1 Series and sequences

1. Recognize the following sequence and write it in concise form. Compute a close-form summation of the first n terms.
   $5, 7, 9, 11, 13, 15, 17, ...$

   Solution: Arithmetic progression $2x + 3$ for $x = 1, 2, 3, 4, 5...$

$$\sum_{k=1}^{n}(2k+3) = 2\sum_{k=1}^{n} k + 3n = 2\frac{n(n+1)}{2} + 3n = n(n+1) + 3n = n(n+4)$$

2. Recognize the following sequence and write it in concise form. Compute a close-form summation of the first n terms.
   $3, 9, 19, 33, 51, 73, 99, ...$

   Solution: Quadratic progression $2x^2 + 1$ for $x = 1, 2, 3, 4, 5...$

$$\sum_{k=1}^{n}(2k^2+1) = 2\sum_{k=1}^{n} k^2 + n = 2\frac{n(n+1)(2n+1)}{6} + n =$$

$$= \frac{n(n+1)(2n+1)+3n}{3} = \frac{n(2n^2+3n+1)+3n}{3} = \frac{n(2n^2+3n+4)}{3}$$

3. Recognize the following sequence and write it in concise form. Compute a close-form summation of the first n terms.
$48, 96, 192, 384, 768, 1536, ...$

<u>*Solution:*</u> Geometric progression $24 * 2^x$ for $x = 1, 2, 3, 4, 5...$

$$\sum_{k=1}^{n}(24 * 2^k) = 24\sum_{k=1}^{n} 2^k = 24(\sum_{k=0}^{n} 2^k - 1) = 24(\frac{2^{n+1} - 1}{2 - 1} - 1) = 24(2^{n+1} - 2) = 48(2^n - 1)$$

4. (More challenging) Prove that $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} ... + \frac{1}{n^2} < 1$ for any natural number n.

<u>*Solution:*</u>

$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} ... + \frac{1}{n^2} < \frac{1}{1 * 2} + \frac{1}{2 * 3} + \frac{1}{3 * 4} ... + \frac{1}{(n-1) * n} < \sum_{k=1}^{n}(\frac{1}{k} - \frac{1}{k+1}) = 1 - \frac{1}{n+1}$$

6. $S_n = \sum_{k=1}^{n} (2k-1)^2 = n(4n^2-1)/3$

Solution: Base case $n = 1 : S_1 = (2-1)^2 = 1 = 1(4 * 1^2 - 1)/3$

Induction step: give S_n formula, we want to prove that $S_{n+1} = (n+1)(4(n+1)^2-1)/3$

$$S_{n+1} = S_n + (2(n+1)-1)^2 = n(4n^2-1)/3 + (2n+1)^2 = n(2n-1)(2n+1)/3 + 3(2n+1)^2/3 =$$

$$= \frac{1}{3}(2n+1)(n(2n-1)+3(2n+1)) = \frac{1}{3}(2n+1)(2n^2-n+6n+3) = \frac{1}{3}(2n+1)(2n^2+2n+3n+3) =$$

$$= \frac{1}{3}(2n+1)(2n(n+1)+3(n+1)) = \frac{1}{3}(n+1)(2n+1)(2n+3) = \frac{1}{3}(n+1)(4n^2+8n+3) =$$

$$= \frac{1}{3}(n+1)(4n^2+4n+4-1) = \frac{1}{3}(n+1)(4(n+1)^2-1)$$

7. $S_n = \sum_{i=1}^{n} (-1)^i * i^2 = (-1)^n \frac{1}{2} n(n+1)$

<u>*Solution:*</u> Base case $n = 1 : (-1)1^2 = (-1)\frac{1*(1+1)}{2}$

Induction step: assuming S_n true, we want to prove $S_{n+1} = (-1)^{n+1}\frac{1}{2}(n+1)(n+2)$

$$S_{n+1} = S_n + (-1)^{n+1}(n+1)^2 = (-1)^n \frac{1}{2} n(n+1) + (-1)^{n+1}(n+1)^2 =$$

$$= (-1)^n \frac{1}{2}(n+1)(n-2(n+1)) = (-1)^n \frac{1}{2}(n+1)(-n-2) = (-1)^{n+1}\frac{1}{2}(n+1)(n+2)$$

8. Prove that $n! > 3^n > 2^n > n^2 > n\log_2(n) > n > \log_2(n)$ for $n \geq 7$

<u>*Solution:*</u> Base case $n = 7$:
$7! = 5040 > 3^7 = 2187 > 2^7 = 128 > 7^2 = 49 > 7 * \log_2(7) = 19.65 > 7 > \log_2(7) = 2.81$

Induction step: using inequalities for n, we want to prove that
$(n+1)! > 3^{n+1} > 2^{n+1} > (n+1)^2 > (n+1)\log_2(n+1) > n+1 > \log_2(n+1)$
We start from the left side:

$$(n+1)! = n!(n+1) > 3^n(n+1) > 3^n * 3 = 3^{n+1} = 3^n * 3 > 2^n * 3 > 2^n * 2 = 2^{n+1}$$

$$2^{n+1} = 2^n * 2 > n^2 * 2 = n^2 + n^2 \geq n^2 + 7n = n^2 + 2n + 5n > n^2 + 2n + 1 = (n+1)^2$$

We have proved so far the first three inequalities. Here is the proof for the last one:
$2^{n+1} > n^2 \Rightarrow n+1 > \log_2(n^2) \geq \log_2(7n) = \log_2(n+6n) > \log_2(n+1)$

Finally the inequalities fourth and fifth:
$n+1 > \log_2(n+1) \Rightarrow (n+1)^2 > (n+1)\log_2(n+1) > (n+1)\log_2(7) > n+1$

3 Induction proofs

PB 1 Show that 5 divides $8^n - 3^n$ for any natural number n.

Solution: Base case $n = 0 : 8^0 - 3^0 = 1 - 1 = 0$ is a multiple of 5
Induction Step : Assuming $8^n - 3^n = 5k$ we want to prove that $5|(8^{n+1} - 3^{n+1})$
$8^{n+1} - 3^{n+1} = 8 * 8^n - 3 * 3^n = 5 * 8^n + 3(8^n - 3^n) = 5 * 8^n + 5k = 5(8^n + k)$ thus multiple of 5

Solution without induction:
$8^{n+1} - 3^{n+1} = (8-3)\sum_{k=0}^{n} 8^k * 3^{n-k}$
which is a multiple of 5 due to the first factor.

PB 2 Binary trees height Prove that depth (height) of a binary tree with n nodes is at least $\lfloor \log_2(n) \rfloor$ (depth is the max number of edges on a path from root to a leaf).

Solution: Base case $n = 1, depth = 0 \geq log(1) = 0$
Base case $n = 2, depth = 1 \geq log(2) = 1$

Strong Induction Step: Will assume the property is true for any $k < n$, and will prove it for n. In particular the k-s for which we are going to need it are the number of nodes in the Left and Right subtrees.

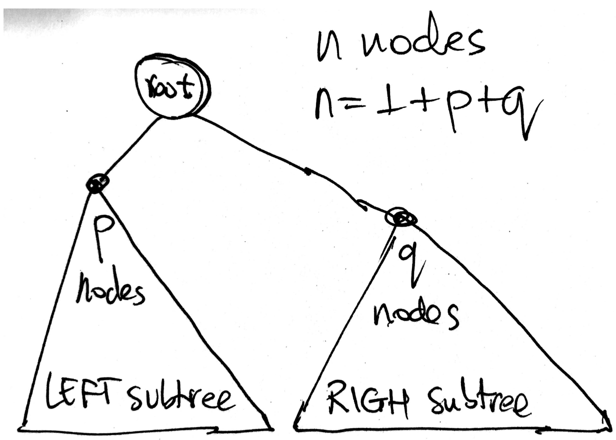
Lets say the root of the binary tree has a left subtree with p nodes and a right subtree with q nodes. Then $n = 1 + p + q$. Lets assume (without loss of generality) that $p \geq q$.
$p < n$ so by induction hypothesis we know $depth_L \geq \lfloor \log_2(p) \rfloor$

$$depth = 1 + \max(depth_L, depth_R) \geq 1 + depth_L \geq$$

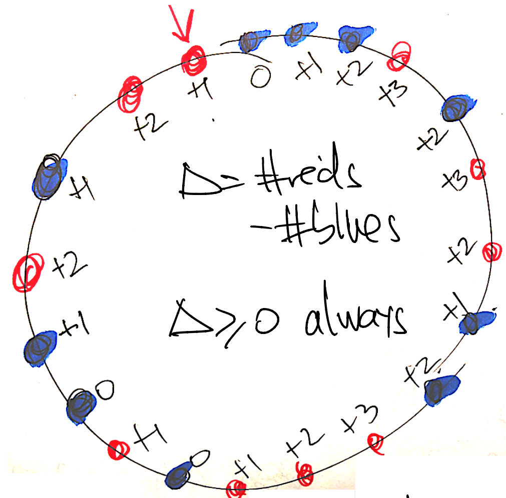
$$1 + \lfloor \log_2(p) \rfloor = 1 + \lfloor \log_2(\tfrac{2p}{2}) \rfloor = 1 + \lfloor \log_2(2p) - 1 \rfloor = \lfloor \log_2(2p) \rfloor$$

If n is even then $p \geq q = n - p - 1 \Rightarrow 2p > n \Rightarrow \lfloor \log_2(2p) \rfloor \geq \lfloor \log_2(n) \rfloor$

If n is odd then $\lfloor \log_2(n-1) \rfloor = \lfloor \log_2(n) \rfloor$
and $p \geq q = n - p - 1 \Rightarrow 2p \geq n - 1 \Rightarrow \lfloor \log_2(2p) \rfloor \geq \lfloor \log_2(n-1) \rfloor = \lfloor \log_2(n) \rfloor$



PB 10 ★★

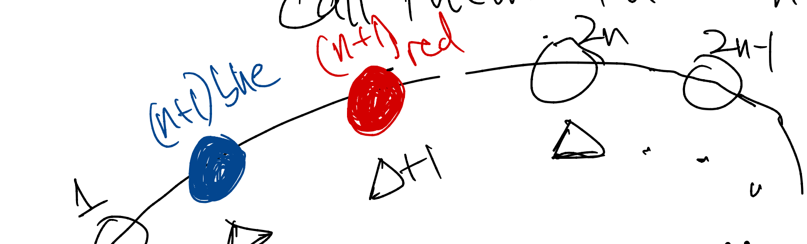


2n dots are placed around the outside of the circle; n of them are colored red and the remaining n are colored blue. Going around the circle anti-clockwise, you keep a count of how many red and blue dots you have passed. If at all times the number of red dots you have passed is at least the number of blue dots, you consider it a successful trip around the circle. Prove that no matter how the dots are placed on the circle, it is possible to have a successful trip around the circle if you start at the right point.

__Solution__: We want at all times #red dot ⩾ #blue dots, and we can choose the start dot (red) to go around anti-clockwise.

__ind step__ if its always possible for (n red, n blue) ⟹ its possible for (n+1 red, n+1 blue)

__proof__: given (n+1 red, n+1 blue) dots we find a pair (red blue) anticlock ordered. and call them the n+1 pair. This is possible by starting at a red and going anticlockwise until we find a blue. **We now remove this pair (red, blue)** resulting in 2n dots (n red + n blue). By induction hypoth there is a start such that a succesfull anticlock run has, at all times Δ = #reds − #blues ⩾ 0



That start will work for (n+1 reds + n+1 blues) ∵ Δ⩾0 at 2n, Δ+1⩾0 at n+1 red, Δ⩾0 at n+1 blue, the rest same Δ.