$$\neg\left[\boxed{(a \Rightarrow b)} \wedge \boxed{(b \Rightarrow c)}\right] = \neg(a \Rightarrow b) \vee \neg(b \Rightarrow c)$$

$$\underbrace{\qquad X \qquad \wedge \qquad Y \qquad}$$ $$\underbrace{\neg(a \Rightarrow b)}_{X} \qquad \underbrace{\neg(b \Rightarrow c)}_{Y}$$

OR

$$\neg\left[(\neg a \vee b) \wedge (\neg b \vee c)\right]$$

$$\neg\left[(\neg a \wedge \neg b) \vee (\neg a \wedge c) \vee (b \wedge \neg b) \vee (b \wedge c)\right]$$

$\boxed{a \Rightarrow b \Rightarrow c}$ ambiguous:

$(a \Rightarrow b) \Rightarrow c$

or

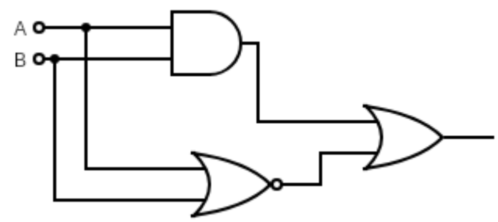
$a \Rightarrow (b \Rightarrow c)$

or

$(a \Rightarrow b) \wedge (b \Rightarrow c)$

Problem 2 [Easy]: Truth Tables, Statements, and Circuits

Each row of this table corresponds to an equivalent truth table, boolean statement, and circuit. (10 points) Fill out the following table. For this problem, you may hand write/draw the table and put that picture into your PDF.

<table>
<tr><th>Truth Table</th><th>Boolean Statement</th><th>Circuit</th></tr>
<tr><td><table><tr><th>A</th><th>B</th><th>Out</th></tr><tr><td>0</td><td>0</td><td>1</td></tr><tr><td>0</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td><td>1</td></tr></table></td><td></td><td></td></tr>
<tr><td></td><td>$\neg(A \vee B) \wedge (\neg A \vee B)$</td><td></td></tr>
<tr><td></td><td></td><td></td></tr>
</table>

Problem 3 [Medium]: Logical Equivalence

Determine if each pair is logically equivalent. If so, show the steps to transform one into the other. If not, show on what inputs they differ.

i. (2 points) $\neg(a \vee T) \vee b$, b

ii. (2 points) $(\neg a \vee b) \wedge a$, $a \wedge b$

iii. (2 points) $(a \implies b) \wedge \neg b$, $\neg a$

iv. (2 points) $\neg((a \wedge \neg b) \vee (\neg a \wedge b))$, $\neg b$

v. (2 points) $(\neg a \wedge (\neg b \vee c)) \wedge (\neg a \vee c)$, $\neg(a \vee (\neg b \wedge \neg c))$

same

Problem 4 [Medium]: NAND: The universal gate

In this problem we'll explore the fact that all logical circuits can be implemented using just NAND gates.

i. (5 points) Let's denote p NAND q as $p \barwedge q$. Write a logical expression for the three circuits corresponding to AND, OR, and NOT.

ii. (5 points) Validate your three logical expressions with three truth tables. For clarity and full credit, show each variable and distinct statement in a separate column, culminating in your final formula. For instance, if we wanted a table for the statement $(p \wedge q) \vee q$, we would need one column for p, one for q, one for $p \wedge q$ and one for the whole statement.

★ problems: no credit, no deadline, no formal grading, and possibly no solutions. If you work on these and need help, let Virgil know by email.

Problem 7 ★ (no credit) Implication is transitive

If a, b, c are boolean variables, show that $(a \Rightarrow b) \wedge (b \Rightarrow c) \Rightarrow (a \Rightarrow c)$.
Do this both using a truth table (easy), and by using boolean algebra derivation laws (not so easy)

fun! ☺ ★

Problem 8 ★ (no credit) Josephine's Problem

In Josephine's Kingdom every woman has to pass a logic exam before being allowed to marry. Every married woman knows about the fidelity of every man in the Kingdom except for her own husband, and etiquette demands that no woman should be told about the fidelity of her husband. Also, a gunshot fired in any house in the Kingdom will be heard in any other house. Queen Josephine announced that at least one unfaithful man had been discovered in the Kingdom, and that any woman knowing her husband to be unfaithful was required to shoot him at midnight following the day after she discovered his infidelity. How did the wives manage this?

extremely hard (IMO)

Problem 9 ★★★ (no credit) 4-hemisphere cover doable 3D trigonometry

A sphere is covered with some number of "caps" which are hemispheres. Prove that it is possible to choose four hemispheres, and remove all others, while still keeping the sphere covered.

Problem 10 ★★★ (no credit) 4k+1 primes characterization

Prove that a prime number p can be written as the sum of two squares $p = a^2 + b^2$ if and only if $p = 1 \pmod 4$.
For example $p = 13 = 4 * 3 + 1$ is such number, therefore is the sum of two squares $13 = 2^2 + 3^2$.
But the prime $p = 23 = 4 * 5 + 3 = 3 \bmod 4$ is not, so it cannot be writen as sum of two sqaured integers.

~~II. Express the following in English. Then write the equivalent set expressed and its cardinality.~~

A	B	$A \Rightarrow B$	$\neg B \Rightarrow \neg A$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

~~1. $x = 5 \cdot \{x, x^2, x^3, x^4\}$~~

write as OR

$\Rightarrow$

¬cond ∨ premise

$\boxed{A \Rightarrow B \equiv (\neg B \Rightarrow \neg A)}$

$\neg A \vee B \equiv (B \vee \neg A)$

conclusion ∨ ¬premise

$\neg A \vee (\neg\neg B)$

$X \vee Y \equiv Y \vee X$

~~2. $\{x \mid x \in \mathbb{Z}$ and $x < 4\}$~~

IMPLICATION (last module)

implication

$A \Rightarrow B$ same as $\neg A \vee B$

works always except when $A = 1 \wedge B = 0$

$B \Rightarrow A$ converse

NOT THE SAME AS $A \Rightarrow B$

$\neg B \vee A$

$\rightarrow$ fails when $A = 0 \wedge B = 1$

contrapositive of $A \Rightarrow B$

$\neg B \Rightarrow \neg A$

SAME AS $A \Rightarrow B$

$\neg A \vee B$

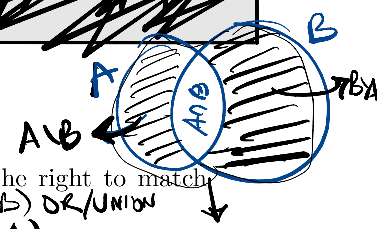
↳ if conclusion is FALSE, then the premise is FALSE

[redacted]

Sets

in A or B but not in both ← sym. diff $(A \cup B) \setminus (A \cap B)$

$= \boxed{(A \setminus B) \cup (B \setminus A)}$



i. Set Notation: Draw an arrow from the box on the left to the box on the right to match notation or phrases to the equivalent definition.

in (A but not in B) OR/UNION in (B but not in A)

1.

Notation		Definition
$A \subset B$		Intersection
$A \subseteq B$		Subset and not equal (i.e. a proper subset such that A is inside B but not equal. A has less elements)
$A \triangle B$		~~The complement~~ complement
$A \cup B$		Union
$A \cap B$		Subset and may be equal (i.e. A = B)
$\overline{A}$ → complement U − A (Univers)		The union of A and B without the intersection of A and B (i.e. $A \oplus B$ disjoint (Also a gate we have learned about))
$\emptyset$		The empty set

$A - B = A \cap \overline{B}$ → $\boxed{\text{diff}: A \cap \overline{B}}$ (and) in A AND not in B

ii. Express the following in English. Then write the equivalent set expressed and its cardinality.

1. $x = 5 : \{x, x^2, x^3, x^4\}$

2. $\{x \mid x \in \mathbb{Z} \text{ and } x < 4\}$

iii. Compute the following set given: $A = \{1, 2, 3, 4, 5\}, B = \{-1, 2, -3, 4, -5\}, C = \{-4, -2, 0, 2, 4\}$. Draw a Venn Diagram to visually represent each resulting set (Note you do not need to add in the numbers). Let U = set of all integers

1. $A \cap B$

2. $A \cup B$

3. $(A \cup B) \cap C$

4. $C \cap U$

$A = \{\cancel{\emptyset}, 1\}$

$B = \{\emptyset, x, z\}$

$A \times B = \{(x,y), (x,x), (x,z), (1,y), (1,y), (1,z)\}$

iv. Compute the Cartesian product of the given sets to generate all of the ordered tuples. A = $\{a, b, c\}, B = \{1, 2, 3\}, \boxed{C = \{6\}}$.

1. $\boxed{A \times B}$

size

$|A \times B| = |A| \times |B| = 3 \times 3 = 9$

2. $A \times C$ $= \{(a,6), (b,6), (c,6)\}$

all pairs $(a \in A, c \in C)$

$Y = \{A\}$

empty set $= A$

v. Compute the power sets below given the following. $\boxed{X = \{1, 2, 3\}}$, $Y = \{\emptyset\}$. Note to yourself, if given a set of 3 elements, how many subsets total will you have in your power set?

1. $\mathcal{P}(X)$ = set of subsets of X

$|\mathcal{P}(x)| = 2^3 = 8$

$= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

2. $\mathcal{P}(Y)$

$= \{\emptyset, \{\underset{Y}{A}\}\} = \{\emptyset, \{\emptyset\}\}$

vi. Given the following Set, compute the computer representation(i.e. a bit-string) using 0s and 1s.

1. $A = \{0, 1, 2, 3, 4, 5, 7\}$

2. $\bar{A} = \{6\}$

3. $B = \{0, 2, 4, 6\}$

4. $B \cup \{1, 3, 5\}$

Set Operations

i. Given
U = {pink, red, blue, white, yellow, black}
A = {red, blue, yellow, black}
B= {pink, red, white, yellow}
C = {white}

Determine:
A-B B-A A△B B-A-C $\mathcal{P}(A)$ {V ∈ $\mathcal{P}(A)$: $|V| = 1$}

OR $\left\{\begin{array}{l}\boxed{(B \setminus A) \setminus C} \quad \text{correct} \\ \cancel{B \setminus (A \setminus C)}\end{array}\right.$

A: all integers multiples of 26

B: all integers multiples of 13

26 divides x

Proof with sets

A B

Q: A is a subset of B

i. Prove that { x ∈ ℤ: 26|x } ⊆ { x ∈ ℤ: 13|x } $\iff \forall x\ 26|x \Rightarrow 13|x$

any integer divisible with 26 is also divisible with 13

$\forall x \in A$

$k \in \mathbb{Z}$

$26|x \Rightarrow x = 26 \cdot k$

$\Rightarrow x = 2 \cdot 13 \cdot k$

$\Rightarrow 13|x$

$\Rightarrow x \in B$ means $A \subset B$

why is this?

26 is a multiple of 13

$26 = 13 \cdot 2$

Counting

i. Crete plans to register for one course at MyWay University. The school has 4 departments: