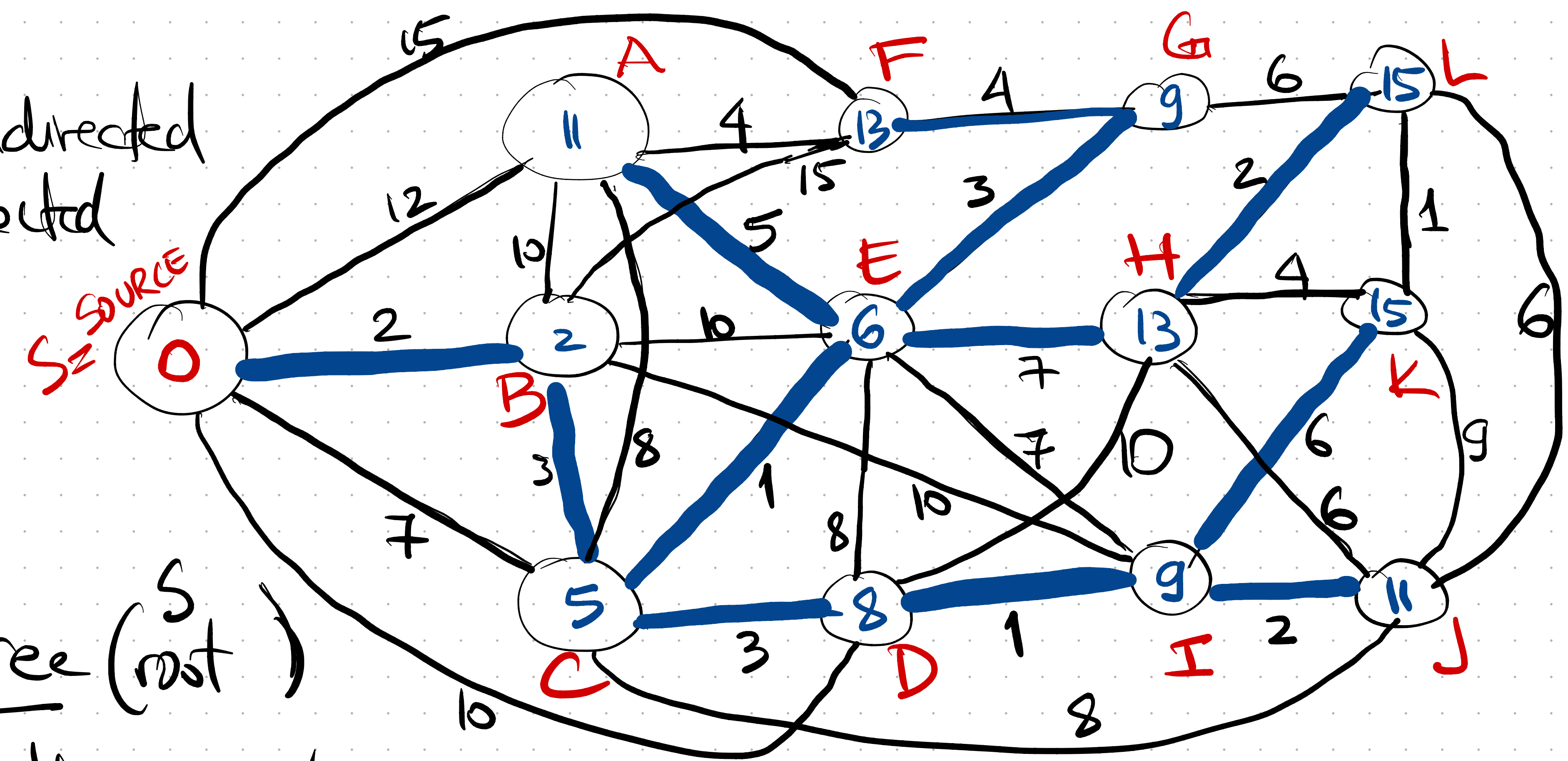
Dijkstra Alg for Single-Source Shortest Path in Graphs (Dir or unDir)

Given: Graph(V,E) directed or undirected
Source vertex S | assume connected
Each edge has positive weight
$W(u,v) \geq 0$



Goal: Produce the Dijkstra Tree (root S) to all reachable vertices in the graph such that the tree-path to every vertex (u) Dijkstra(S, u) is a SP(S, u) in the graph.
Shortest path

Dijkstra Algorithm: at all times vertices are partitioned into 3 groups:

TREE	QUEUE	NOT YET
already in Dij tree (SP done)	candidates (with ESP) to add to Dij-tree	not connected yet to tree, ESP = ∞

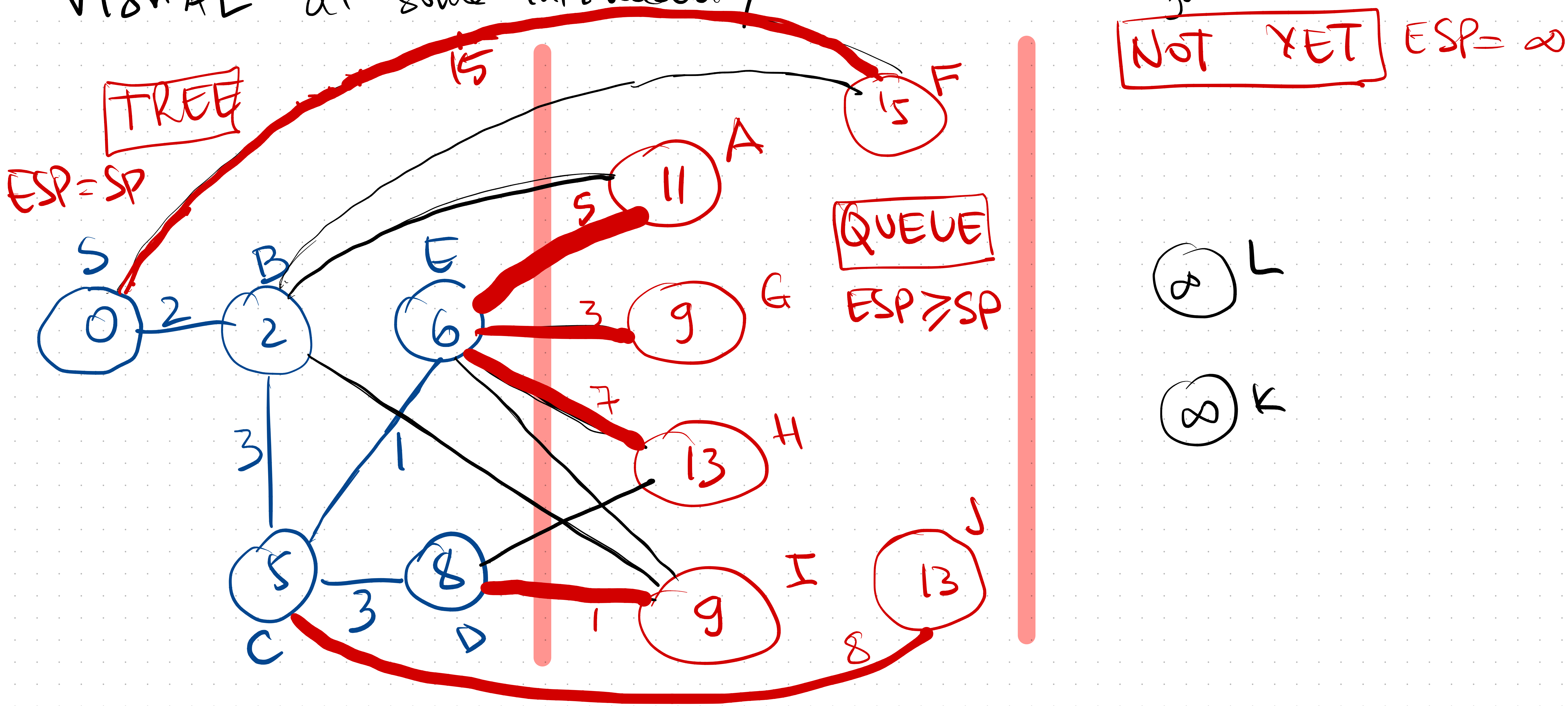
At any time $SP(s,u)$ = true SP in the graph (unknown)

written value inside each node

$\boxed{ESP(s,u)}$ = the SP found so far. $\boxed{ESP(s,u) \geq SP(s,u)}$

TREE vertices: $SP(s,u) = ESP(s,u)$

Visual at some intermediary time in the Algorithm:



Dijkstra Theorem: • The min-ESP in the queue is actually SP (in the example G and I have min ESP=9)

• Then the node in queue with min ESP is added to Tree with parent/edge that give the SP=ESP. (in example G can be added with parent E (edge EG) or I can be added with parent D (edge DI))

Algorithm: • maintain the queue, the Tree and update all ESPs.

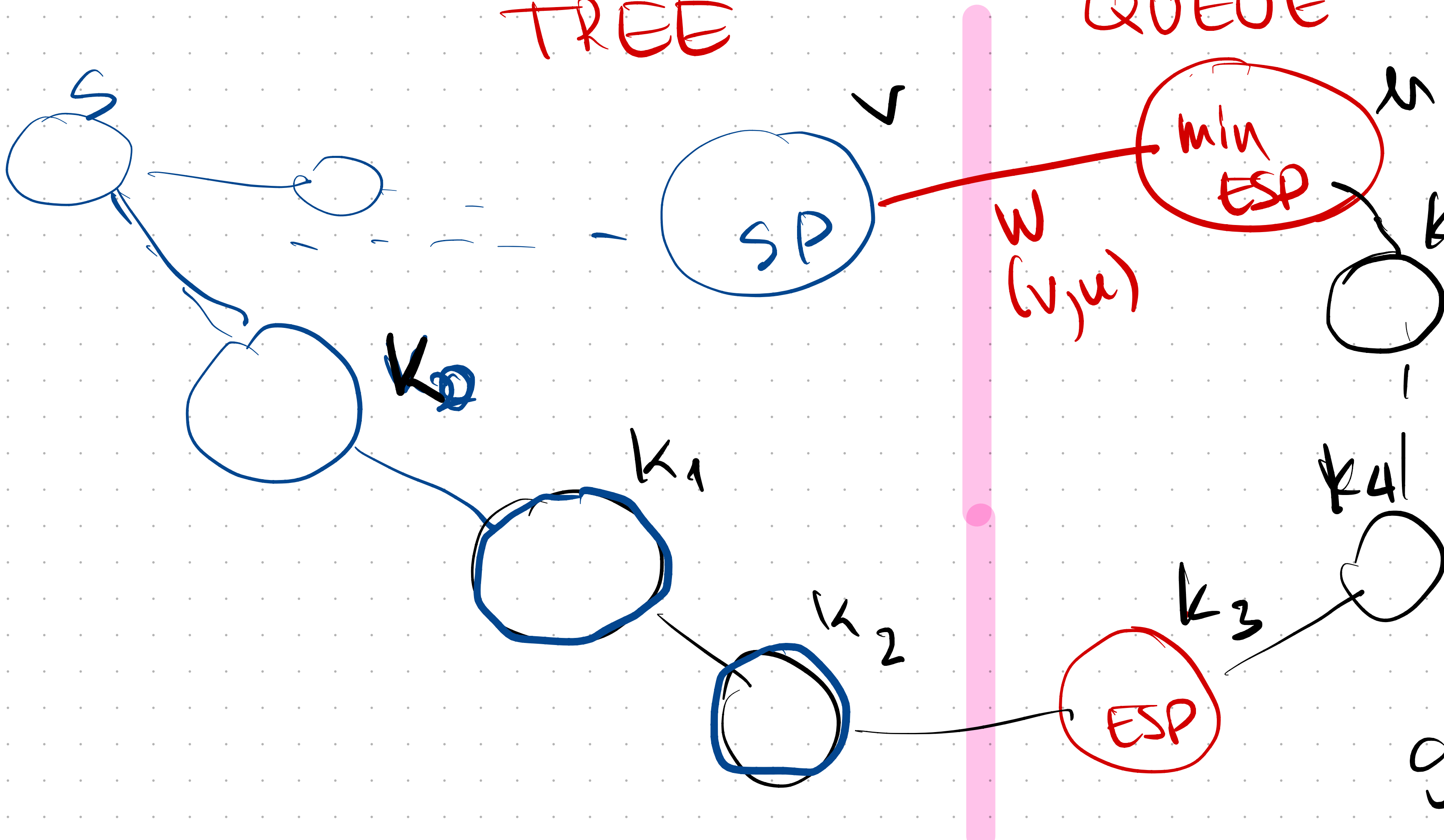
• When min-ESP in queue is added (ex. edge E→G)

- G.SP = G.ESP ; G.parent = E
- For all u ∈ G.adj-list: update u.ESP if necessary:

  if u.ESP > G.SP + w(G→u)

  then u.ESP = G.SP + w(G→u) ; u.parent = G

Proof: Assume (hypothetical) that min ESP in the queue is not SP node u.



Let v be the node in the tree that connects u for min ESP

so $v.SP + w(v \to u) = u.ESP$

Since $u.ESP > u.SP$ lets look at the path $s \to u$ that give the SP = shortest path:

$$s \to k_0 \to k_1 \to \dots \to k_n \to u$$

then $u.SP = w(s \to k_0) + w(k_0 \to k_1) + \dots + w(k_{n-1} \to k_n) + w(k_n \to u)$

$\boxed{\text{Key}}$: at some point this path crosses from $\boxed{\text{TREE}}$ group to $\boxed{\text{QUEUE}}$

In example thats $k_2 \to k_3$. Then look at $k_3.ESP$

$$k_3.ESP < u.SP < u.ESP$$

CONTRADICTION: $u.ESP$ is not the min-ESP in QUEUE.