REC 9: Counting

Problem 1

A delivery driver has 10 **unique** packages to deliver.

i How many ways can the delivery driver select 6 of these packages to deliver?

ii How many ways can the delivery driver select 4 of these packages to discard while delivering the rest? Explain how this problem relates to the first subproblem.

iii How many different routes may the delivery truck drive to deliver 8 of the 10 packages? A route is an ordered sequence of destinations. You may assume each package may only be delivered to its unique destination.

iv Before leaving for the day, the delivery trucks are loaded with packages. How many ways can 120 **unique** packages be loaded into 10 delivery trucks where some trucks may have no packages?

v How many ways can 120 **identical** packages be loaded into 10 delivery trucks where some trucks may have no packages?

vi The delivery truck driver union is concerned that the workload is unequal. How many ways can 120 **identical** packages be loaded into 10 delivery trucks where each truck must have at least 5 packages?

vii How many ways can 120 **unique** packages be loaded into 10 delivery trucks where each truck must have the same number of packages?

Problem 2

A) Count all passwords of exactly 8 capital letters that have a letter occurring at least 5 times. Examples: SAATARAA, TUUURUUE, ABABABBB. There are 26 capital letters.

B)★ (optional, no credit) Count all passwords of exactly 12 capital letters that have a letter occurring at least 5 times.

Problem 3 Problem 9: Help Jack Sparrow count sequence chopping

In how many ways one can arrange symbols (a,b,c,d,e,f,g) into 5 bins preserving relative symbols order ? Solution: It comes down to choosing 4 spots for "bin separators" is a sequence of 7 symbols + 4 separators = 11. So the answer is $\binom{11}{4}$
Jack Sparrow sees this as balls in bins, sort of, but he uses partition/sum rule to break the problem into several disjoint cases, count them separately, and add up.
A case corresponds to exactly how many bins are non-empty; for each case, there are exactly 3(non-empty bins)-1 separators which can be anywhere in between the symbols, that is in 8 possible spots: $|a|b|c|d|e|f|g|$. Here are the cases:
- all symbols to 1 bin, thus 0 separators. Choosing the bin $\binom{5}{1}$; choosing the separators $\binom{8}{0}$
- all symbols to 2 bins thus 1 separators. Choosing the bins $\binom{5}{2}$; choosing the separators $\binom{8}{1}$
- all symbols to 3 bins so 2 separators. Choosing the bins $\binom{5}{3}$; choosing the separators$\binom{8}{2}$
- all symbols to 4 bins so 3 separators. Choosing the bins $\binom{5}{4}$; choosing the separators$\binom{8}{3}$
- all symbols to 5 bins so 4 separator. Choosing the bins $\binom{5}{5}$; choosing the separators$\binom{8}{4}$
Applying product rule for each case then partition rule across cases gives $\sum_{k=1}^{5}\binom{5}{k}\binom{8}{k-1}$.
This is incorrect, why ?
How can it be fixed following initial idea to break into cases by number of non-empty bins?

Problem 4 Problem 6: How many valid dates?

10 men who are pairs of brothers $(a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5)$ are to blind-date 10 women who are pairs of sisters $(x_1y_1, x_2y_2, x_3y_3, x_4y_4, x_5y_5)$ such that any two brothers do not date two corresponding sisters, that is for example if (a_2, x_4) is a date then b_2 cannot date y_4. In how many ways can the dates be arranged? Explain why these solutions are wrong:

i. Jack Sparrow 's solution: There are 10! ways to arrange the dates without restrictions. There are $5! * 2^5$ ways to arrange dates that violates the restriction since it comes down permuting the pairs and then choosing for each pair which brother dates which sister (2 possibilites per pair). So the answer is $10! - 5! * 2^5$. Why is this wrong?

ii. (difficulty ★) Virgil has a solution, also wrong: We need the number of derangements = permutations without fix point for n=5. Examples: 21453, 41253. Not a derangement : 52134 because 2 is in original position. For n=5 there are $D_5 = 44$ derangements which can be counted by brute force or by Inclusion-Exclusion (next exercise). Then the answer is 5! (choose a permutation of the 5 men $a_1..a_5$) * 2^5 (choose which sister to date) * D_5 (choose a derangement for brothers $b_1..b_5$).
Why is Virgil's solution wrong?

Problem 5 Problem 7 Derangements ★ (optional)

A *derangement* of 1 2 3 $\ldots n$ is a permutation that leaves none of these numbers in place. By inspection, the derangements of 123 are 312 and 231. Find the number of derangements of 1 2 3 4 5 using Inclusion-Exclusion

Problem 6 Problem 8 Counting Schedules (optional)

Students are picking their schedules for next semester and need to figure out their NUPath requirements. The overall course counts for each of the attributes are as follows:

- Societies/Institutions 271
- Interpreting Culture 243
- Creative Expression 157
- Difference/Diversity 182
- Natural/Designed World 136
- Ethical Reasoning 100
- Formal/Quantitative Reasoning 50
- Analyzing/Using Data 135

Two schedules are the same if they have the same courses.

i. Sarah is planning to take four courses next semester. She is planning to take 2 Formal/Quantitative Reasoning courses. She needs Societies/Institutions, Interpreting Culture, Creative Expression, and Difference/Diversity. Assuming no course counts for more than one NUPath requirement and that she wants to do as many NUPath requirements as possible. How many schedules can she build?

ii. A course can have 1 or 2 predefined NUPath attributes. Rick is planning to take 4 courses. He wants to maximize the number of possible NUPath requirements he can complete this semester. Rick wants to know how many schedules he can build. How many *cases* does Rick need to consider to do so?

iii. Sam needs to take 2 courses for their major. These two courses each have one of the following attributes: Formal/Quantitative Reasoning, Natural/Designed World, or Analyzing/Using Data. They want to maximize the other attributes for the two courses, but they don't want to take any Creative Expression courses if they take an Interpreting Culture course. Sam wants to know how many schedules they can build. How many *cases* does Sam need to consider to do so?