Last time

- ~~Finish Perm. & Comb.~~
  - ~~examples~~
  - ~~balls-in-bins~~

Today

$\boxed{\text{Binomial Theorem}}$ → Binomial Distribution

Binomial Expansion

$x, y \in \mathbb{R}$

Next time

- Counting Problem Examples

$$(x+y)^n = \underbrace{(x+y)}_{\substack{1\text{st pair} \\ (\;)}}\underbrace{(x+y)}_{\substack{2\text{nd} \\ (\;)\text{ pair}}}\underbrace{(x+y)}_{3\text{rd}} \cdots \underbrace{(x+y)}_{(n-1)^{th}}\underbrace{(x+y)}_{n^{th}}$$

open
( ) ( ) ...
make products

pick	1	2	3	...		n-1	n	product	
	x	x	x	...		x	x	x^n	$\binom{n}{n}_x$ $\binom{n}{0}_y$
1y	x	x	x	...		x	y	$x^{n-1}y$	
	x	x	x	...	x	y	x	$x^{n-1}y$	$\binom{n}{1}_y = \binom{n}{n-1}_x$
	x	x	x	... x	y	x	x	$x^{n-1}y$	
	⋮								
	y	x	x	x ...	x	x	x	$x^{n-1}y$	
2y	x	x	x	...		x y	y	$x^{n-2}y^2$	$\binom{n}{n-2}_x = \binom{n}{2}_y$

x x x - - - y x y | $x^{n-2}y^2$
x x x - - - x y y x | $x^{n-2}y^2$
x x y y x - - - x y x | $x^{n-2}y^2$
y y x x x - - - - x | $x^{n-2}y^2$

3y
x x x - - - - x y y y | $x^{n-3}y^3$ $\binom{n}{3}$ y $\binom{n}{n-3}$ x
x x x - - - y x y y | $x^{n-3}y^3$
y y - - - - x x x y | $x^{n-3}y^3$
y y y x x - - - - x | $x^{n-3}y^3$

4y
x x x - - - - x y y y y | $x^{n-4} \cdot y^4$
⋮

⋮

2x
x x y y y - - - - y | x^2y^{n-2}
x y x y y - - - y | x^2y^{n-2}
y y - - - - y x x | x^2y^{n-2}

2^n total product lines

1x	$xyyy\cdots - - - y$	x^1y^{n-1}
	$yxyy\cdots - - - y$	x^1y^{n-1}
	$\vdots$	
	$yyy\cdots - - - yx$	x^1y^{n-1}
0x	$yyy\cdots - - - y$	y^n

times $\binom{n}{1} = \binom{n}{n-1}$

$x \qquad y$

$\binom{n}{0} = \binom{n}{n}$

$x \qquad y$

General "k" "x" "n-k" y $\boxed{x^k y^{n-k}}$

choose x from k ()
choose y from the other n-k ()

How many times do I see $x^k y^{n-k}$ term? $\binom{n}{k} = \binom{n}{n-k}$

$x \qquad y$

Sum it up

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

informal proved

Later this term: proof by induction over n.

$$(x+y)^2 = x^2 + 2xy + y^2 \xRightarrow{2\to 3} (x+y)^3 = (x+y)^2(x+y)$$

$$= (x^2+2xy+y^2)(x+y) = x^3 + 3x^2y + 3xy^2 + y^3$$

Binomial theorem: $(x+y)^n$

$(x+y)^0 = 1$

$(x+y)^1 = x + y$

$\boxed{(x+y)^2} = x^2 + 2xy + y^2 = (x+y)(x+y)$

choose x or y choose x or y

	x	y
x	x^2	xy
y	yx	y^2

$(x+y)(x+y)$

$(x+y)^3 = (x+y)(x+y)(x+y)$

0 y's $\leftarrow$	F	x	x	$\rightarrow$	x^2
1 y $\leftarrow$	O	x	y	$\rightarrow$	xy
	I	y	x	$\rightarrow$	yx
2 y's $\leftarrow$	L	y	y	$\rightarrow$	y^2

$\Rightarrow x^2 + 2xy + y^2$

Q: How many y's do you choose $\rightarrow$ dictates term, e.g., xy^2 vs. x^2y

How many ways to do so? $\rightarrow$ dictates coefficient in front of term.

A:

	# ways	term
0 y's	1	x^3
1 y	$3 = \binom{3}{1}$	x^2y
2 y's	$3 = \binom{3}{2}$	xy^2
3 y's	1	y^3

$\Rightarrow$

1   3   3   1

$\boxed{x^3 + 3x^2y + 3xy^2 + y^3}$

$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$

2 y's $\Rightarrow \binom{4}{2}$ options to get

$\boxed{\text{How many } y\text{'s ?}}$

	# ways	term
0	$1 = \binom{4}{0}$	x^4
1	$4 = \binom{4}{1}$	$x^3 y$
y 2	$6 = \binom{4}{2}$	$x^2 y^2$
3	$4 = \binom{4}{3}$	$x y^3$
y 4	$1 = \binom{4}{4}$	y^4

1 4 6 4 1

$\Rightarrow x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x+y)^n = (x+y)(x+y)\cdots(x+y)$ ← n terms

# y's	# ways	term
0	$\binom{n}{0}$	x^n
1	$\binom{n}{1}$	$x^{n-1}y$
2	$\binom{n}{2}$	$x^{n-2}y^2$
⋮	⋮	⋮
n	$\binom{n}{n}$	y^n

$\Rightarrow$

$$\binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n$$

$$= \boxed{\sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j}$$

index by j instead of k

Application:

$$\left(x + 2y^{-2}\right)^6$$

Q: What is the term that includes y^{-8}?

$$= \overbrace{(x+2y^{-2})(x+2y^{-2})\cdots(x+2y^{-2})}^{6}$$

$\Rightarrow$ to get y^{-8}, need to expand by $2y^{-2}$ $\boxed{4 \text{ times}}$

y term

$$\binom{6}{4} x^2 \left(2y^{-2}\right)^4$$

$$= \binom{6}{2} x^2 \left(2y^{-2}\right)^4$$

$$= \frac{6 \cdot 5}{\cancel{2} \cdot 1} x^2 \cdot 2^{\cancel{4}\,3} y^{-8}$$

$$= 240\, x^2 y^{-8}$$

$x=1 \quad y=1$

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \quad =1$$

$$\boxed{2^n = \sum_{k=0}^{n} \binom{n}{k}}$$

$|set| = n$

subsets of size k

$$\iff 2^n = \sum_{k=0}^{n} (\#\text{subsets of size } k)$$

$|size| = 0$

$|size| = 1$

$2^n =$

$|size| = n$

$2^n = \#$ all subsets

$2^n = |\mathcal{P}(set)|$

powerset size

$x=1 \quad y=-1$

$$(1+(-1))^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k$$

$(1-1)^n$

$$0 = +\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots (-1)^n \binom{n}{n}$$

alternate the sign $\Rightarrow$ sum of binomial coef with alternate signs $\pm$

$= 0$

$$(x+y)^4 = (x+y)(x+y)^3$$

$$= (x+y)(x^3+3x^2y+3xy^2+y^3)$$

$n=4$

$$= \boxed{x^4 + 3x^3y + 3x^2y^2 + xy^3}$$

$$+ \boxed{x^3y + 3x^2y^2 + 3xy^3 + y^4}$$

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$n=3$: ~~1~~ 1 3 3 1

1 3 3 1

1 4 6 4 1

$\binom{3}{2}$ $\binom{3}{1}$ $\binom{4}{2}$

$$\boxed{\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}}$$

exercise
- with combinations
- with !!

$$\binom{4}{2} = \binom{3}{2} + \binom{3}{1}$$

need 2 y's

expand by x in first term; need 2 y's from 2nd term

expand by y in first term, need 1 y in 2nd term

Choose 3 out of 8 people

$$\binom{8}{3} = \binom{7}{3} + \binom{7}{2}$$

Pascal's Triangle

$(x+y)^n$

$j = \#\, y\text{'s}$

	0	1	2	3	4 ...
$n=0$	1				
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	1	4	6	4	1
5					
⋮					

$$\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}$$

1 $\quad (x+y)^0$

1 1 $\quad (x+y)$

1 2 1 $\quad (x+y)^2$

$x^2+2xy+y^2$

1 3 3 1

$x^3+3x^2y+3xy^2+y^3$

1 4 6 4 1

$x^4 \quad 4x^3y \quad 6x^2y^2 \quad 4xy^3 \quad y^4$

$n=5$ 1 5 10 10 5 1

next row coef for $(x+y)^5$

$n=6$ 1 6 15 20 15 6 1

$(x+y)^6 \quad x^6 + 6x^5y + 15x^4y^2 \dots$

Applications & Consequences

① What is $\boxed{\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n} = 2^n}$

Consider $S = \{a, b, c\}$

$P(S)$

0	$\emptyset$	1	$= \binom{3}{0}$
1	$\{a\}\ \{b\}\ \{c\}$	3	$= \binom{3}{1}$
2	$\{a,b\}\ \{b,c\}\ \{a,c\}$	3	$= \binom{3}{2}$
3	$\{a,b,c\}$	1	$= \binom{3}{3}$
		8	

$$\binom{3}{0}+\binom{3}{1}+\binom{3}{2}+\binom{3}{3} = 2^3$$

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$1 \quad 1$

$$2^n = (1+1)^n = \sum_{j=0}^{n} \binom{n}{j} \cdot 1^{n-j} \cdot 1^j = \sum_{j=0}^{n} \binom{n}{j}$$

$$11^0 = 1$$
$$11^1 = 11$$
$$11^2 = 121$$
$$11^3 = 1331$$
$$11^4 = 14641$$

$$11^n = (\overset{x}{1} + \overset{y}{10})^n = \sum_{j=0}^{n} \binom{n}{j} \cdot 1^{n-j} \cdot 10^j$$

$$= \sum_{j=0}^{n} \binom{n}{j} \cdot 10^j$$

$$= \binom{n}{n} \cdot 10^n + \binom{n}{n-1} 10^{n-1} \cdots \binom{n}{0} \cdot 10$$

$$11^3 = (1+10)^3 = \binom{3}{3} \cdot 10^3 + \binom{3}{2} 10^2 + \binom{3}{1} \cdot 10^1 + \binom{3}{0} \cdot 10^0$$

$$= 1 \cdot 10^3 + 3 \cdot 10^2 + 3 \cdot 10 + 1$$

$$= 1331$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

IND 12 Binomial Th by induction over $n \to n+1$ $\quad \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

ind step

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}} \overset{IH}{\Longrightarrow} (x+y)^{n+1} = \boxed{\sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}}$$

old customer $\qquad$ new customer

proof: $(x+y)^{n+1} = (x+y)^n (x+y) \overset{IH}{=} (x+y) \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} =$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k}$$

separate $k=0$

$$= x^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1} x^k y^{n+1-k} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n+1-k} + y^{n+1}$$

$k=n$ separate

indexed from $1:n$ instead of $0:n-1$

$k=1$: $\binom{n}{0} x^1 y^n$ | $k=n$: $\binom{n}{n-1} x^n y^1$

same prev $k=n-1$

$$= \underset{k=n+1}{x^{n+1}} + \sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k}\right] x^k y^{n+1-k} + \underset{k=0}{y^{n+1}}$$

$\checkmark$ $k=1:n$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k \, y^{n+1-k}$$

✓ Base case $n=1$

$$(x+y)^1 = \sum_{k=0}^{1} \binom{1}{k} x^k y^{n-k}$$

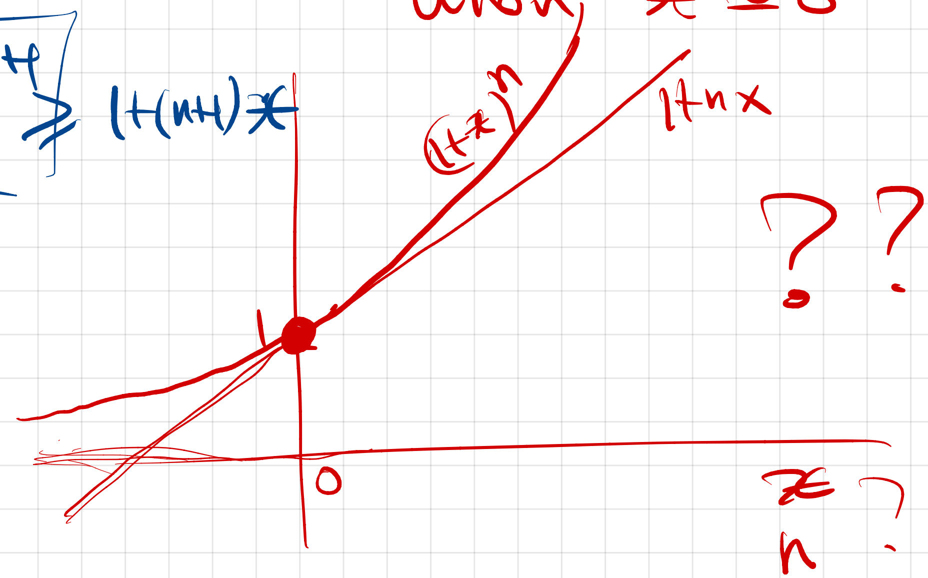
INDB $x > -1$ $n \geq 0$ integer $\Rightarrow (1+x)^n \geq 1 + nx$

$x+1 > 0$

$x \in \mathbb{R}$

useful approx $(1+x)^n \simeq 1+nx$ when $x \simeq 0$

ind step $n \to n+1$ $(1+x)^n \geq 1+nx \Rightarrow \boxed{(1+x)^{n+1} \geq 1+(n+1)x}$



proof

$$(1+x)^{n+1} = (1+x)^n \cdot (1+x)$$

$$\underset{\substack{\text{IH} \\ 1+x>0}}{\geq} (1+nx)(1+x) = 1 + nx + x + nx^2$$

$$= 1 + (n+1)x + \underbrace{nx^2}_{\geq 0} \geq 1 + (n+1)x \;✓$$

"YB" application

wanted last time: $\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$ (a_n) mon increasing

$a_n \qquad a_{n+1}$

Binomial Th $x=1$ $y=\frac{1}{n}$

$$(x+y)^n = \left(1+\frac{1}{n}\right)^n = \underbrace{1}_{k=0} + \sum_{k=1}^{n} \binom{n}{k}\frac{1}{n^k} = 1 + \sum_{k=1}^{n} \frac{n!}{k!(n-k)!}\cdot\frac{1}{n^k}$$

$x=1$ $y=\frac{1}{n+1}$

$$(x+y)^{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} = \underbrace{1}_{k=0} + \sum_{k=1}^{n}\binom{n+1}{k}\frac{1}{(n+1)^k} + \underbrace{\frac{1}{(n+1)^{n+1}}}_{k=n+1}$$

$$= 1 + \boxed{\sum_{k=1}^{n} \frac{(n+1)!}{k!(n+1-k)!}\frac{1}{(n+1)^k}} + \frac{1}{(n+1)^{n+1}}$$

want:

$$\cancel{1}+\sum_{k=1}^{n}\frac{n!}{k!(n-k+1)!}\cdot\frac{n-k+1}{n^k} \overset{?}{\leq} \cancel{1}+\sum_{k=1}^{n}\frac{n!}{k!(n-k+1)!}\cdot\frac{n+1}{(n+1)^k} + \underbrace{\frac{1}{(n+1)^{n+1}}}_{\text{extra}}$$

Sufficient to

$$\sum_{k=1}^{n} \frac{n!}{k!\,(n-k+1)!} \cdot \frac{n-k+1}{n^k} \overset{?}{\leq} \sum_{k=1}^{n} \frac{n!}{k!\,(n-k+1)!} \cdot \frac{n+1}{(n+1)^k}$$

$$\sum_{k=1}^{n} \frac{n!}{k!\,(n-k+1)!} \left(\frac{n-k+1}{n^k} - \frac{n+1}{(n+1)^k} \right) \leq 0$$

Lucky? Sufficient

$$\frac{n-k+1}{n^k} \overset{?}{\leq} \frac{n+1}{(n+1)^k}$$

$$\left(\frac{n+1}{n}\right)^k \overset{?}{\leq} \frac{n+1}{n-k+1}$$

$$\left(\frac{n}{n+1}\right)^k \overset{?}{\geq} \frac{n-k+1}{n+1}$$

$$\left(1-\frac{1}{n+1}\right)^{k} \overset{?}{\geq} 1-k\cdot\frac{1}{n+1} \checkmark$$

proved

$$(1+x)^{k} \geq 1+kx$$

$$x=-\frac{1}{n+1} \Rightarrow \left(1-\frac{1}{n+1}\right)^{k} \geq 1-k\cdot\frac{1}{n+1}$$

$$x>-1$$

IND 14

p prime $a \neq 0 \mod p$ then $a^{p-1} = 1 \mod p$

(ind $a \geq 0$ then $\boxed{a^p = a}^{IH} \mod p \ \forall a$)

Fermat's Little Th

induction by $a \to a+1$ $a \neq 0 \mod p$

$a+1 \neq 0 \mod p$

$a^{p-1} = 1 \mod p \Rightarrow (a+1)^{p-1} = 1 \mod p$

prod p new customer

$$(a+1)^p = \sum_{k=0}^{p} a^k \cdot \binom{p}{k} = 1 + a^p + \sum_{k=1}^{p-1} \binom{p}{k} a^k$$

IH

$$\underset{\mod p}{\overset{IH}{=}} 1 + a + p(\text{something})$$

exercise $p \mid \binom{p}{k}$ $\forall\, 1 \leq k \leq p-1$

$$= 1 + a \mod p$$

if $\exists (1+a)^{-1}$: $(a+1)^{p-1} = (a+1)^p (a+1)^{-1} = (1+a)(1+a)^{-1} = 1 \mod p$