CS1800

Discrete Structures

Fall 2019

Lecture 21

11/19/19

Last time

- Order notation

Today

- Comparing growth of functions

- Graphs

Next time

- Continue Graphs

Functional Relations

		E.g.
constant	c	3
poly-logarithmic	$\log^c n$	$\log n$, $\log^2 n = (\log n)^2$, ...
polynomial	n^c	$\sqrt{n} = n^{1/2}$, n, n^2, n^3 $n \log n$
exponential	c^n	2^n, 3^n, 1.1^n
factorial	$n!$	

Claim: $n^2 = O(2^n)$

means: $\exists$ constants $c, n_0 > 0$ s.t. $\underline{n^2 \le c \cdot 2^n}$ $\forall\, n \ge n_0$

Claim 1: $\forall n \ge 4 \quad n^2 \le 2^n$

$c = 1$

$n_0 = 4$ ✓

B.C. $n = 4$ $\quad 4^2 = 16$, $2^4 = 16$ $\quad 16 \le 16$ ✓

I.S. assume that if true at $n = k$, $\quad k^2 \le 2^k$ ind. hyp.

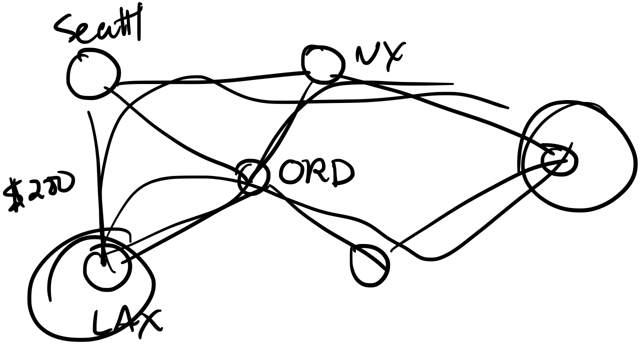
show that true at $n = k+1$ $\quad (k+1)^2 \le 2^{k+1}$

$$
\begin{aligned}
(k+1)^2 &= k^2 + 2k + 1 \\
&\le k^2 + k^2 \qquad k \ge 3 \\
&= 2 \cdot k^2 \\
&\le 2 \cdot 2^k \qquad \text{by ind. hyp.} \\
&= 2^{k+1} \ ✓
\end{aligned}
$$

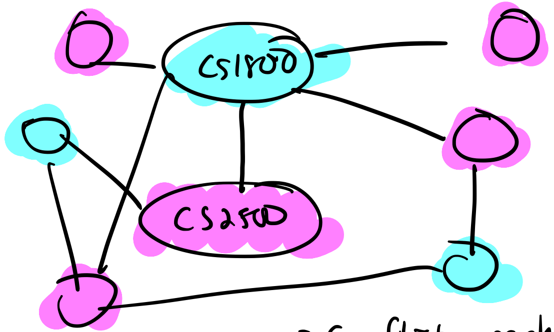
Claim 2:
$$
\begin{aligned}
2k+1 &\le 2k + k \qquad k \ge 1 \\
&= 3k \\
&\le k^2 \qquad k \ge 3
\end{aligned}
$$

$f(n)$	$g(n)$	$f(n)$ is -- $\overset{<}{o(g(n))}$, $\overset{\leq}{O(g(n))}$, $\overset{=}{\Theta(g(n))}$, $\underset{\geq}{\Omega(g(n))}$, $\underset{>}{\omega(g(n))}$
$2n$	$3n$	$2n = \Theta(3n)$ $2n = O(3n)$ $2n = \Omega(3n)$ <u>not</u> $o(\ )$ $\omega(\ )$
n	2^n	$n = o(2^n)$ $n = O(2^n)$
n^2	$3n^2 + n\lg n + 4n + 6$	$n^2 = O(3n^2 + n\lg n + 4n + 6)$ $3n^2 + n\lg n + 4n + 6 = O(n^2)$

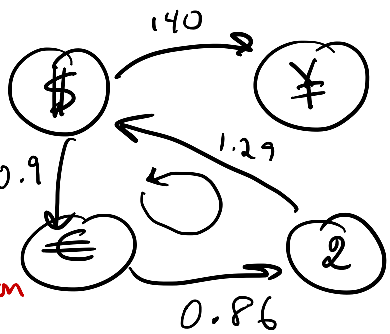
Graph Examples



- shortest paths
  - distances
  - prices



- conflict graphs
  - exam scheduling
  - graph coloring

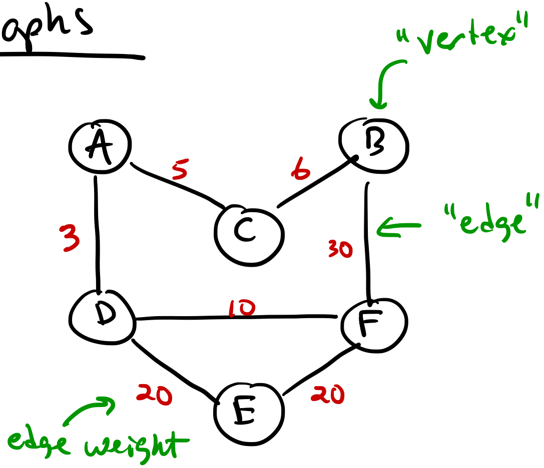


conversion rate

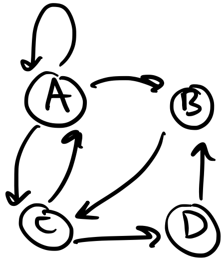
- arbitrage opportunities
  - if product of weights on a cycle > 1
- $0.9 \times 0.86 \times 1.29 = 0.99846$
  < 1

Graphs

G:



Directed graph: ... edges are ordered pairs of vertices.



$E = \{ (A,A), (A,B), (A,C), (B,C), (C,A), \ldots \}$

Formally: An <u>undirected</u> graph is a set of <u>vertices</u> and a set of <u>edges</u>, $G = (V, E)$, where each edge is a set of two vertices

$G = (V, E)$

$G = (V, E, w)$

$w : E \to \mathbb{R}$

$V = \{ A, B, C, D, E, F \}$

$E = \{ \{A, C\}, \{A, D\}, \{B, C\}, \ldots \}$

By convention:

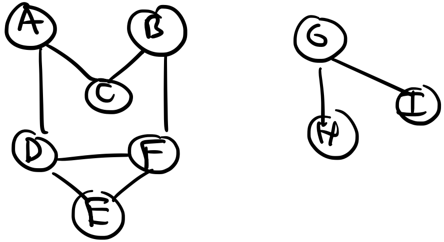
undirected graphs:

- no self loops
- no multiple edges between pairs of vertices

directed graphs:

- do allow self loops
- no multiple directed edges between vertex pairs

Graph properties & notation



- **degree** of vertex:

  # edges connected to it

$\deg(D) = 3$

$\deg(I) = 1$

directed graphs
- in-degree
- out-degree

- **path** from u to v :

  a sequence of edges from u to v

A → B   A — C — B $(\{A, C\}, \{C, B\})$

A → B   A - D — F — B $(\{A, D\}, \{D, F\}, \ldots)$

- **cycles** : path from v back to v

- simple path or cycle

  ↳ no repeated vertices

- Reachability : can you get from u to v?

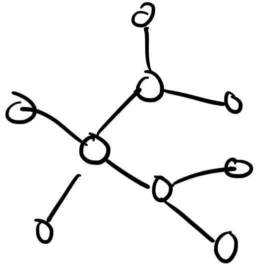
  B is reachable from A

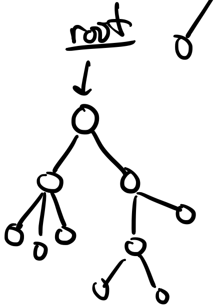
  H is not reachable from A

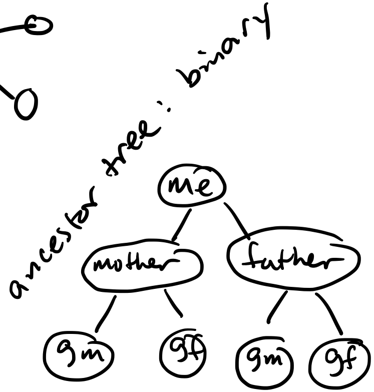
Special graphs

Trees

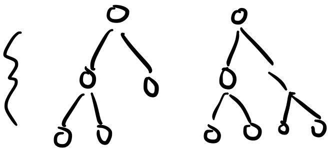
- acyclic graphs
- rooted trees
- multiple trees
  - forest

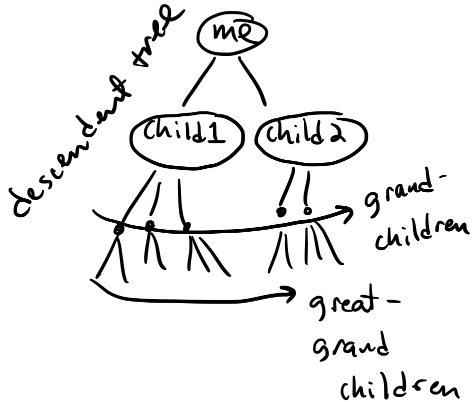




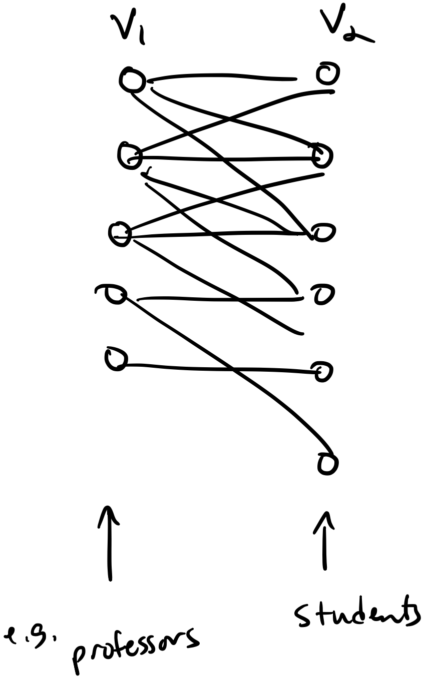


forest





- Bipartite graphs
  - 2 sets of vertices V_1 & V_2
  - all edges are between V_1 & V_2



- edge if student taking professor's class
- weight might be grade or TRACE eval.