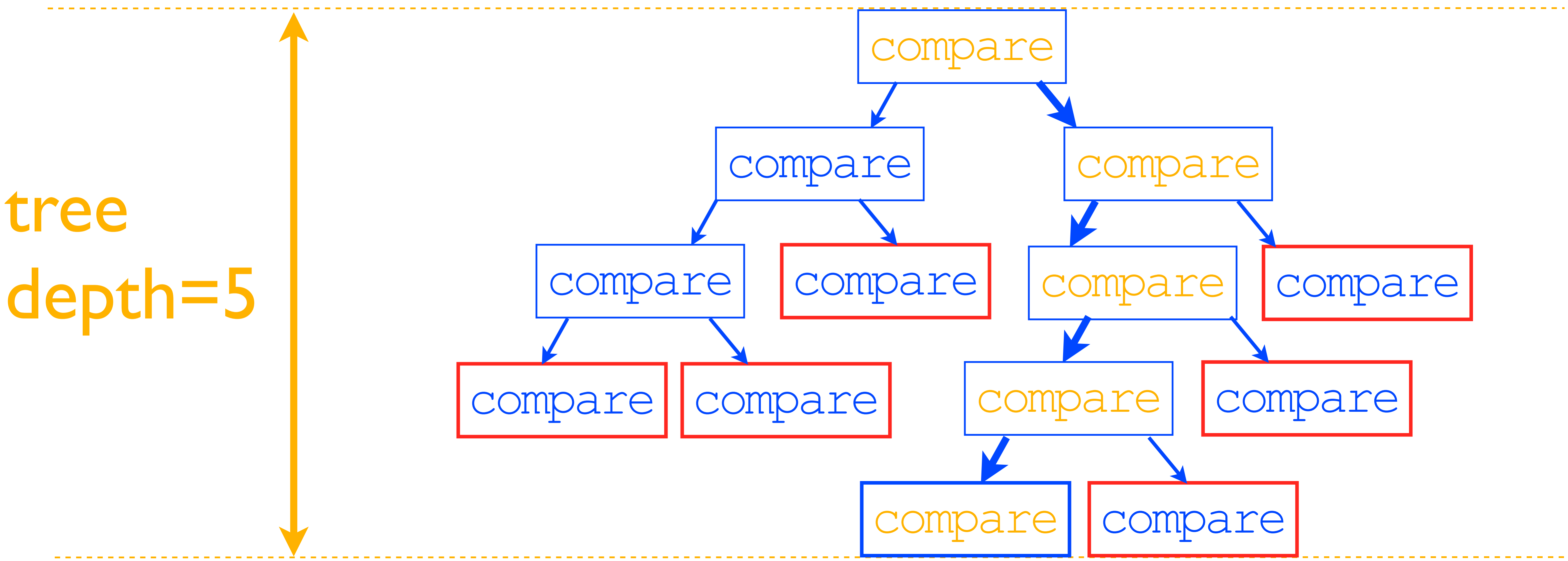


- tree of comparisons : essentially what the algorithm does
  - each program execution follows a certain path
  - red nodes are terminal / output
  - the algorithm has to have n output nodes... why ?
  - if tree is balanced, longest path = tree depth = log(n)

# Bubble Sort

- Simple idea: as long as there is an inversion, swap the bubble
  - inversion = a pair of indices i<j with A[i]>A[j]
  - swap A[i]<->A[j]
    - directly `swap (A[i], A[j]);`
    - code it yourself: `aux = A[i]; A[i]=A[j];A[j]=aux;`
- how long does it take?
  - worst case : how many inversions have to be swapped?
  - $O(n^2)$

# Insertion Sort

- partial array is sorted

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 8 | 20 | 49 | | | | | |

- get a new element V=9

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- get a new element V=9
- find correct position with binary search i=3

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- get a new element V=9
- find correct position with binary search i=3
- move elements to make space for the new element

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| 1 | 5 | 8 | | 20 | 49 | | | | |

- insert into the existing array at correct position

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 8 | 9 | 20 | 49 | | | | |

# Insertion Sort - variant

- partial array is sorted

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
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# Insertion Sort - variant

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- get a new element V=9; put it at the end of the array

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- Move in V=9 from the back until reaches correct position

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# Insertion Sort – variant

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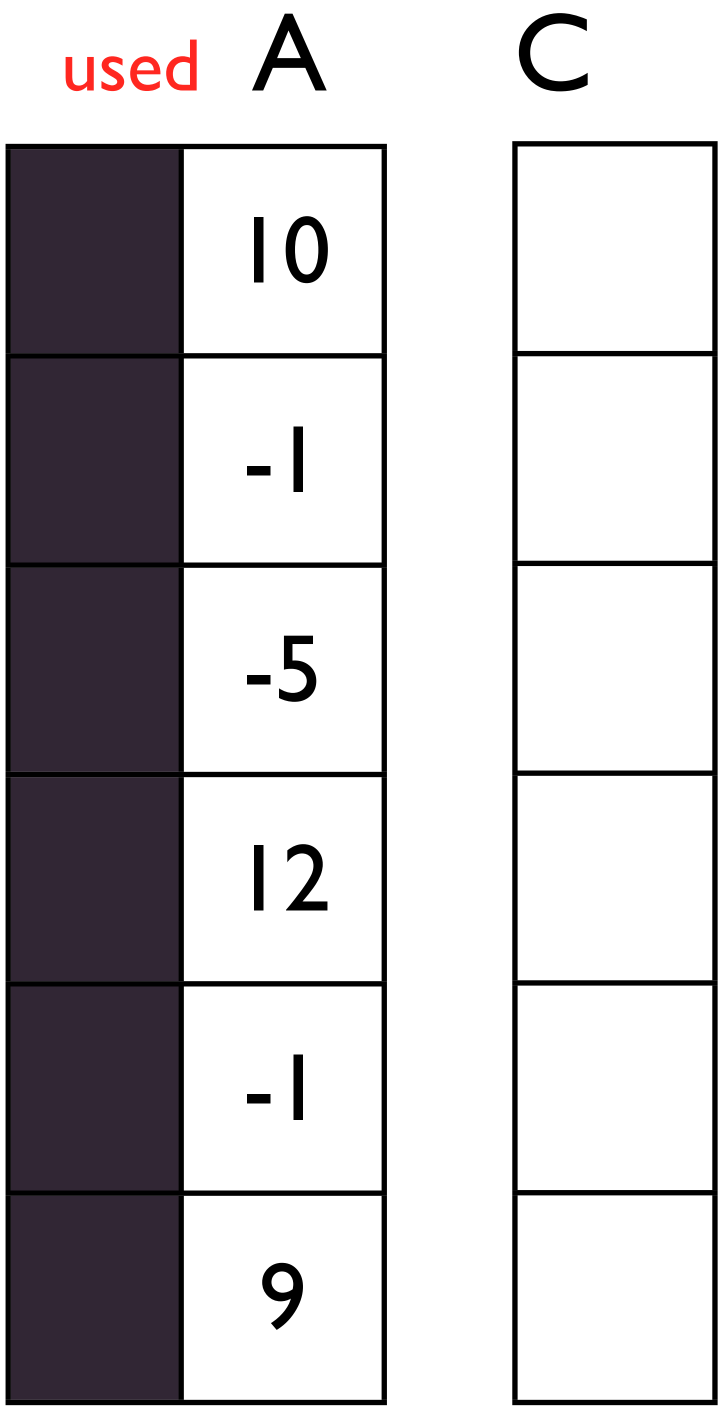
# Insertion Sort Running Time

- For one element, there might be required to move $O(n)$ elements (worst case $\Theta(n)$)
  - $O(n)$ insertion time
- Repeat insertion for each element of the n elements gives $n*O(n) = O(n^2)$ running time

# Selection Sort

- sort array A[] into a new array C[]
- while (condition)
  - find minimum element x in A at index i, ignore "used" elements
  - write x in next available position in C
  - mark index i in A as "used" so it doesn't get picked up again
- Insertion/Selection Running Time = $O(n^2)$



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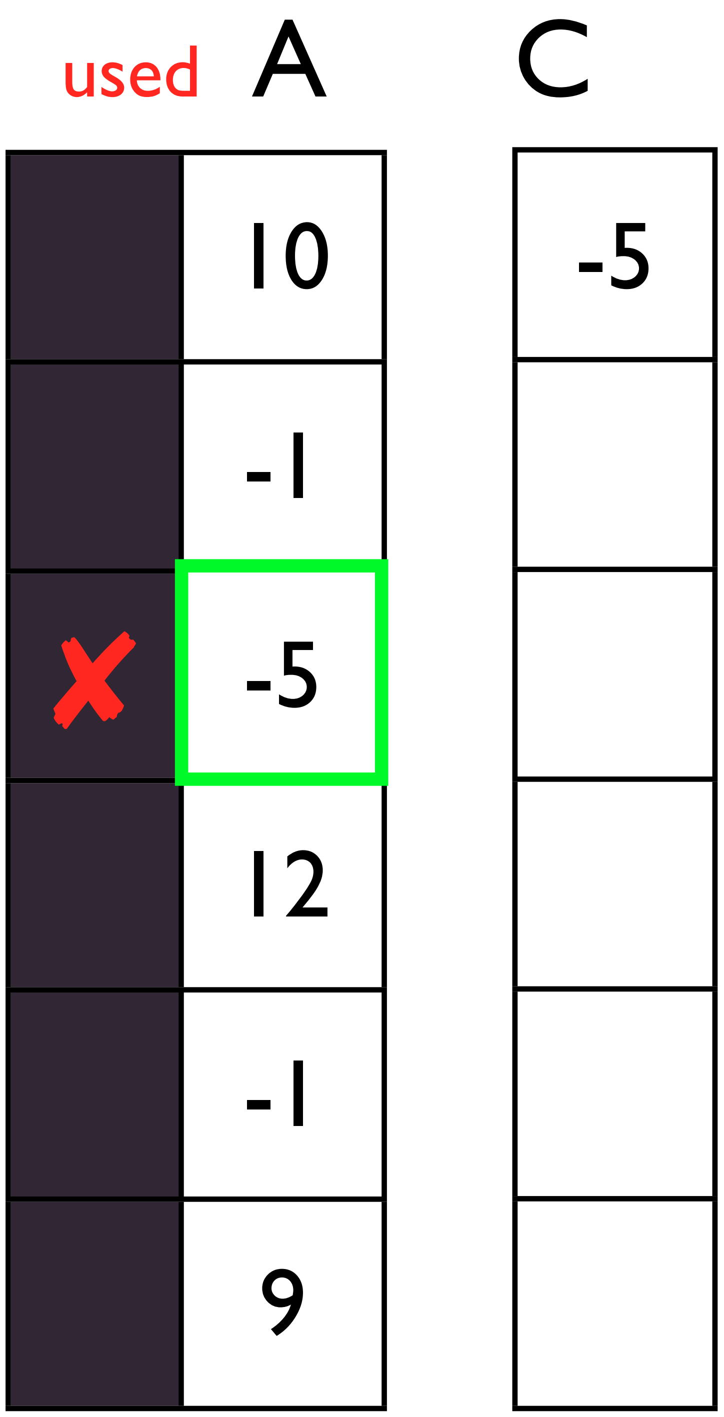

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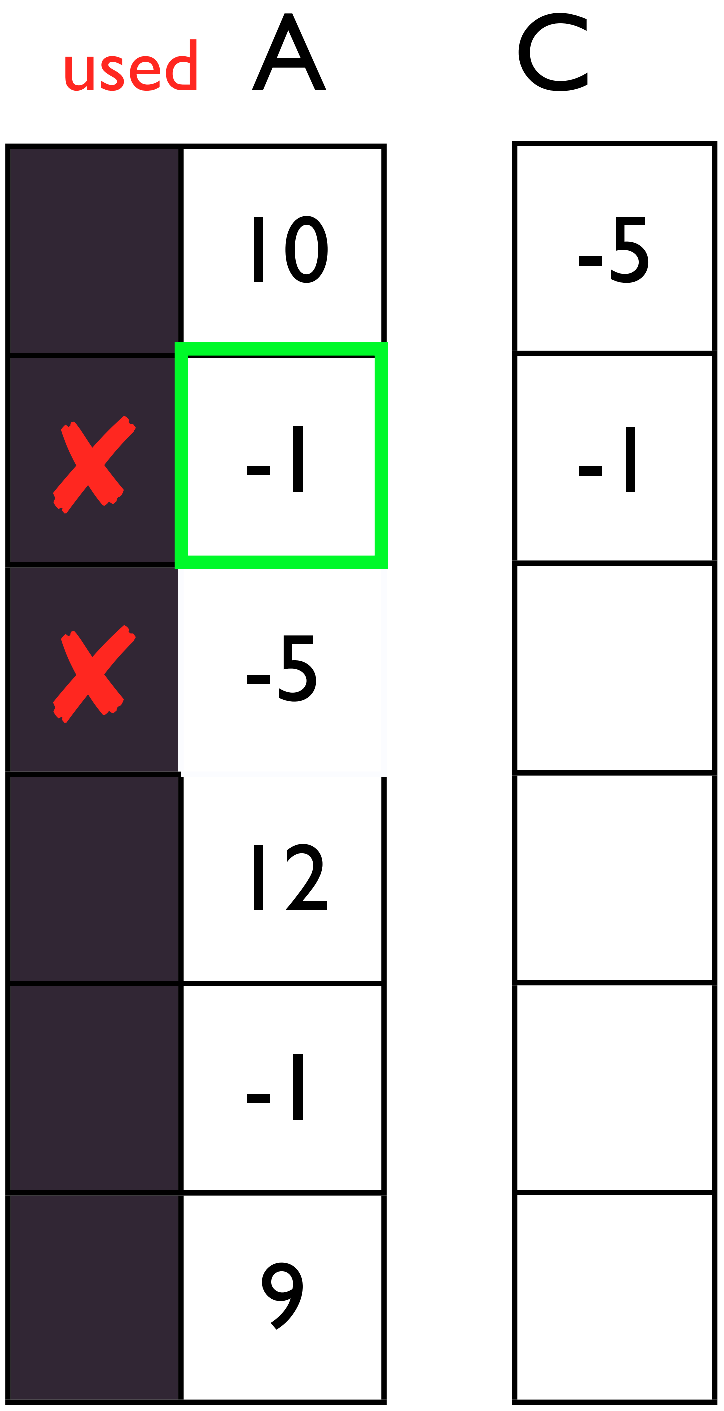

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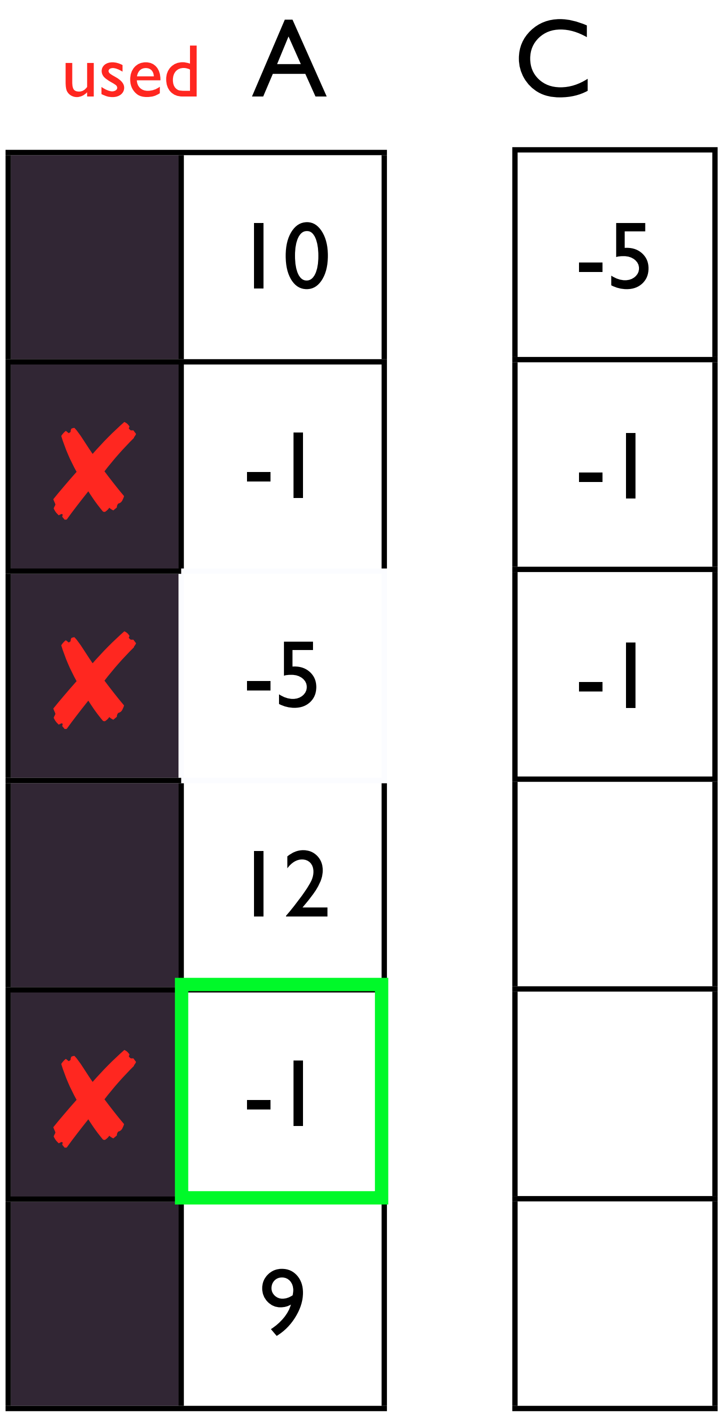

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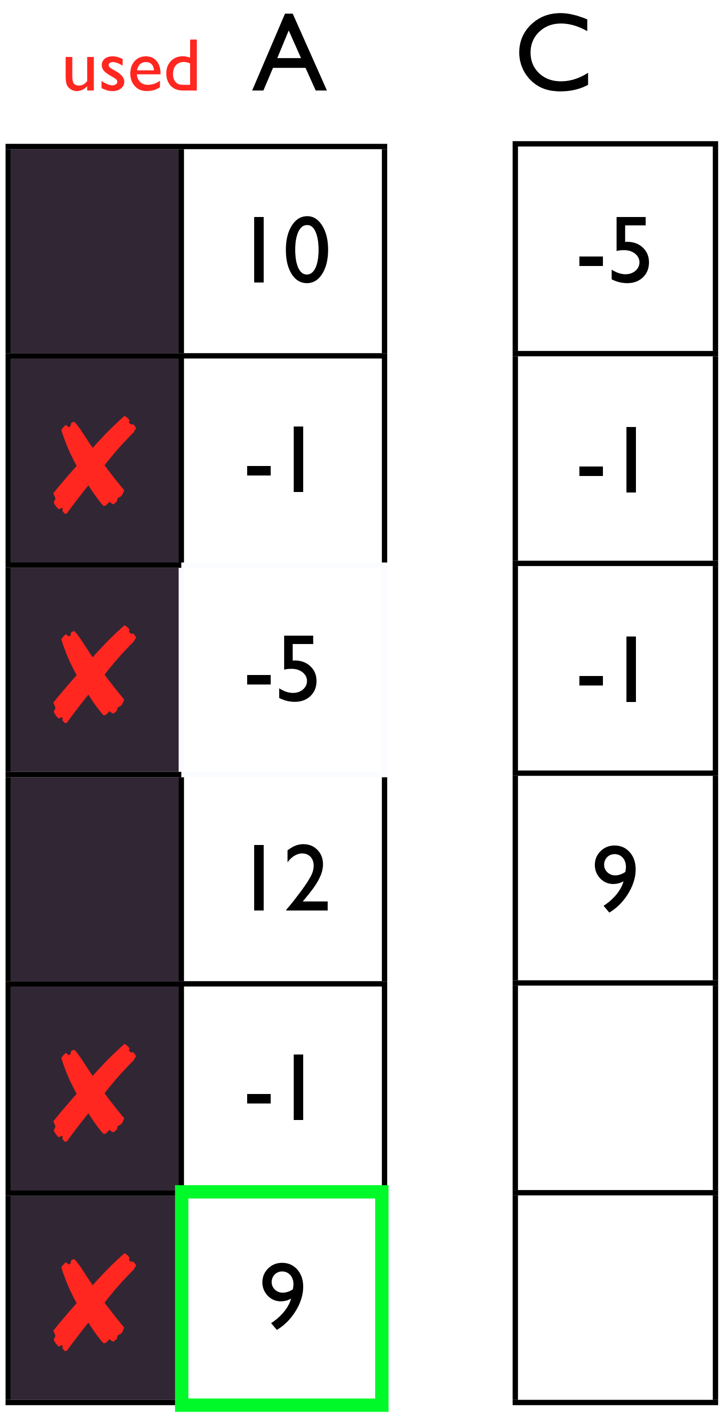

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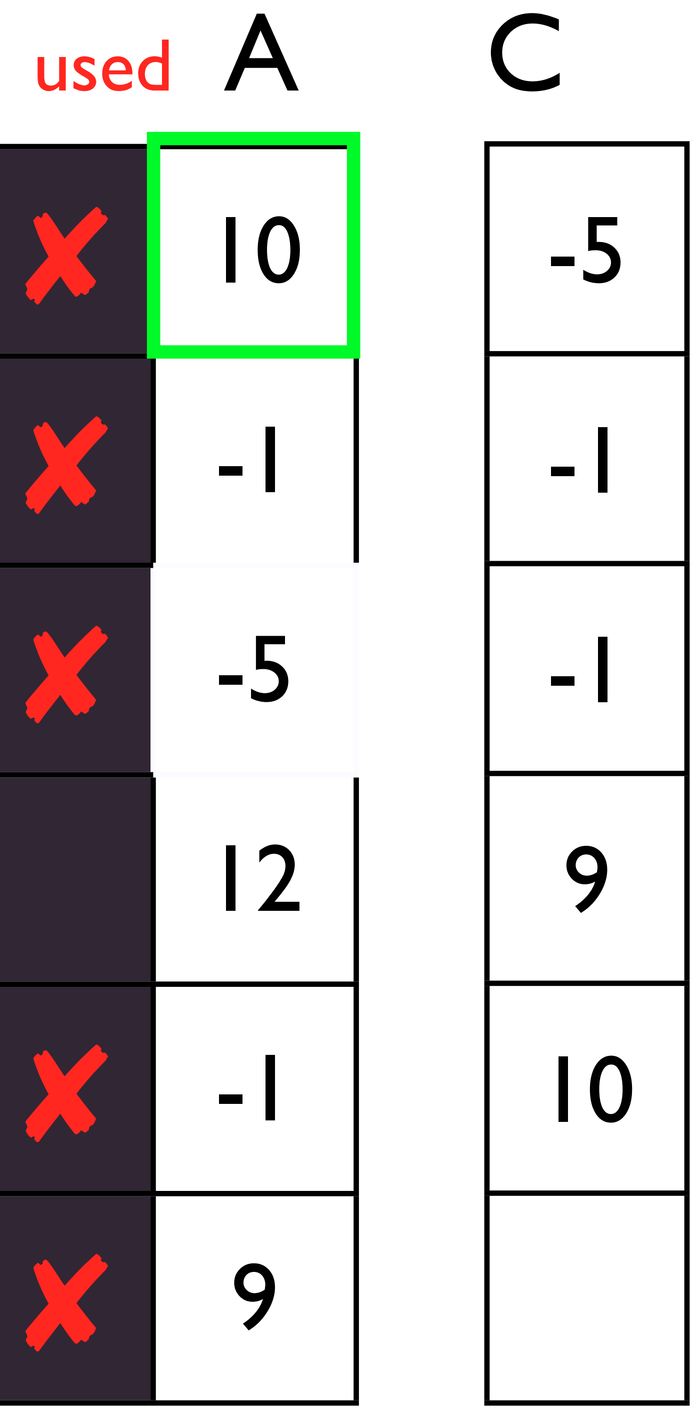

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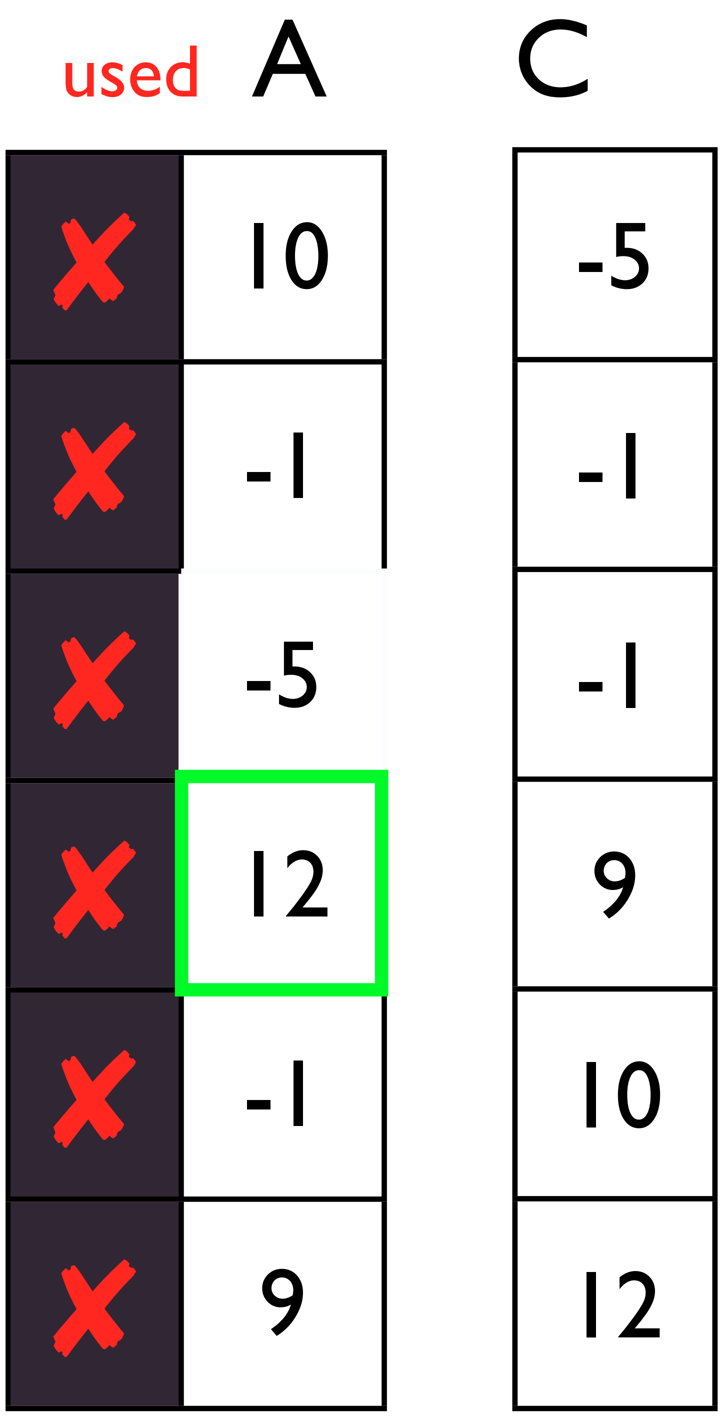


# Merge two sorted arrays

- two sorted arrays
  - A[] = { 1, 5, 10, 100, 200, 300}; B[] = {2, 5, 6, 10};
- merge them into a new array C
  - index i for array A[], j for B[], k for C[]
  - init i=j=k=0;
  - while (what_condition_?)
    - if (A[i] <= B[j]) { C[k]=A[i], i++ } //advance i in A
    - else {C[k]=B[j], j++} // advance j in B
    - advance k
  - end_while

# Merge two sorted arrays

- complete pseudocode
  - `index i for array A[], j for B[], k for C[]`
  - `init i=j=k=0;`
  - `while (`k < size(A)+size(B)+1`)`
    - `if(i>size(A) {C[k]=B[j], j++}` // copy elem from B
    - `else if (j>size(B) {C[k]=A[i], i++}` // copy elem from A
    - `else if (A[i] <= B[j]) { C[k]=A[i], i++ }` //advance i
    - `else {C[k]=B[j], j++}` // advance j
    - `k++` //advance k
  - `end_while`

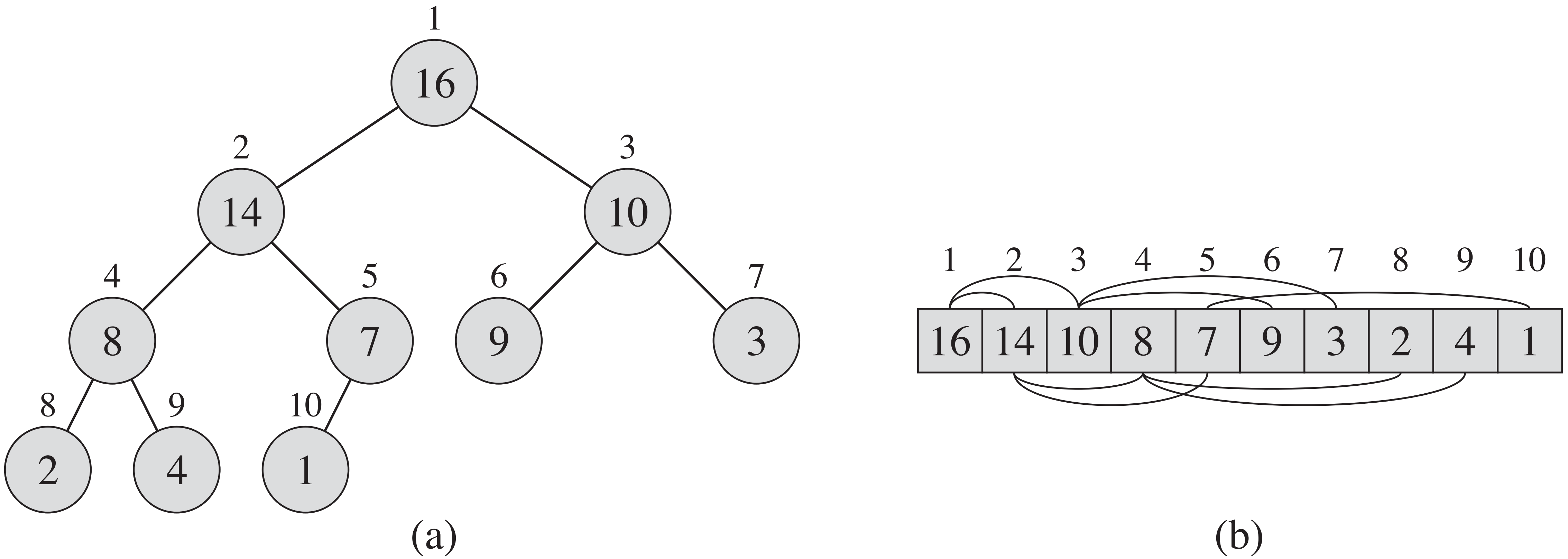
# MergeSort

- divide and conquer strategy
- MergeSort array A
  - divide array A into two halves A-left,  A-right
  - MergeSort A-left (recursive call)
  - MergeSort A-right (recursive call)
  - Merge (A-left, A-right) into a fully sorted array
- running time : O(nlog(n))

# MergeSort running time

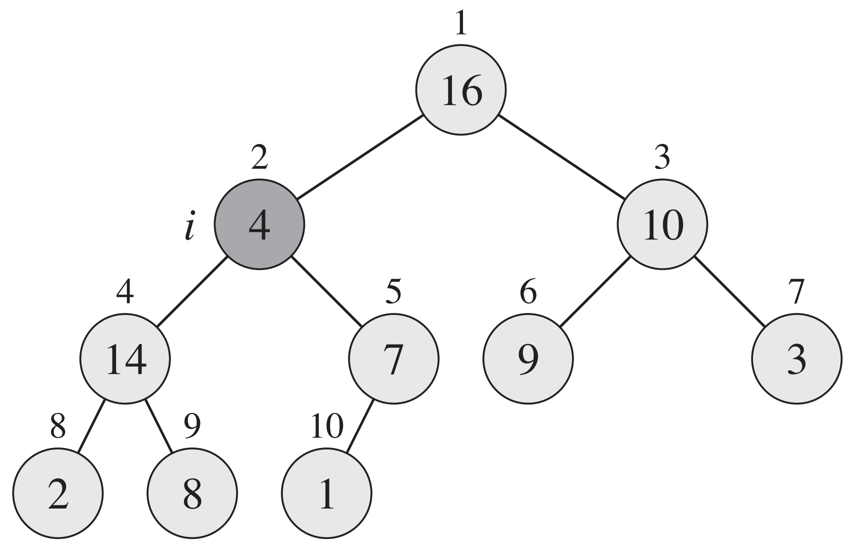
- $T(n) = 2T(n/2) + \Theta(n)$
  - 2 sub-problems of size n/2 each, and a linear time to combine results
  - Master Theorem case 2 (a=2, b=2, c=1)
  - Running time $T(n) = \Theta(n \log n)$

# Heap DataStructure

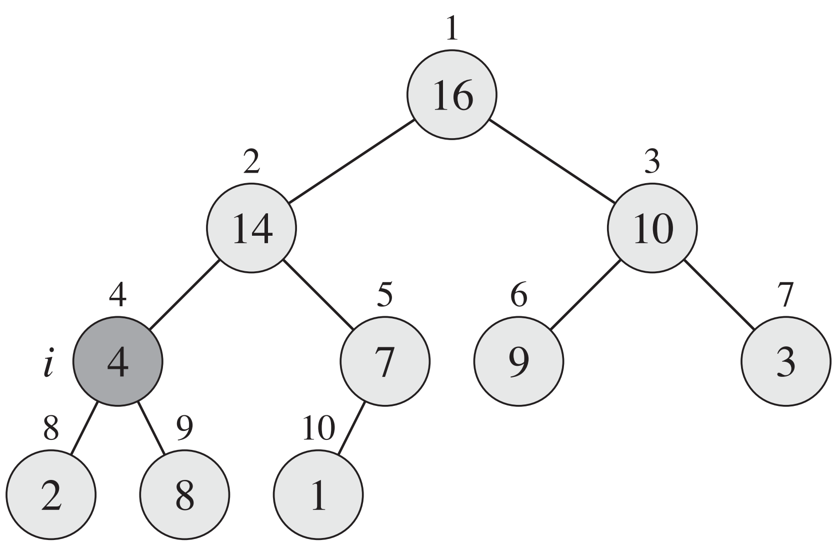



- binary tree
- max-heap property : parent > children

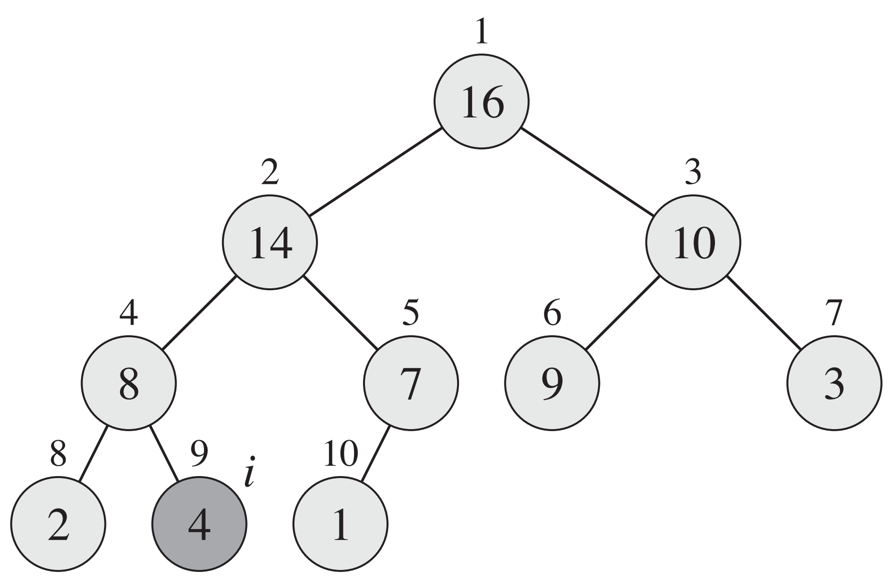
# Max Heap property



(a)



(b)



(c)

- Assume the Left and Right subtrees satisfy the Max-Heap property, but the top node does not
- Float down the node by consecutively swapping it with higher nodes below it.

# Building a heap

- Representing the heap as array datastructure
  - Parent(i) = i/2
  - Left_child(i)=2i
  - Right_child(i) = 2i+1
- A = input array has the last half elements leafs
- MAX-HEAPIFY the first half of A, reverse order

```
for i=size(A)/2 downto 1
    MAX-HEAPIFY (A,i)
```

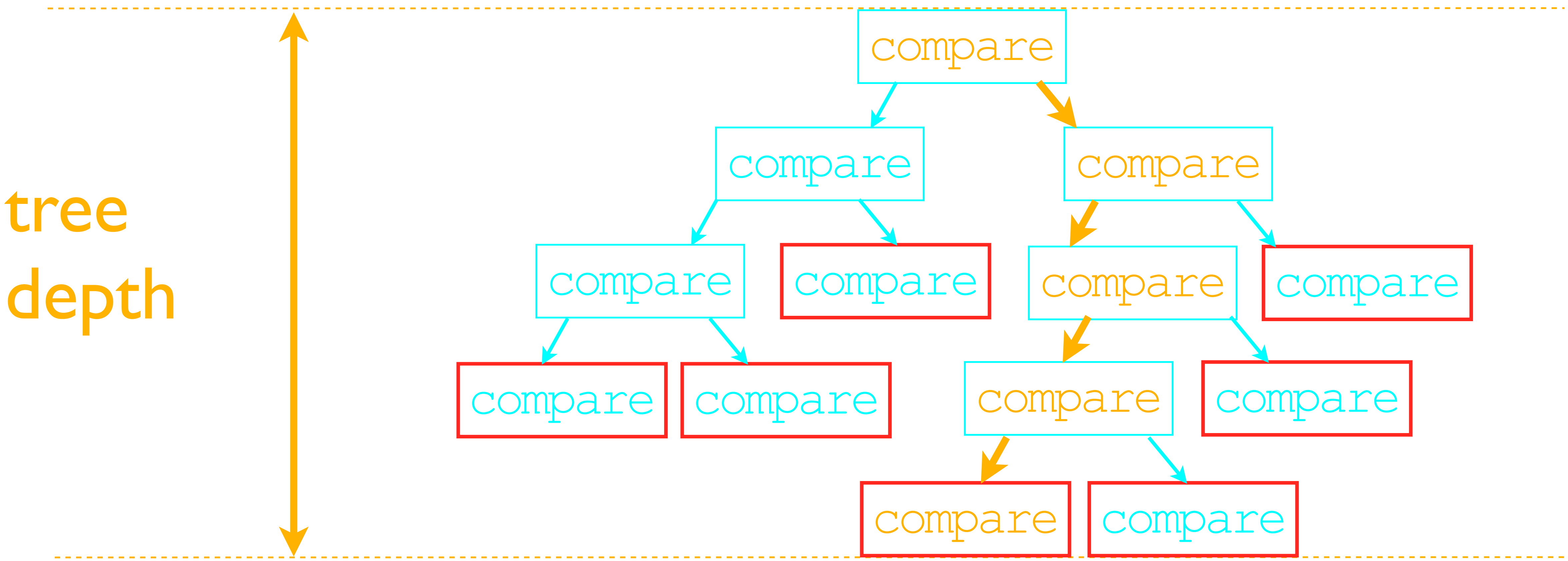
# Heapsort

- Build a Max-Heap from input array
- LOOP
  - swap heap_root (max) with a leaf
  - output (take out) the max element; reduce size
  - MAX-HEAPIFY from the root to maintain the heap property
- END LOOP
- the output is in order

# HeapSort running time

- Max-Heapify procedure time is given by recurrence
  - $T(n) \leq T(2n/3) + \Theta(1)$
  - master Theorem $T(n)=O(\log n)$
- Build Max-Heap : running n times the Max-Heapify procedure gives the running time $O(n\log n)$
- Extracting values: again run n times the Max-Heapify procedure gives the running time $O(n\log n)$
- Total $O(n\log n)$

# Sorting : tree of comparisons




- tree of comparisons : essentially what the algorithm does
  - each program execution follows a certain path
  - red nodes are terminal / output
  - the algorithm has to have n! output nodes... why ?
  - if tree is balanced, longest path = tree depth = n log(n)

# QuickSort - pseudocode

- QuickSort(A,b,e) *//array A , sort between indices b and e*
  - q = Partition(A,b,e) *// returns pivot q, b<=q<=e*
  - *// Partition also rearranges A so that if* `i<q then A[i]<=A[q]`
  - *//* *and if* `i>q then A[i]>=A[q]`
  - if(b<q-1) QuickSort(A,b,q-1)
  - if(q+1<e) QuickSort(A,q+1,e)
- After Partition the pivot index contains the right value:

b=0 q=3 e=9

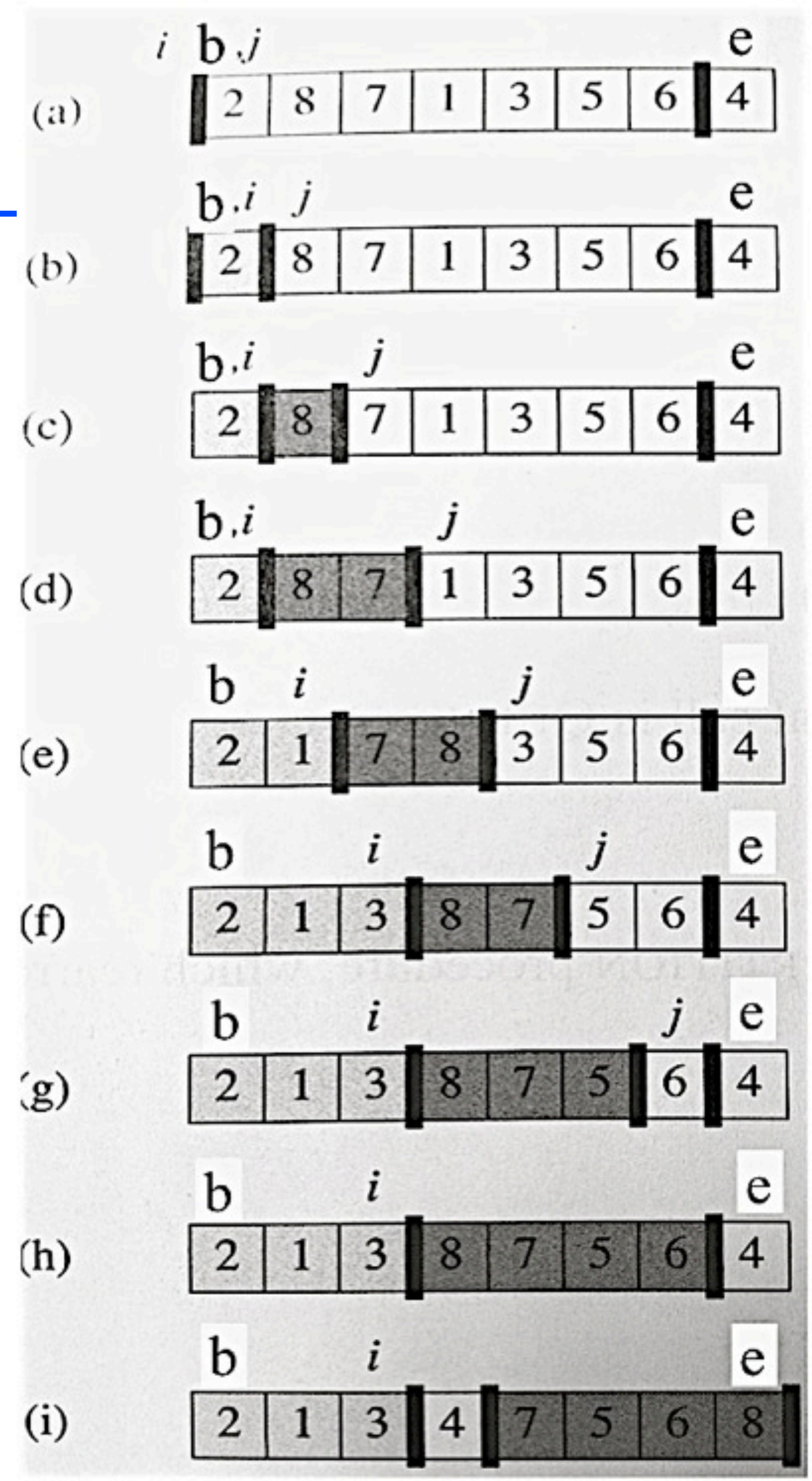
| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| -3 | 0 | 5 | 7 | 18 | 8 | 7 | 29 | 21 | 10 |

# QuickSort Partition

- TASK: rearrange A and find pivot q, such that
  - all elements before q are smaller than A[q]
  - all elements after q are bigger than A[q]
- Partition (A, b, e)
  - x=A[e]//pivot value
  - i=b-1
  - for j=b TO e-1
    - if A[j]<=x then
    - i++; swap A[i]<->A[j]
  - swap A[i+1]<->A[e]
  - q=i+1; return q

# Partition Example

- set pivot value x = A[e], `// x=4`
  - i =index of last value < x
  - i+1 = index of first value > x
- run j through array indices b to e-1
  - if A[j] <= x `//see steps (d),(e)`
    - swap (A[j] , A[i+1]);
    - i++; `//advance i`
- move pivot in the right place
  - swap (pivot=A[e] , A[i+1])
- return pivot index
  - return i+1



# QuickSort time

- Partition runs in linear time
  - If pivot position is q, the QuickSort recurrence is $T(n) = n + T(q) + T(n-q)$
- Best case q is always in the middle
  - $T(n)=n+2T(n/2)$, overall $\Theta(n*\log n)$
- Worst case: q is always at extreme, 1 or n
  - $T(n) = n + T(1) + T(n-1)$, overall $\Theta(n^2)$

# QuickSort Running Time

- Depends on the Partition balance
- Worst case: Partition produces unbalanced split n = (1, n-1) most of the time
  - results in $O(n^2)$ running time
- Average case: most of the time split balance is not worse than n = (cn, (1-c)n) for a fixed c
  - for example c=0.99 means balance not worse than (1/100*n, 99/100*n)
  - results in O(n*logn) running time
  - can prove that on expectation (average), if pivot value is chosen randomly, running time is Θ(n*logn), see book.

# Median Stats

- Task: find k-th element
  - k=n is same as "find MAX", or "find highest"
  - k=2 means "find second-smalles"
  - k=1 is same as "finding MIN"
- naive approach, based on selection sort:
  - find first smallest (MIN)
  - then find second smallest, third smallest, etc; until the k-th smallest element
  - Running Time: average case $k=\Theta(n)$, and each "finding" min takes $\Theta(n)$ time, so total $\Theta(n^2)$

# Median Stats

- "find k-th element"
- better approach, based on QuickSort
- Median(A,b,e,k) *//find k-th greatest in array A , sort between indices b=1 and e=n*
  - q = Partition(A,b,e) *//returns pivot index q, b<=q<=e*
  - *//Partition also rearranges A so that if* `i<q then A[i]<=A[q]`
  - *//* *and if* `i>q then A[i]>=A[q]`
  - if(q==k) return A[q] *//found the k-th greatest*
  - if(q>k) Median(A,b,q-1,k)
    - else Median(A,q+1,e,q-k)
- Not like Quiksort, Median recursion goes only on one side, depending on the pivot
- why the second Median call has $k_{(new)}=q-k_{(old)}$ ?

# Median Stats

- Running Time of Median
- the recursive calls makes $T(n) = n + \max(T(q), T(n-q))$
  - "max" : assuming the recursion has to call the longer side
  - just like QuickSort, average case is when q is "balanced", i.e. $cn<q<(1-c)n$ for some constant $0<c<1$
  - balanced case: $T(n) = n + T(cn)$; Master Theorem gives linear time $\Theta(n)$
  - expected (average) case can be proven linear time (see book); worst case $\Theta(n^2)$
- worst case can run in linear time with a rather complicated choice of the pivot value before each partition call (see book)

# Linear-time Sorting: Counting Sort

- Counting Sort (A[]) : count values, NO comparisons
- STEP 1 : build array C that counts A values
  - `init C[]=0 ;`
  - `run index i through A`
    - `value = A[i]`
    - `C[value] ++;` *//counts each value occurrence*
- STEP 2: assign values to counted positions
  - `init position=0;`
  - `for value=0:RANGE`
    - `for i=1:C[value]`
      - `position = position+1;`
    - `OUTPUT[position]=value;`

# Counting Sort

- n elements with values in k-range of $\{v_1, v_2, \ldots v_k\}$
  - for example: 100,000 people sorted by age: n=100,000; k = {1,2,3,...170} since 170 is maximum reasonable age in years.
- Linear Time $\Theta(n+k)$
  - Beats the bound? YES, linear $\Theta(n)$, not $\Theta(n*\log n)$, if k is a constant
  - Definitely appropriate when k is constant or increases very slowly
  - Not good when k can be large. Example: sort pictures by their size; n=10000 (typical picture collection), size range k can be any number from 200Bytes to 40MBytes.
- Stable (equal input elements preserve original order)