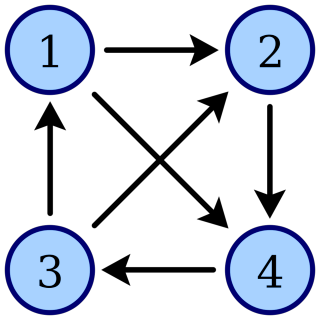
REC 12: Graphs

Problem 1 Graph Representation



For each graph representation below, describe a modification of the undirected graph representation which allows it to store each edge's direction above. In other words, by observing your new representation scheme, one learns not only which vertices share an edge but the direction of that edge (e.g. from node 2 to node 4 above). To demonstrate your method, express the directed graph shown above using your method. There may be more than one way to do this, choose a solution which is simple and uses the least computer memory.

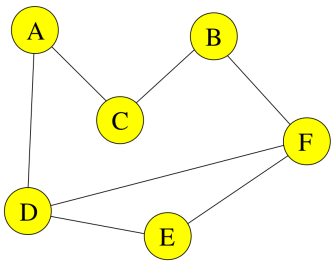
Note: By convention we number rows from top to bottom and columns from left to right. (So the "first" row of a matrix are all the numbers on top side while the "first" column are all the values on its left side)

i Adjacency Matrix

ii Adjacency List

Problem 2 Graph quiz

Refer to the graph below for the following problems.



i How many vertices, n, does the graph have?

ii How many edges, m, does the graph have?

iii How many connected components does the graph have?

iv Give a path of length 5 which does not repeat any edges from vertex **B** to vertex **E**.

v Give the shortest path from vertex **B** to vertex **E**. (Assume each edge has equal cost).

vi Is this graph a tree? Explain why or why not.

vii Show the *adjacency list* representation for this graph, where the vertices are ordered alphabetically and each adjacency list is also ordered alphabetically.

viii Show the *Adjacency Matrix* of the graph above. Order the vertices alphabetically (so the first row and column correspond to A, the second row and column correspond to B, ...).

Problem 3 Bipartite graph

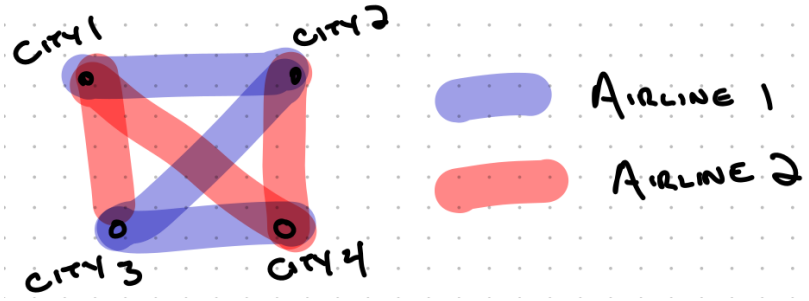
A town with a particularly has 20 doctors and 100 patients. The physicians association puts together its annual report which claims, 'The doctors have seen, on average, 30 patients this year'. The insurance association which puts together its own annual report claims, 'The patients have seen, on average, 5 doctors this year'. (Assume that no doctor visits another (that is, nobody is both a doctor and a patient) Someone a particular interest in Graph Theory, correctly observes, 'At least one of these reports must contain an error!'. Explain why this resident knows one report is in error.

Hint: try representing the network of appointments which have taken place as a graph. All doctors are nodes on the left while all patients are nodes on the right. An edge connects the two if the doctor has treated the patient in the past year (a patient may have seen multiple doctors). As with many math problems, if you're stuck, try drawing a few small, concrete examples do some quick computations and see if you notice any patterns. This one is fun, we hope you enjoy it!

Problem 4 connectivity: graph vs its complement

Consider a graph which represents the flights made by two airlines among N cities. Between every pair of cities exactly one airline flies the route. Show that there exists an airline which may carry a passenger from any city to any other city using only their own routes.

For example, consider the particular graph:



Airline 1 is able to carry passengers from any city to any other using only their routes. Notice that to achieve this Airline 1 may use indirect flights.

Problem 5 Draw the graph

If possible, draw an example of each graph as described. Otherwise, describe why such a graph does not exist. Unless otherwise specified, each graph is undirected and has exactly 5 nodes. Please use uppercase letters starting at A to index the nodes of your graph.

Remember, a path is a sequence of unique, adjacent edges.

i An acyclic graph where every pair of nodes has an edge.

ii A rooted tree where B is the root[1]

iii A graph which contains no cycles but is not a tree.

iv A weighted graph where every path from A to E has weight 2.

v A rooted tree with the minimal number of leafs.

vi A strongly connected, directed graph with 5 nodes which has the minimal number of edges.

A strongly connected directed graph has a directed path to and from every pair of nodes. A directed path is a sequence of adjacent edges which moves with the direction of the edges. Note that because a directed path can't go 'backwards', we may not use the same path to connect A to B as we would to connect B to A.

[1] Remember, any node of a tree may be a root. To be clear here, please label your root. (Conventionally we draw them at the top, but this isn't universally followed so you'd do well to label your root, clarifying to your reader this isn't any old tree but a rooted tree).

Problem 6 Solve Recurrence

Solve eac of the following recurrences by substitution. Assume a base case of $T(1) = 1$. As part of your solution, you will need to establish a pattern for what the recurrence looks like after the k-th substitution. Check that this pattern is consistent with your substitutions, but you do not need to formally prove it is correct via induction.

i $T(n) = T(n-2) * 7$