

Complete Monitoring for Gradual Types: Supplementary Material

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Key concepts: path-based ownership (N 2.2, T 3.2, A 4.2), heap-based ownership (T 3.3), type soundness (N 5.1, A 7.2, T 6.6), complete monitoring (N 5.4), blame soundness + completeness (N 5.5, A 7.5), heap-based blame soundness (T 6.4).

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1 COMMON DEFINITIONS

1.1 Surface Definitions

Surface Language

$$\begin{aligned}
e &= x \mid n \mid i \mid \lambda x. e \mid \lambda(x:\tau). e \mid \langle e, e \rangle \mid \text{app}\{\tau?\} e e \mid \text{unop}\{\tau?\} e \mid \text{binop}\{\tau?\} e e \mid \text{dyn } b e \mid \text{stat } b e \\
\tau &= \text{Nat} \mid \text{Int} \mid \tau \times \tau \mid \tau \Rightarrow \tau \\
\tau? &= \tau \mid \mathcal{U} \\
b &= (\ell \blacktriangleleft \tau \blacktriangleleft \ell) \\
\ell &= \text{countable set of names} \\
\text{unop} &= \text{fst} \mid \text{snd} \\
\text{binop} &= \text{sum} \mid \text{quotient} \\
\Gamma &= \cdot \mid (x:\tau?), \Gamma \\
L &= \cdot \mid (x:\ell), L \\
n &= \mathbb{N} \\
i &= \mathbb{Z} \\
\bar{b} &= \cdot \mid b, \bar{b}
\end{aligned}$$

$e:\tau?$ wf

$$\begin{aligned}
e_0:\tau_0 \text{ wf} &\text{ iff } \exists \ell_0. \ell_0 \Vdash e_0 \text{ and } \vdash e_0:\tau_0 \\
e_0:\mathcal{U} \text{ wf} &\text{ iff } \exists \ell_0. \ell_0 \Vdash e_0 \text{ and } \vdash e_0:\mathcal{U}
\end{aligned}$$

$L; \ell \Vdash e$ well-named components

$$\begin{array}{c}
\frac{(x_0:\ell_0) \in L_0}{L_0; \ell_0 \Vdash x_0} \quad \frac{}{L_0; \ell_0 \Vdash i_0} \quad \frac{(x_0:\ell_0), L_0; \ell_0 \Vdash e_0}{L_0; \ell_0 \Vdash \lambda x_0. e_0} \quad \frac{(x_0:\ell_0), L_0; \ell_0 \Vdash e_0}{L_0; \ell_0 \Vdash \lambda(x_0:\tau_0). e_0} \quad \frac{L_0; \ell_0 \Vdash e_0 \quad L_0; \ell_0 \Vdash e_1}{L_0; \ell_0 \Vdash \langle e_0, e_1 \rangle} \\
\\
\frac{L_0; \ell_0 \Vdash e_0 \quad L_0; \ell_0 \Vdash e_1}{L_0; \ell_0 \Vdash \text{app}\{\tau?\} e_0 e_1} \quad \frac{L_0; \ell_0 \Vdash e_0}{L_0; \ell_0 \Vdash \text{unop}\{\tau?\} e_0} \quad \frac{L_0; \ell_0 \Vdash e_0 \quad L_0; \ell_0 \Vdash e_1}{L_0; \ell_0 \Vdash \text{binop}\{\tau?\} e_0 e_1} \quad \frac{L_0; \ell_1 \Vdash e_0}{L_0; \ell_0 \Vdash \text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) e_0} \\
\\
\frac{L_0; \ell_1 \Vdash e_0}{L_0; \ell_0 \Vdash \text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) e_0}
\end{array}$$

$\Gamma \vdash e : \tau$ static typing

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\end{array}$$

$$\begin{array}{c}
\frac{(x_0 : \tau_0) \in \Gamma_0}{\Gamma_0 \vdash x_0 : \tau_0} \quad \frac{}{\Gamma_0 \vdash n_0 : \text{Nat}} \quad \frac{}{\Gamma_0 \vdash i_0 : \text{Int}} \quad \frac{(x_0 : \tau_0), \Gamma_0 \vdash e_0 : \tau_1}{\Gamma_0 \vdash \lambda(x_0 : \tau_0). e_0 : \tau_0 \Rightarrow \tau_1} \quad \frac{\Gamma_0 \vdash e_0 : \tau_0 \quad \Gamma_0 \vdash e_1 : \tau_1}{\Gamma_0 \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1} \\
\\
\frac{\Gamma_0 \vdash e_0 : \tau_0 \Rightarrow \tau_1 \quad \Gamma_0 \vdash e_1 : \tau_0}{\Gamma_0 \vdash \text{app}\{\tau_1\} e_0 e_1 : \tau_1} \quad \frac{\Gamma_0 \vdash e_0 : \tau_0 \quad \Delta(\text{unop}, \tau_0) = \tau_1 \quad \tau_1 \leqslant: \tau_2}{\Gamma_0 \vdash \text{unop}\{\tau_2\} e_0 : \tau_2} \quad \frac{\Gamma_0 \vdash e_0 : \tau_0 \quad \Gamma_0 \vdash e_1 : \tau_1 \quad \Delta(\text{binop}, \tau_0, \tau_1) = \tau_2 \quad \tau_2 \leqslant: \tau_3}{\Gamma_0 \vdash \text{binop}\{\tau_3\} e_0 e_1 : \tau_3} \quad \frac{\Gamma_0 \vdash e_0 : \mathcal{U}}{\Gamma_0 \vdash \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) e_0 : \tau_0} \\
\\
\frac{\Gamma_0 \vdash e_0 : \tau_0 \quad \tau_0 \leqslant: \tau_1}{\Gamma_0 \vdash e_0 : \tau_1}
\end{array}$$

$\Gamma \vdash e : \mathcal{U}$ dynamic typing

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\end{array}$$

$$\begin{array}{c}
\frac{x_0 \in \Gamma_0}{\Gamma_0 \vdash x_0 : \mathcal{U}} \quad \frac{}{\Gamma_0 \vdash i_0 : \mathcal{U}} \quad \frac{(x_0 : \mathcal{U}), \Gamma_0 \vdash e_0 : \mathcal{U}}{\Gamma_0 \vdash \lambda x_0. e_0 : \mathcal{U}} \quad \frac{\Gamma_0 \vdash e_0 : \mathcal{U} \quad \Gamma_0 \vdash e_1 : \mathcal{U}}{\Gamma_0 \vdash \langle e_0, e_1 \rangle : \mathcal{U}} \quad \frac{\Gamma_0 \vdash e_0 : \mathcal{U} \quad \Gamma_0 \vdash e_1 : \mathcal{U}}{\Gamma_0 \vdash \text{app}\{\mathcal{U}\} e_0 e_1 : \mathcal{U}} \\
\\
\frac{\Gamma_0 \vdash e_0 : \mathcal{U}}{\Gamma_0 \vdash \text{unop}\{\mathcal{U}\} e_0 : \mathcal{U}} \quad \frac{\Gamma_0 \vdash e_0 : \mathcal{U} \quad \Gamma_0 \vdash e_1 : \mathcal{U}}{\Gamma_0 \vdash \text{binop}\{\mathcal{U}\} e_0 e_1 : \mathcal{U}} \quad \frac{\Gamma_0 \vdash e_0 : \tau_0}{\Gamma_0 \vdash \text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) e_0 : \mathcal{U}}
\end{array}$$

$\tau \leqslant: \tau$

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\end{array}$$

$$\begin{array}{c}
\frac{}{\text{Nat} \leqslant: \text{Int}} \quad \frac{\tau_0 \leqslant: \tau_2 \quad \tau_1 \leqslant: \tau_3}{\tau_0 \times \tau_1 \leqslant: \tau_2 \times \tau_3} \quad \frac{\tau_2 \leqslant: \tau_0 \quad \tau_1 \leqslant: \tau_3}{\tau_0 \Rightarrow \tau_1 \leqslant: \tau_2 \Rightarrow \tau_3} \quad \frac{}{\tau_0 \leqslant: \tau_0}
\end{array}$$

$b \leqslant: b$

$$\begin{array}{c}
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\end{array}$$

$$\frac{\tau_0 \leqslant: \tau_1}{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \leqslant: (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)}$$

$\Delta : \text{unop} \times \tau \longrightarrow \tau$

$$\Delta(\text{fst}, \tau_0 \times \tau_1) = \tau_0$$

$$\Delta(\text{snd}, \tau_0 \times \tau_1) = \tau_1$$

157 $\Delta : \text{binop} \times \tau \times \tau \longrightarrow \tau$
 158 $\Delta(\text{sum}, \text{Nat}, \text{Nat}) = \text{Nat}$
 159 $\Delta(\text{sum}, \text{Int}, \text{Int}) = \text{Int}$
 160 $\Delta(\text{quotient}, \text{Nat}, \text{Nat}) = \text{Nat}$
 161 $\Delta(\text{quotient}, \text{Int}, \text{Int}) = \text{Int}$

1.2 Evaluation Definitions

177 **Base Evaluation Language**
 178
 179 $v = n \mid i \mid \langle v, v \rangle \mid \lambda x. e \mid \lambda(x:\tau). e$
 180 $E = [] \mid \langle E, e \rangle \mid \langle v, E \rangle \mid \text{unop}\{\tau?\} E \mid \text{app}\{\tau?\} E e \mid \text{app}\{\tau?\} v E \mid \text{binop}\{\tau?\} E e \mid \text{binop}\{\tau?\} v E \mid \text{dyn } b E \mid$
 181 $\text{stat } b E$
 182 $\text{Err} = \text{TagErr} \bullet \mid \text{TagErr} \circ \mid \text{DivErr} \mid \text{BndryErr}(\bar{b}, v)$
 183 $e = \text{Err} \mid x \mid n \mid i \mid \lambda x. e \mid \lambda(x:\tau). e \mid \langle e, e \rangle \mid \text{app}\{\tau?\} e e \mid \text{unop}\{\tau?\} e \mid \text{binop}\{\tau?\} e e \mid \text{dyn } b e \mid \text{stat } b e$
 184 $K = \text{Nat} \mid \text{Int} \mid \text{Pair} \mid \text{Fun}$
 185 $\tau = \text{Nat} \mid \text{Int} \mid \tau \times \tau \mid \tau \Rightarrow \tau$
 186 $\tau? = \tau \mid \mathcal{U}$
 187 $b = (\ell \blacktriangleleft \tau \blacktriangleleft \ell)$
 188 $\ell = \text{countable set of names}$
 189
 190 $\text{unop} = \text{fst} \mid \text{snd}$
 191 $\text{binop} = \text{sum} \mid \text{quotient}$
 192 $\Gamma = \cdot \mid (x:\tau?), \Gamma$
 193 $L = \cdot \mid (x:\ell), L$
 194 $n = \mathbb{N}$
 195 $i = \mathbb{Z}$
 196 $\bar{b} = \cdot \mid b, \bar{b}$
 197 $b^* = \mathcal{P}(b)$
 198 $\ell^* = \mathcal{P}(\ell)$
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$$\begin{array}{l}
209 \quad \boxed{\text{tag-match} : K \times v \longrightarrow \mathcal{B}} \\
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\end{array}
\quad \text{tag-match}(K_0, v_0) = \left\{ \begin{array}{l}
\text{True} \\
\text{if } K_0 = \text{Nat and } v_0 \in n \\
\text{or } K_0 = \text{Int and } v_0 \in i \\
\text{or } K_0 = \text{Pair and} \\
\quad v_0 \in \langle v, v \rangle \cup \\
\quad (\text{mon}(\ell \blacktriangleleft (\tau \times \tau) \blacktriangleleft \ell) v) \\
\text{or } K_0 = \text{Fun and} \\
\quad v_0 \in (\lambda x. e) \cup (\lambda(x:\tau). e) \cup \\
\quad (\text{mon}(\ell \blacktriangleleft (\tau \Rightarrow \tau) \blacktriangleleft \ell) v) \\
\text{tag-match}(K_0, v_1) \\
\text{if } v_0 = \text{trace}_v \bar{b}_0 v_1 \\
\text{False} \\
\text{otherwise}
\end{array} \right.$$

$$\begin{array}{l}
228 \quad \boxed{[\cdot] : \tau \longrightarrow K} \\
229 \\
230 \\
231 \\
232 \\
233 \\
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235 \\
\end{array}
\quad [\tau_0] = \left\{ \begin{array}{l}
\text{Nat} \quad \text{if } \tau_0 = \text{Nat} \\
\text{Int} \quad \text{if } \tau_0 = \text{Int} \\
\text{Pair} \quad \text{if } \tau_0 \in \tau \times \tau \\
\text{Fun} \quad \text{if } \tau_0 \in \tau \Rightarrow \tau
\end{array} \right.$$

$$\begin{array}{l}
236 \quad \boxed{\text{rev} : \bar{b} \longrightarrow \bar{b}} \\
237 \\
238 \\
239 \\
\end{array}
\quad \text{rev}((\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1), \dots, (\ell_{2n} \blacktriangleleft \tau_n \blacktriangleleft \ell_{2n+1})) = (\ell_{2n+1} \blacktriangleleft \tau_n \blacktriangleleft \ell_{2n}), \dots, (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0)$$

$$\begin{array}{l}
240 \quad \boxed{\delta : \text{unop} \times v \longrightarrow e} \\
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\end{array}
\quad \delta(\text{unop}, \langle v_0, v_1 \rangle) = \left\{ \begin{array}{l}
v_0 \\
\text{if } \text{unop} = \text{fst}\{\tau?\} \\
v_1 \\
\text{if } \text{unop} = \text{snd}\{\tau?\}
\end{array} \right.$$

$$\begin{array}{l}
248 \quad \boxed{\delta : \text{binop} \times v \times v \longrightarrow e} \\
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258 \\
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260 \\
\end{array}
\quad \delta(\text{binop}, i_0, i_1) = \left\{ \begin{array}{l}
i_0 + i_1 \\
\text{if } \text{binop} = \text{sum}\{\tau?\} \\
\text{DivErr} \\
\text{if } \text{binop} = \text{quotient}\{\tau?\} \\
\text{and } i_1 = 0 \\
[i_0/i_1] \\
\text{if } \text{binop} = \text{quotient}\{\tau?\} \\
\text{and } i_1 \neq 0
\end{array} \right.$$

1.3 Ownership Evaluation Definitions

Ownership Evaluation Language

261	ℓ	= countable set of labels, with 1-1 correspondence to names
262	ℓ_\bullet	= label for the “top” of an expression
263	$\bar{\ell}$	= $\cdot \mid \bar{\ell}, \ell$
264	v	= $n \mid i \mid \langle v, v \rangle \mid \lambda x. e \mid \lambda(x:\tau). e$
265	E	= $[\] \mid (E)^\ell \mid \langle E, e \rangle \mid \langle v, E \rangle \mid \text{unop}\{\tau?\} E \mid \text{app}\{\tau?\} E e \mid \text{app}\{\tau?\} v E \mid \text{binop}\{\tau?\} E e \mid \text{binop}\{\tau?\} v E \mid$ $\text{dyn } b E \mid \text{stat } b E$
266	Err	= $\text{TagErr} \bullet \mid \text{TagErr} \circ \mid \text{DivErr} \mid \text{BndryErr}(\bar{b}, v)$
267	e	= $(e)^\ell \mid \text{Err} \mid x \mid n \mid i \mid \lambda x. e \mid \lambda(x:\tau). e \mid \langle e, e \rangle \mid \text{app}\{\tau?\} e e \mid \text{unop}\{\tau?\} e \mid \text{binop}\{\tau?\} e e \mid \text{dyn } b e \mid \text{stat } b e$
268	K	= $\text{Nat} \mid \text{Int} \mid \text{Pair} \mid \text{Fun}$
269	τ	= $\text{Nat} \mid \text{Int} \mid \tau \times \tau \mid \tau \Rightarrow \tau$
270	$\tau?$	= $\tau \mid \mathcal{U}$
271	b	= $(\ell \blacktriangleleft \tau \blacktriangleleft \ell)$
272	ℓ	= countable set of names
273	unop	= $\text{fst} \mid \text{snd}$
274	binop	= $\text{sum} \mid \text{quotient}$
275	Γ	= $\cdot \mid (x:\tau?), \Gamma$
276	L	= $\cdot \mid (x:\ell), L$
277	n	= \mathbb{N}
278	i	= \mathbb{Z}
279	\bar{b}	= $\cdot \mid b, \bar{b}$
280	b^*	= $\mathcal{P}(b)$
281	ℓ^*	= $\mathcal{P}(\ell)$

$e:\tau? \overline{\text{wf}}$ well-formed expression

$(e_0)^{\ell_0}:\tau_0 \overline{\text{wf}}$ iff $\ell_0 \Vdash (e_0)^{\ell_0}$ and $\Vdash (e_0)^{\ell_0}:\tau_0$

$(e_0)^{\ell_0}:\mathcal{U} \overline{\text{wf}}$ iff $\ell_0 \Vdash (e_0)^{\ell_0}$ and $\Vdash (e_0)^{\ell_0}:\mathcal{U}$

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$L; \ell \Vdash e$ (selected rules)

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\end{array}$$

$$\begin{array}{c}
\frac{L_0; \ell_0 \Vdash e_0}{L_0; \ell_0 \Vdash (e_0)^{\ell_0}} \quad \frac{(x_0 : \ell_0) \in L_0}{L_0; \ell_0 \Vdash x_0} \quad \frac{}{L_0; \ell_0 \Vdash i_0} \quad \frac{(x_0 : \ell_0), L_0; \ell_0 \Vdash e_0}{L_0; \ell_0 \Vdash \lambda x_0. e_0} \quad \frac{(x_0 : \ell_0), L_0; \ell_0 \Vdash e_0}{L_0; \ell_0 \Vdash \lambda(x_0 : \tau_0). e_0} \\
\\
\frac{L_0; \ell_0 \Vdash e_0}{L_0; \ell_0 \Vdash \langle e_0, e_1 \rangle} \quad \frac{L_0; \ell_0 \Vdash e_0 \quad L_0; \ell_0 \Vdash e_1}{L_0; \ell_0 \Vdash \text{app}\{\tau?\} e_0 e_1} \quad \frac{L_0; \ell_0 \Vdash e_0}{L_0; \ell_0 \Vdash \text{unop}\{\tau?\} e_0} \quad \frac{L_0; \ell_0 \Vdash e_0 \quad L_0; \ell_0 \Vdash e_1}{L_0; \ell_0 \Vdash \text{binop}\{\tau?\} e_0 e_1} \\
\\
\frac{L_0; \ell_1 \Vdash e_0}{L_0; \ell_0 \Vdash \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)(e_0)^{\ell_1}} \quad \frac{L_0; \ell_1 \Vdash e_0}{L_0; \ell_0 \Vdash \text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)(e_0)^{\ell_1}} \quad \frac{L_0; \ell_1 \Vdash v_0}{L_0; \ell_0 \Vdash \text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)(v_0)^{\ell_1}} \\
\\
\frac{L_0; \ell_0 \Vdash v_0}{L_0; \ell_0 \Vdash \text{trace}_v \bar{b}_0 v_0} \quad \frac{L_0; \ell_0 \Vdash e_0}{L_0; \ell_0 \Vdash \text{trace} \bar{b}_0 e_0}
\end{array}$$

$\Gamma \Vdash e : \tau$

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\end{array}$$

$$\begin{array}{c}
\frac{\Gamma_0 \Vdash e_0 : \tau_0}{\Gamma_0 \Vdash (e_0)^{\ell_0} : \tau_0} \quad \frac{(x_0 : \tau_0) \in \Gamma_0}{\Gamma_0 \Vdash x_0 : \tau_0} \quad \frac{}{\Gamma_0 \Vdash n_0 : \text{Nat}} \quad \frac{}{\Gamma_0 \Vdash i_0 : \text{Int}} \quad \frac{(x_0 : \tau_0), \Gamma_0 \Vdash e_0 : \tau_1}{\Gamma_0 \Vdash \lambda(x_0 : \tau_0). e_0 : \tau_0 \Rightarrow \tau_1} \\
\\
\frac{\Gamma_0 \Vdash e_0 : \tau_0 \quad \Gamma_0 \Vdash e_1 : \tau_1}{\Gamma_0 \Vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1} \quad \frac{\Gamma_0 \Vdash e_0 : \tau_0 \Rightarrow \tau_1 \quad \Gamma_0 \Vdash e_1 : \tau_0}{\Gamma_0 \Vdash \text{app}\{\tau_1\} e_0 e_1 : \tau_1} \quad \frac{\Gamma_0 \Vdash e_0 : \tau_0 \quad \Delta(\text{unop}, \tau_0) = \tau_1}{\Gamma_0 \Vdash \text{unop}\{\tau_1\} e_0 : \tau_1} \quad \frac{\Gamma_0 \Vdash e_0 : \tau_0 \quad \Gamma_0 \Vdash e_1 : \tau_1 \quad \Delta(\text{binop}, \tau_0, \tau_1) = \tau_2}{\Gamma_0 \Vdash \text{binop}\{\tau_2\} e_0 e_1 : \tau_2} \\
\\
\frac{\Gamma_0 \Vdash e_0 : \mathcal{U}}{\Gamma_0 \Vdash \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) e_0 : \tau_0} \quad \frac{\Gamma_0 \Vdash e_0 : \tau_0 \quad \tau_0 \leqslant \tau_1}{\Gamma_0 \Vdash e_0 : \tau_1}
\end{array}$$

$\Gamma \Vdash e : \mathcal{U}$

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\end{array}$$

$$\begin{array}{c}
\frac{\Gamma_0 \Vdash e_0 : \mathcal{U}}{\Gamma_0 \Vdash (e_0)^{\ell_0} : \mathcal{U}} \quad \frac{x_0 \in \Gamma_0}{\Gamma_0 \Vdash x_0 : \mathcal{U}} \quad \frac{}{\Gamma_0 \Vdash i_0 : \mathcal{U}} \quad \frac{(x_0 : \mathcal{U}), \Gamma_0 \Vdash e_0 : \mathcal{U}}{\Gamma_0 \Vdash \lambda x_0. e_0 : \mathcal{U}} \quad \frac{\Gamma_0 \Vdash e_0 : \mathcal{U} \quad \Gamma_0 \Vdash e_1 : \mathcal{U}}{\Gamma_0 \Vdash \langle e_0, e_1 \rangle : \mathcal{U}} \\
\\
\frac{\Gamma_0 \Vdash e_0 : \mathcal{U} \quad \Gamma_0 \Vdash e_1 : \mathcal{U}}{\Gamma_0 \Vdash \text{app}\{\mathcal{U}\} e_0 e_1 : \mathcal{U}} \quad \frac{\Gamma_0 \Vdash e_0 : \mathcal{U}}{\Gamma_0 \Vdash \text{unop}\{\mathcal{U}\} e_0 : \mathcal{U}} \quad \frac{\Gamma_0 \Vdash e_0 : \mathcal{U} \quad \Gamma_0 \Vdash e_1 : \mathcal{U}}{\Gamma_0 \Vdash \text{binop}\{\mathcal{U}\} e_0 e_1 : \mathcal{U}} \quad \frac{\Gamma_0 \Vdash e_0 : \tau_0}{\Gamma_0 \Vdash \text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) e_0 : \mathcal{U}}
\end{array}$$

$\text{fst} : \tau \longrightarrow \tau$

$$\text{fst}(\tau_0 \times \tau_1) = \tau_0$$

365 $\boxed{snd : \tau \longrightarrow \tau}$
 366 $snd(\tau_0 \times \tau_1) = \tau_1$
 367 $\boxed{dom : \tau \longrightarrow \tau}$
 368 $dom(\tau_0 \Rightarrow \tau_1) = \tau_0$
 369 $\boxed{cod : \tau \longrightarrow \tau}$
 370 $cod(\tau_0 \Rightarrow \tau_1) = \tau_1$
 371 $\boxed{\dots : \bar{\ell} \times \bar{\ell} \longrightarrow \bar{\ell}}$ append labels, but remove consecutive duplicates
 372 $(\ell_0, \dots, \ell_n)(\ell_m, \dots, \ell_k) = \ell_0, \dots, \ell_n, \ell_m, \dots, \ell_k$
 373 if $\ell_n \neq \ell_m$
 374 $(\ell_0, \dots, \ell_n)(\ell_n, \dots, \ell_k) = \ell_0, \dots, \ell_n, \dots, \ell_k$
 375 $\boxed{\dots : \bar{b} \times \bar{b} \longrightarrow \bar{b}}$ append boundaries
 376 $\bar{b}_1 = \bar{b}_1$
 377 $(b_0, \bar{b}_0)\bar{b}_1 = b_0, (\bar{b}_0\bar{b}_1)$
 378 $\boxed{rev : \bar{\ell} \longrightarrow \bar{\ell}}$ reverse a list of ownership labels
 379 $rev(\ell_0, \dots, \ell_n) = \ell_n, \dots, \ell_0$
 380 $\boxed{\bar{b} \simeq \bar{\ell}}$
 381
$$\frac{}{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1), \cdot \simeq \cdot, \ell_1, \ell_0} \qquad \frac{\bar{b}_1 \simeq \bar{\ell}_2, \ell_1}{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1), \bar{b}_1 \simeq \bar{\ell}_2, \ell_1, \ell_0}$$

 382 $\boxed{hole-owner : E \longrightarrow \ell}$ return the ownership label that surrounds the hole
 383 $hole-owner(E_0) = h-own(\ell_\bullet, E_0)$
 384 $\boxed{h-own : \ell \times E \longrightarrow \ell}$
 385 $h-own(\ell_0, []) = \ell_0$
 386 $h-own(\ell_0, \langle E_0, e_1 \rangle) = h-own(\ell_0, E_0)$
 387 $h-own(\ell_0, \langle v_0, E_1 \rangle) = h-own(\ell_0, E_1)$
 388 $h-own(\ell_0, \text{app}\{\tau?\} E_0 e_1) = h-own(\ell_0, E_0)$
 389 $h-own(\ell_0, \text{app}\{\tau?\} v_0 E_1) = h-own(\ell_0, E_1)$
 390 $h-own(\ell_0, \text{unop}\{\tau?\} E_0) = h-own(\ell_0, E_0)$
 391 $h-own(\ell_0, \text{binop}\{\tau?\} E_0 e_1) = h-own(\ell_0, E_0)$
 392 $h-own(\ell_0, \text{binop}\{\tau?\} v_0 E_1) = h-own(\ell_0, E_1)$
 393 $h-own(\ell_0, \text{dyn}(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) E_0) = h-own(\ell_0, E_0)$
 394 $h-own(\ell_0, \text{stat}(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) E_0) = h-own(\ell_0, E_0)$
 395 $h-own(\ell_0, (E_0)^{\ell_1}) = h-own(\ell_1, E_0)$
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417 $\boxed{\text{has-boundary}(e, b)}$ check if a boundary appears in an expression

418

419
$$\frac{\text{has-boundary}(e_0, b_0)}{\text{has-boundary}(\lambda x_0. e_0, b_0)}$$

420
$$\frac{\text{has-boundary}(e_0, b_0)}{\text{has-boundary}(\lambda(x_0 : \tau_0). e_0, b_0)}$$

421
$$\frac{\text{has-boundary}(e_0, b_0)}{\text{has-boundary}(\langle e_0, e_1 \rangle, b_0)}$$

422

423
$$\frac{\text{has-boundary}(e_1, b_0)}{\text{has-boundary}(\langle e_0, e_1 \rangle, b_0)}$$

424
$$\frac{\text{has-boundary}(e_0, b_0)}{\text{has-boundary}(\text{app}\{\tau?\} e_0 e_1, b_0)}$$

425
$$\frac{\text{has-boundary}(e_1, b_0)}{\text{has-boundary}(\text{app}\{\tau?\} e_0 e_1, b_0)}$$

426

427
$$\frac{\text{has-boundary}(e_0, b_0)}{\text{has-boundary}(\text{unop}\{\tau?\} e_0, b_0)}$$

428
$$\frac{\text{has-boundary}(e_0, b_0)}{\text{has-boundary}(\text{binop}\{\tau?\} e_0 e_1, b_0)}$$

429
$$\frac{\text{has-boundary}(e_1, b_0)}{\text{has-boundary}(\text{binop}\{\tau?\} e_0 e_1, b_0)}$$

430

431
$$\frac{\text{has-boundary}(e_0, b_0)}{\text{has-boundary}(\text{dyn } b_0 e_0, b_0)}$$

432
$$\frac{\text{has-boundary}(e_0, b_0)}{\text{has-boundary}(\text{dyn } b_1 e_0, b_0)}$$

433
$$\frac{\text{has-boundary}(e_1, b_0)}{\text{has-boundary}(\text{stat } b_0 e_0, b_0)}$$

434

435
$$\frac{\text{has-boundary}(e_0, b_0)}{\text{has-boundary}(\text{stat } b_1 e_0, b_0)}$$

436

437

438 $\boxed{\text{add-trace} : \bar{b} \times v \longrightarrow v}$ extend existing trace (if any), otherwise start a new one

439
$$\text{add-trace}(\cdot, v_1) = v_1$$

440
$$\text{add-trace}(\bar{b}_0, ((\text{trace}_v \bar{b}_1 v_1))^{\bar{\ell}_2}) = \text{trace}_v \bar{b}_0 \bar{b}_1 ((v_1))^{\bar{\ell}_2}$$

441 if $\bar{b}_0 \neq \cdot$

442
$$\text{add-trace}(\bar{b}_0, v_1) = \text{trace}_v \bar{b}_0 v_1$$

443 if $\bar{b}_0 \neq \cdot$ and $v_1 \notin ((\text{trace}_v \bar{b} v))^{\bar{\ell}}$

444

445

446 $\boxed{\text{get-trace} : v \longrightarrow \bar{b}}$ get trace (if any) from a value

447
$$\text{get-trace}(\text{trace}_v \bar{b}_0 ((v_0))^{\bar{\ell}_1}) = \bar{b}_0$$

448
$$\text{get-trace}(v_0) = \cdot$$

449 if $v_0 \notin \text{trace}_v \bar{b} ((v))^{\bar{\ell}}$

450

451 $\boxed{\text{rem-trace} : v \longrightarrow v}$ remove trace (if any) from a value

452
$$\text{rem-trace}(((\text{trace}_v \bar{b}_0 ((v_0))^{\bar{\ell}_1}))^{\bar{\ell}_2}) = ((v_0))^{\bar{\ell}_1 \bar{\ell}_2}$$

453
$$\text{rem-trace}(v_0) = v_0$$

454 if $v_0 \notin ((\text{trace}_v \bar{b} ((v))^{\bar{\ell}}))^{\bar{\ell}}$

455

456

457 $\boxed{\text{owners} : v \longrightarrow \bar{\ell}}$

458
$$\text{owners}(v_0) = \{\ell_0\} \cup \text{owners}(v_1)$$

459 if $v_0 = (v_1)^{\ell_0}$

460
$$\text{owners}(v_0) = \text{owners}(v_1)$$

461 if $v_0 = \text{trace}_v \bar{b}_0 v_1$

462
$$\text{owners}(v_0) = \{\}$$

463 otherwise

464

465

466

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469 $\boxed{\text{senders} : \bar{b} \rightarrow \bar{\ell}}$
 470 $\text{senders}(\cdot) = \{\}$
 471 $\text{senders}(\bar{b}_0) = \{\ell_1\} \cup \text{senders}(\bar{b}_1)$
 472 $\text{if } \bar{b}_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1), \bar{b}_1$
 473
 474 $\boxed{\text{forget} : e \rightarrow e}$
 475
 476 $\text{forget}((e_0)^{\ell_0}) = \text{forget}(e_0)$
 477 $\text{forget}(x_0) = x_0$
 478 $\text{forget}(i_0) = i_0$
 479 $\text{forget}(\lambda x_0. e_0) = \lambda x_0. \text{forget}(e_0)$
 480 $\text{forget}(\lambda(x_0 : \tau_0). e_0) = \lambda(x_0 : \tau_0). \text{forget}(e_0)$
 481 $\text{forget}(\langle e_0, e_1 \rangle) = \langle \text{forget}(e_0), \text{forget}(e_1) \rangle$
 482 $\text{forget}(\text{app}\{\tau?\} e_0 e_1) = \text{app}\{\tau?\} \text{forget}(e_0) \text{forget}(e_1)$
 483 $\text{forget}(\text{unop}\{\tau?\} e_0) = \text{unop}\{\tau?\} \text{forget}(e_0)$
 484 $\text{forget}(\text{binop}\{\tau?\} e_0 e_1) = \text{binop}\{\tau?\} \text{forget}(e_0) \text{forget}(e_1)$
 485 $\text{forget}(\text{dyn } b_0 e_0) = \text{dyn } b_0 \text{forget}(e_0)$
 486 $\text{forget}(\text{stat } b_0 e_0) = \text{stat } b_0 \text{forget}(e_0)$
 487
 488 $\boxed{\text{mon-depth} : v \rightarrow n}$ count monitors around a value
 489
 490 $\text{mon-depth}(i) = 0$
 491 $\text{mon-depth}(\langle v_0, v_1 \rangle) = 0$
 492 $\text{mon-depth}(\lambda x_0. e_0) = 0$
 493 $\text{mon-depth}(\lambda(x_0 : \tau_0). e_0) = 0$
 494 $\text{mon-depth}(\text{trace}_v \bar{b}_0 v_0) = \text{mon-depth}(v_0)$
 495 $\text{mon-depth}(\text{mon } b_0 v_0) = *1 + \text{mon-depth}(v_0)$
 496
 497 $\boxed{\text{last} : \bar{\ell} \rightarrow \ell}$ count monitors around a value
 498
 499 $\text{last}(\ell_0 \cdots \ell_n) = \ell_n$
 500
 501

1.4 Abbreviations

502
 503 $e_0 = ((e_1)^{\bar{\ell}_0}) \iff e_0 = (\cdots (e_1)^{\ell_n} \cdots)^{\ell_1}$
 504 $(\text{trace}_v^? \bar{b}_0 v_1) = v_0 \iff \text{rem-trace}(v_0) = v_1 \text{ and } \text{get-trace}(v_0) = \bar{b}_0$
 505 $(\text{mon}^? b_0 v_1) = v_0 \iff \text{if } v_0 = \text{mon } b_0 v_1 \text{ or } v_0 \notin \text{mon } b v$
 506 $(\text{mon}^{+?} b_0 \dots b_n v_1) = v_0 \iff v_0 = \text{mon } b_0 (\dots \text{mon } b_n v_0 \dots)$
 507 $E_0[e_0]^{\ell_0} \iff \text{hole-owner}(E_0) = \ell_0$
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2 Natural

2.1 Normal Natural

Natural Language extends Base Evaluation Language

$v = n \mid i \mid \langle v, v \rangle \mid \lambda x. e \mid \lambda(x:\tau). e \mid \text{mon}(\ell \blacktriangleleft \tau \Rightarrow \tau \blacktriangleleft \ell) v$

$\Gamma \vdash_{\mathcal{N}} e : \tau$ extends $\Gamma \vdash e : \tau$

$$\frac{\Gamma \vdash_{\mathcal{N}} v_1 : \mathcal{U}}{\Gamma \vdash_{\mathcal{N}} \text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_1 : \tau_0} \qquad \frac{}{\Gamma \vdash_{\mathcal{N}} \text{Err} : \tau_0}$$

$\Gamma \vdash_{\mathcal{N}} e : \mathcal{U}$ extends $\Gamma \vdash e : \mathcal{U}$

$$\frac{\Gamma \vdash_{\mathcal{N}} v_1 : \tau_0}{\Gamma \vdash_{\mathcal{N}} \text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_1 : \mathcal{U}} \qquad \frac{}{\Gamma \vdash_{\mathcal{N}} \text{Err} : \mathcal{U}}$$

$\rightarrow_{\mathcal{N}}^*$ reflexive-transitive closure of $\rightarrow_{\mathcal{N}}$

$\rightarrow_{\mathcal{N}}$ compatible closure of $\triangleright_{\mathcal{N}} \cup \blacktriangleright_{\mathcal{N}}$

$e \triangleright_{\mathcal{N}} e$

$\text{unop}\{\tau_0\} v_0 \quad \triangleright_{\mathcal{N}} \text{TagErr} \circ$
 if $\delta(\text{unop}, v_0)$ is undefined

$\text{unop}\{\tau_0\} v_0 \quad \triangleright_{\mathcal{N}} \delta(\text{unop}, v_0)$
 if $\delta(\text{unop}, v_0)$ is defined

$\text{binop}\{\tau_0\} v_0 v_1 \quad \triangleright_{\mathcal{N}} \text{TagErr} \circ$
 if $\delta(\text{binop}, v_0, v_1)$ is undefined

$\text{binop}\{\tau_0\} v_0 v_1 \quad \triangleright_{\mathcal{N}} \delta(\text{binop}, v_0, v_1)$
 if $\delta(\text{binop}, v_0, v_1)$ is defined

$\text{app}\{\tau_0\} v_0 v_1 \quad \triangleright_{\mathcal{N}} \text{TagErr} \circ$
 if $v_0 \notin (\lambda x. e) \cup (\text{mon } b v)$

$\text{app}\{\tau_0\} (\lambda(x_0:\tau_1). e_0) v_1 \quad \triangleright_{\mathcal{N}} e_0[x_0 \leftarrow v_1]$

$\text{app}\{\tau_0\} (\text{mon}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_0) v_1 \quad \triangleright_{\mathcal{N}} \text{dyn } b_0 (\text{app}\{\mathcal{U}\} v_0 (\text{stat } b_1 v_1))$
 where $b_0 = (\ell_0 \blacktriangleleft \text{cod}(\tau_1) \blacktriangleleft \ell_1)$ and $b_1 = (\ell_1 \blacktriangleleft \text{dom}(\tau_1) \blacktriangleleft \ell_0)$

$\text{dyn}(\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1) v_0 \quad \triangleright_{\mathcal{N}} \text{mon}(\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1) v_0$
 if $\text{tag-match}(\lfloor \tau_0 \Rightarrow \tau_1 \rfloor, v_0)$

$\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \langle v_0, v_1 \rangle \quad \triangleright_{\mathcal{N}} \langle \text{dyn } b_0 v_0, \text{dyn } b_1 v_1 \rangle$
 if $\text{tag-match}(\lfloor \tau_0 \rfloor, \langle v_0, v_1 \rangle)$ and $b_0 = (\ell_0 \blacktriangleleft \text{fst}(\tau_0) \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \text{snd}(\tau_0) \blacktriangleleft \ell_1)$

$\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) i_0 \quad \triangleright_{\mathcal{N}} i_0$
 if $\text{tag-match}(\lfloor \tau_0 \rfloor, i_0)$

$\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \quad \triangleright_{\mathcal{N}} \text{BdryErr}((\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1), v_0)$
 if $\neg \text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$

573	$e \blacktriangleright_{\mathbb{N}} e$	
574		
575	$unop\{\mathcal{U}\} v_0$	$\blacktriangleright_{\mathbb{N}} \text{TagErr} \bullet$
576	if $\delta(unop, v_0)$ is undefined	
577	$unop\{\mathcal{U}\} v_0$	$\blacktriangleright_{\mathbb{N}} \delta(unop, v_0)$
578	if $\delta(unop, v_0)$ is defined	
579		
580	$binop\{\mathcal{U}\} v_0 v_1$	$\blacktriangleright_{\mathbb{N}} \text{TagErr} \bullet$
581	if $\delta(binop, v_0, v_1)$ is undefined	
582	$binop\{\mathcal{U}\} v_0 v_1$	$\blacktriangleright_{\mathbb{N}} \delta(binop, v_0, v_1)$
583	if $\delta(binop, v_0, v_1)$ is defined	
584		
585	$app\{\mathcal{U}\} v_0 v_1$	$\blacktriangleright_{\mathbb{N}} \text{TagErr} \bullet$
586	if $v_0 \notin (\lambda x. e) \cup (\text{mon } b \ v)$	
587	$app\{\mathcal{U}\} (\lambda x_0. e_0) v_1$	$\blacktriangleright_{\mathbb{N}} e_0[x_0 \leftarrow v_1]$
588	$app\{\mathcal{U}\} (\text{mon } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0) v_1$	$\blacktriangleright_{\mathbb{N}} \text{stat } b_0 (\text{app}\{\tau_1\} v_0 (\text{dyn } b_1 \ v_1))$
589	where $b_0 = (\ell_0 \blacktriangleleft \text{cod}(\tau_0) \blacktriangleleft \ell_1)$ and $b_1 = (\ell_1 \blacktriangleleft \text{dom}(\tau_0) \blacktriangleleft \ell_0)$ and $\tau_1 = \text{cod}(\tau_0)$	
590		
591	$\text{stat } (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1) v_0$	$\blacktriangleright_{\mathbb{N}} \text{mon } (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1) v_0$
592	if $\text{tag-match}(\lfloor \tau_0 \Rightarrow \tau_1 \rfloor, v_0)$ and $v_0 \in (\lambda(x:\tau). e) \cup (\text{mon } b \ v)$	
593	$\text{stat } (\ell_0 \blacktriangleleft (\tau_0 \times \tau_1) \blacktriangleleft \ell_1) \langle v_0, v_1 \rangle$	$\blacktriangleright_{\mathbb{N}} \langle \text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0, \text{stat } (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_1 \rangle$
594		
595	$\text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) i_0$	$\blacktriangleright_{\mathbb{N}} i_0$
596	if $\text{tag-match}(\lfloor \tau_0 \rfloor, i_0)$	
597	$\text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0$	$\blacktriangleright_{\mathbb{N}} \text{TagErr} \circ$
598	if $\neg \text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$	
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2.2 Natural Ownership Lifting

Natural Ownership Language extends Ownership Evaluation Language

$v = \dots \mid \text{mon } (\ell \blacktriangleleft \tau \Rightarrow \tau \blacktriangleleft \ell) v$

$\xrightarrow{\mathbb{N}}^*$ reflexive-transitive closure of $\xrightarrow{\mathbb{N}}$

$\xrightarrow{\mathbb{N}}$ compatible closure of $\triangleright_{\mathbb{N}} \cup \blacktriangleright_{\mathbb{N}}$

$$\begin{array}{l}
625 \quad \boxed{(e)^\ell \triangleright_{\mathbb{N}} (e)^\ell} \\
626 \\
627 \quad (unop\{\tau_0\} \langle (v_0) \bar{\ell}_0 \rangle^{\ell_0}) \quad \triangleright_{\mathbb{N}} \quad (\text{TagErr} \circ)^{\ell_0} \\
628 \quad \text{if } v_0 \notin (v)^\ell \text{ and } \delta(unop, v_0) \text{ is undefined} \\
629 \\
630 \quad (unop\{\tau_0\} \langle (v_0) \bar{\ell}_0 \rangle^{\ell_0}) \quad \triangleright_{\mathbb{N}} \quad (\delta(unop, v_0))^{\bar{\ell}_0 \ell_0} \\
631 \quad \text{if } \delta(unop, v_0) \text{ is defined} \\
632 \\
633 \quad (binop\{\tau_0\} \langle (v_0) \bar{\ell}_0 \rangle^{\ell_0} \langle (v_1) \bar{\ell}_1 \rangle^{\ell_1}) \quad \triangleright_{\mathbb{N}} \quad (\text{TagErr} \circ)^{\ell_0} \\
634 \quad \text{if } v_0 \notin (v)^\ell \text{ and } v_1 \notin (v)^\ell \text{ and } \delta(binop, v_0, v_1) \text{ is undefined} \\
635 \\
636 \quad (binop\{\tau_0\} \langle (v_0) \bar{\ell}_0 \rangle^{\ell_0} \langle (v_1) \bar{\ell}_1 \rangle^{\ell_1}) \quad \triangleright_{\mathbb{N}} \quad (\delta(binop, v_0, v_1))^{\ell_0} \\
637 \quad \text{if } \delta(binop, v_0, v_1) \text{ is defined} \\
638 \\
639 \quad (app\{\tau_0\} \langle (v_0) \bar{\ell}_0 \rangle^{\ell_0} v_1) \quad \triangleright_{\mathbb{N}} \quad (\text{TagErr} \circ)^{\ell_0} \\
640 \quad \text{if } v_0 \notin (v)^\ell \cup (\lambda x. e) \cup (\text{mon } b \ v) \\
641 \\
642 \quad (app\{\tau_0\} \langle (\lambda(x_0 : \tau_1). e_0) \bar{\ell}_0 \rangle^{\ell_0} v_1) \quad \triangleright_{\mathbb{N}} \quad (\langle e_0[x_0 \leftarrow \langle (v_1) \bar{\ell}_0 \rangle^{\ell_0} \text{rev}(\bar{\ell}_0)] \rangle)^{\bar{\ell}_0 \ell_0} \\
643 \\
644 \quad (app\{\tau_0\} \langle (\text{mon}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) \langle (v_0) \bar{\ell}_0 \rangle^{\ell_2}) \bar{\ell}_0 \rangle^{\ell_3} v_1) \quad \triangleright_{\mathbb{N}} \quad (\langle \text{dyn } b_0 \langle \text{app}\{\mathcal{U}\} v_0 \langle \text{stat } b_1 \langle (v_1) \bar{\ell}_0 \rangle^{\ell_2} \text{rev}(\bar{\ell}_0) \rangle^{\ell_2} \rangle)^{\bar{\ell}_0 \ell_3} \\
645 \quad \text{where } b_0 = (\ell_0 \blacktriangleleft \text{cod}(\tau_1) \blacktriangleleft \ell_1) \text{ and } b_1 = (\ell_1 \blacktriangleleft \text{dom}(\tau_1) \blacktriangleleft \ell_0) \\
646 \\
647 \quad (\text{dyn}(\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1) \langle (v_0) \bar{\ell}_0 \rangle^{\ell_2}) \quad \triangleright_{\mathbb{N}} \quad (\text{mon}(\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1) \langle (v_0) \bar{\ell}_0 \rangle^{\ell_2}) \\
648 \quad \text{if } \text{tag-match}(\lfloor \tau_0 \Rightarrow \tau_1 \rfloor, v_0) \text{ and } v_0 \in (\lambda x. e) \cup (\text{mon } b \ v) \\
649 \\
650 \quad (\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \langle \langle (v_0, v_1) \rangle \bar{\ell}_0 \rangle^{\ell_2}) \quad \triangleright_{\mathbb{N}} \quad (\langle \text{dyn } b_0 \langle (v_0) \bar{\ell}_0 \rangle^{\ell_2}, \text{dyn } b_1 \langle (v_1) \bar{\ell}_0 \rangle^{\ell_2} \rangle)^{\ell_2} \\
651 \quad \text{if } \text{tag-match}(\lfloor \tau_0 \rfloor, \langle v_0, v_1 \rangle) \text{ and } b_0 = (\ell_0 \blacktriangleleft \text{fst}(\tau_0) \blacktriangleleft \ell_1) \text{ and } b_1 = (\ell_0 \blacktriangleleft \text{snd}(\tau_0) \blacktriangleleft \ell_1) \\
652 \\
653 \quad (\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \langle (i_0) \bar{\ell}_0 \rangle^{\ell_2}) \quad \triangleright_{\mathbb{N}} \quad (i_0)^{\ell_2} \\
654 \quad \text{if } \text{tag-match}(\lfloor \tau_0 \rfloor, i_0) \\
655 \\
656 \quad (\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \langle (v_0) \bar{\ell}_0 \rangle^{\ell_2}) \quad \triangleright_{\mathbb{N}} \quad (\text{BndryErr}(\langle \ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1 \rangle, \langle (v_0) \bar{\ell}_0 \rangle^{\ell_2}))^{\ell_2} \\
657 \quad \text{if } \neg \text{tag-match}(\lfloor \tau_0 \rfloor, v_0) \\
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\end{array}$$

677	$(e)^\ell \blacktriangleright_{\mathbb{N}} (e)^\ell$	
678		
679	$(unop\{\mathcal{U}\} \langle\langle v_0 \rangle\rangle^{\bar{\ell}_0})^{\ell_0}$	$\blacktriangleright_{\mathbb{N}} (\text{TagErr } \bullet)^{\ell_0}$
680	if $v_0 \notin (v)^\ell$ and $\delta(unop, v_0)$ is undefined	
681	$(unop\{\mathcal{U}\} \langle\langle v_0 \rangle\rangle^{\bar{\ell}_0})^{\ell_0}$	$\blacktriangleright_{\mathbb{N}} (\delta(unop, v_0))^{\bar{\ell}_0 \ell_0}$
682	if $\delta(unop, v_0)$ is defined	
683	$(binop\{\mathcal{U}\} \langle\langle v_0 \rangle\rangle^{\bar{\ell}_0} \langle\langle v_1 \rangle\rangle^{\bar{\ell}_1})^{\ell_0}$	$\blacktriangleright_{\mathbb{N}} (\text{TagErr } \bullet)^{\ell_0}$
684	if $v_0 \notin (v)^\ell$ and $v_1 \notin (v)^\ell$ and $\delta(binop, v_0, v_1)$ is undefined	
685	$(binop\{\mathcal{U}\} \langle\langle v_0 \rangle\rangle^{\bar{\ell}_0} \langle\langle v_1 \rangle\rangle^{\bar{\ell}_1})^{\ell_0}$	$\blacktriangleright_{\mathbb{N}} (\delta(binop, v_0, v_1))^{\ell_0}$
686	if $\delta(binop, v_0, v_1)$ is defined	
687	$(app\{\mathcal{U}\} \langle\langle v_0 \rangle\rangle^{\bar{\ell}_0} v_1)^{\ell_0}$	$\blacktriangleright_{\mathbb{N}} (\text{TagErr } \bullet)^{\ell_0}$
688	if $v_0 \notin (v)^\ell \cup (\lambda x. e) \cup (\text{mon } b \ v)$	
689	$(app\{\mathcal{U}\} \langle\langle \lambda x_0. e_0 \rangle\rangle^{\bar{\ell}_0} v_1)^{\ell_0}$	$\blacktriangleright_{\mathbb{N}} (\langle\langle e_0[x_0 \leftarrow \langle\langle v_1 \rangle\rangle^{\ell_0 rev(\bar{\ell}_0)}] \rangle\rangle)^{\bar{\ell}_0 \ell_0}$
690	$(app\{\mathcal{U}\} \langle\langle \text{mon } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \langle\langle v_0 \rangle\rangle^{\ell_2} \rangle\rangle^{\bar{\ell}_0} v_1)^{\ell_3}$	$\blacktriangleright_{\mathbb{N}} (\langle\langle \text{stat } b_0 \ (app\{\tau_1\} \ v_0 \ (\text{dyn } b_1 \ \langle\langle v_1 \rangle\rangle^{\ell_3 rev(\bar{\ell}_0) \ell_2}) \rangle\rangle)^{\bar{\ell}_0 \ell_3}$
691	where $b_0 = (\ell_0 \blacktriangleleft cod(\tau_0) \blacktriangleleft \ell_1)$ and $b_1 = (\ell_1 \blacktriangleleft dom(\tau_0) \blacktriangleleft \ell_0)$ and $\tau_1 = cod(\tau_0)$	
692	$(stat \ (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1) \ \langle\langle v_0 \rangle\rangle^{\bar{\ell}_0})^{\ell_2}$	$\blacktriangleright_{\mathbb{N}} (\text{mon } (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1) \ \langle\langle v_0 \rangle\rangle^{\bar{\ell}_0})^{\ell_2}$
693	if $tag\text{-}match(\lfloor \tau_0 \Rightarrow \tau_1 \rfloor, v_0)$ and $v_0 \in (\lambda x. e) \cup (\text{mon } b \ v)$	
694	$(stat \ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \ \langle\langle \langle v_0, v_1 \rangle \rangle^{\bar{\ell}_0} \rangle\rangle^{\ell_2}$	$\blacktriangleright_{\mathbb{N}} (\langle\langle \text{dyn } b_0 \ \langle\langle v_0 \rangle\rangle^{\bar{\ell}_0}, \text{dyn } b_1 \ \langle\langle v_1 \rangle\rangle^{\bar{\ell}_0} \rangle\rangle)^{\ell_2}$
695	if $tag\text{-}match(\langle\langle v_0, v_1 \rangle\rangle, \tau_0)$ and $b_0 = (\ell_0 \blacktriangleleft fst(\tau_0) \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft snd(\tau_0) \blacktriangleleft \ell_1)$	
696	$(stat \ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \ \langle\langle i_0 \rangle\rangle^{\bar{\ell}_2})^{\ell_3}$	$\blacktriangleright_{\mathbb{N}} (i_0)^{\ell_3}$
697	if $tag\text{-}match(i_0, \tau_0)$	
698	$(stat \ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \ \langle\langle v_0 \rangle\rangle^{\bar{\ell}_2})^{\ell_2}$	$\blacktriangleright_{\mathbb{N}} (\text{TagErr } \circ)^{\ell_2}$
699	if $\neg tag\text{-}match(v_0, \tau_0)$	
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3 Transient

3.1 Normal Transient

$\boxed{\text{Transient Language}}$ extends Base Evaluation Language

p = countable set of heap locations

v = $i \mid n \mid p$

w = $\lambda x. e \mid \lambda(x:\tau). e \mid \langle v, v \rangle$

e = $\dots \mid p \mid \text{check } \tau? e p$

E = $\dots \mid \text{check } \tau? E p$

$\mathcal{H} = \mathcal{P}((p \mapsto w))$

$\mathcal{B} = \mathcal{P}((p \mapsto b^*))$

$\mathcal{T} = \cdot \mid (p:\tau?), \mathcal{T}$

$\boxed{\cdot(\cdot) : \mathcal{H} \times v \longrightarrow v}$ heap dereference

$$\mathcal{H}_0(v_0) = \begin{cases} w_0 & \text{if } v_0 \in p \text{ and } (v_0 \mapsto w_0) \in \mathcal{H}_0 \\ v_0 & \text{if } v_0 \notin p \end{cases}$$

$\boxed{\cdot(\cdot) : \mathcal{B} \times v \longrightarrow b^*}$ blame map dereference

$$\mathcal{B}_0(v_0) = \begin{cases} b_0^* & \text{if } v_0 \in p \text{ and } (v_0 \mapsto b_0^*) \in \mathcal{B}_0 \\ \emptyset & \text{otherwise} \end{cases}$$

$\boxed{\cdot[\cdot \mapsto \cdot] : \mathcal{B} \times v \times b^* \longrightarrow \mathcal{B}}$ blame map replace

$$\mathcal{B}_0[v_0 \mapsto b_0^*] = \begin{cases} \{v_0 \mapsto b_0^*\} \cup (\mathcal{B}_0 \setminus (v_0 \mapsto b_1^*)) & \text{if } v_0 \in p \text{ and } (v_0 \mapsto b_1^*) \in \mathcal{B}_0 \\ \mathcal{B}_0 & \text{otherwise} \end{cases}$$

$\boxed{\cdot[\cdot \cup \cdot] : \mathcal{B} \times v \longrightarrow b^*}$ blame map update

$$\mathcal{B}_0[v_0 \cup b_0^*] = \mathcal{B}_0[v_0 \mapsto b_0^* \cup \mathcal{B}_0(v_0)]$$

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 $\mathcal{T}; \Gamma \vdash_{\mathcal{T}} e : K$

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 $\mathcal{T} \vdash_{\mathcal{T}} \mathcal{H}$

$$\frac{\forall (p_0 \mapsto v_0) \in \mathcal{H}_0 . \mathcal{T}_0; \cdot \vdash_{\mathcal{T}} v_0 : \mathcal{T}_0(p_0)}{\mathcal{T}_0 \vdash_{\mathcal{T}} \mathcal{H}_0}$$

$$\frac{(p_0 : \tau_0) \in \mathcal{T}_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} p_0 : \lfloor \tau_0 \rfloor} \quad \frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_0 : K_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} \text{check } \tau_0 \ e_0 \ p_0 : \lfloor \tau_0 \rfloor} \quad \frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_0 : \mathcal{U}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} \text{check } \tau_0 \ e_0 \ p_0 : \lfloor \tau_0 \rfloor} \quad \frac{(x_0 : \tau_0) \in \Gamma_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} x_0 : \lfloor \tau_0 \rfloor}$$

$$\frac{}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} n_0 : \text{Nat}} \quad \frac{}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} i_0 : \text{Int}} \quad \frac{\mathcal{T}_0; (x_0 : \tau_0), \Gamma_0 \vdash_{\mathcal{T}} e_0 : K_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} \lambda(x_0 : \tau_0). e_0 : \text{Fun}} \quad \frac{\mathcal{T}_0; (x_0 : \mathcal{U}), \Gamma_0 \vdash_{\mathcal{T}} e_0 : \mathcal{U}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} \lambda x_0. e_0 : \text{Fun}}$$

$$\frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_0 : K_0 \quad \mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_1 : K_1}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} \langle e_0, e_1 \rangle : \text{Pair}} \quad \frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_0 : \text{Fun} \quad \mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_1 : K_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} \text{app}\{\tau_1\} \ e_0 \ e_1 : \lfloor \tau_1 \rfloor} \quad \frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_0 : K_0 \quad \Delta(\text{unop}, K_0) = K_1 \quad K_1 \leq \lfloor \tau_2 \rfloor}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} \text{unop}\{\tau_2\} \ e_0 : \lfloor \tau_2 \rfloor}$$

$$\frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_0 : K_0 \quad \mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_1 : K_1 \quad \Delta(\text{binop}, K_0, K_1) = K_2 \quad K_2 \leq \lfloor \tau_3 \rfloor}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} \text{binop}\{\tau_3\} \ e_0 \ e_1 : \lfloor \tau_3 \rfloor} \quad \frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_0 : \mathcal{U}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \ e_0 : \lfloor \tau_0 \rfloor} \quad \frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_0 : K_0 \quad K_0 \leq K_1}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_0 : K_1}$$

$$\frac{(p_0 : \mathcal{U}) \in \mathcal{T}_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} p_0 : \mathcal{U}} \quad \frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_0 : \mathcal{U}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} \text{check } \mathcal{U} \ e_0 \ p_0 : \mathcal{U}} \quad \frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_0 : K_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} \text{check } \mathcal{U} \ e_0 \ p_0 : \mathcal{U}} \quad \frac{(x_0 : \mathcal{U}) \in \Gamma_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} x_0 : \mathcal{U}}$$

$$\frac{}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} n_0 : \mathcal{U}} \quad \frac{}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} i_0 : \mathcal{U}} \quad \frac{\mathcal{T}_0; (x_0 : \tau_0), \Gamma_0 \vdash_{\mathcal{T}} e_0 : K_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} \lambda(x_0 : \tau_0). e_0 : \mathcal{U}} \quad \frac{\mathcal{T}_0; (x_0 : \mathcal{U}), \Gamma_0 \vdash_{\mathcal{T}} e_0 : \mathcal{U}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} \lambda x_0. e_0 : \mathcal{U}}$$

$$\frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_0 : \mathcal{U} \quad \mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_1 : \mathcal{U}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} \langle e_0, e_1 \rangle : \mathcal{U}} \quad \frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_0 : \mathcal{U} \quad \mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_1 : \mathcal{U}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} \text{app}\{\tau_1\} \ e_0 \ e_1 : \mathcal{U}} \quad \frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_0 : \mathcal{U}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} \text{unop}\{\tau_2\} \ e_0 : \mathcal{U}}$$

$$\frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_0 : \mathcal{U} \quad \mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_1 : \mathcal{U}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} \text{binop}\{\tau_3\} \ e_0 \ e_1 : \mathcal{U}} \quad \frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_0 : K_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} \text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \ e_0 : \mathcal{U}}$$

 $\mathcal{T} \vdash_{\mathcal{T}} \mathcal{H}$

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$$\boxed{K \leqslant K}$$

$$\frac{}{\text{Nat} \leqslant \text{Int}}$$

$$\frac{}{K_0 \leqslant K_0}$$

$$\boxed{\rightarrow_{\top}^*} \text{ reflexive-transitive closure of } \rightarrow_{\top}$$

$$\boxed{e; \mathcal{H}; \mathcal{B} \rightarrow_{\top} e; \mathcal{H}; \mathcal{B}}$$

$$E[e_0]; \mathcal{H}_0; \mathcal{B}_0 \rightarrow_{\top} E[e_1]; \mathcal{H}_1; \mathcal{B}_1$$

$$\text{if } e_0; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} e_1; \mathcal{H}_1; \mathcal{B}_1$$

$$E[\text{Err}]; \mathcal{H}_0; \mathcal{B}_0 \rightarrow_{\top} \text{Err}; \mathcal{H}_0; \mathcal{B}_0$$

885 $e; \mathcal{H}; \mathcal{B} \triangleright_{\top} e; \mathcal{H}; \mathcal{B}$

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887 $w_0; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} p_0; (\{p_0 \mapsto w_0\} \cup \mathcal{H}_0); (\{p_0 \mapsto \emptyset\} \cup \mathcal{B}_0)$

888 where p_0 fresh in \mathcal{H}_0 and \mathcal{B}_0

889 $(unop\{\tau_0\} v_0); \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{TagErr } \circ; \mathcal{H}_0; \mathcal{B}_0$

890 if $\delta(unop, \mathcal{H}_0(v_0))$ is undefined

891 $(unop\{\mathcal{U}\} v_0); \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{TagErr } \bullet; \mathcal{H}_0; \mathcal{B}_0$

892 if $\delta(unop, \mathcal{H}_0(v_0))$ is defined

893 $(unop\{\tau?\} p_0); \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} (\text{check } \tau? \delta(unop, \mathcal{H}_0(p_0)) p_0); \mathcal{H}_0; \mathcal{B}_0$

894 if $\delta(unop, \mathcal{H}_0(p_0))$ is defined

895 $(binop\{\tau_0\} v_0 v_1); \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{TagErr } \circ; \mathcal{H}_0; \mathcal{B}_0$

896 if $\delta(binop, v_0, v_1)$ is undefined

897 $(binop\{\mathcal{U}\} v_0 v_1); \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{TagErr } \bullet; \mathcal{H}_0; \mathcal{B}_0$

898 if $\delta(binop, v_0, v_1)$ is defined

899 $(binop\{\tau?\} i_0 i_1); \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \delta(binop, i_0, i_1); \mathcal{H}_0; \mathcal{B}_0$

900 if $\delta(binop, i_0, i_1)$ is defined

901 $(app\{\tau_0\} v_0 v_1); \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{TagErr } \circ; \mathcal{H}_0; \mathcal{B}_0$

902 if $\mathcal{H}_0(v_0) \notin (\lambda x. e) \cup (\lambda(x:\tau). e)$

903 $(app\{\mathcal{U}\} v_0 v_1); \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{TagErr } \bullet; \mathcal{H}_0; \mathcal{B}_0$

904 if $\mathcal{H}_0(v_0) \notin (\lambda x. e) \cup (\lambda(x:\tau). e)$

905 $(app\{\tau?\} p_0 v_0); \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} (\text{check } \tau? e_0[x_0 \leftarrow v_0] p_0); \mathcal{H}_0; \mathcal{B}_0[v_0 \cup rev(\mathcal{B}_0(p_0))]$

906 if $\mathcal{H}_0(p_0) = \lambda(x_0:\tau_0). e_0$ and $tag\text{-}match(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$

907 $(app\{\tau?\} p_0 v_0); \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{BndryErr } (\mathcal{B}_0(p_0), v_0); \mathcal{H}_0; \mathcal{B}_0$

908 if $\mathcal{H}_0(p_0) = \lambda(x_0:\tau_0). e_0$ and $\neg tag\text{-}match(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$

909 $(app\{\tau_0\} p_0 v_0); \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} (\text{check } \tau_0 e_0[x_0 \leftarrow v_0] p_0); \mathcal{H}_0; \mathcal{B}_0[v_0 \cup rev(\mathcal{B}_0(p_0))]$

910 if $\mathcal{H}_0(p_0) = \lambda x_0. e_0$

911 $(app\{\mathcal{U}\} p_0 v_0); \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} (e_0[x_0 \leftarrow v_0]); \mathcal{H}_0; \mathcal{B}_0$

912 if $\mathcal{H}_0(p_0) = \lambda x_0. e_0$

913 $(dyn (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0); \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} v_0; \mathcal{H}_0; (\mathcal{B}_0[v_0 \cup \{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\}])$

914 if $tag\text{-}match(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$

915 $(dyn (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0); \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{BndryErr } (\{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\}, v_0); \mathcal{H}_0; \mathcal{B}_0$

916 if $\neg tag\text{-}match(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$

917 $(stat (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0); \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} v_0; \mathcal{H}_0; (\mathcal{B}_0[v_0 \cup \{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\}])$

918 if $tag\text{-}match(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$

919 $(stat (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0); \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{TagErr } \circ; \mathcal{H}_0; \mathcal{B}_0$

920 if $\neg tag\text{-}match(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$

921 $(check \mathcal{U} v_0 p_0); \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} v_0; \mathcal{H}_0; \mathcal{B}_0$

922 $(check \tau_0 v_0 p_0); \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} v_0; \mathcal{H}_0; (\mathcal{B}_0[v_0 \cup \mathcal{B}_0(p_0)])$

923 if $tag\text{-}match(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$

924 $(check \tau_0 v_0 p_0); \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{BndryErr } (\mathcal{B}_0(v_0) \cup \mathcal{B}_0(p_0), v_0); \mathcal{H}_0; \mathcal{B}_0$

925 if $\neg tag\text{-}match(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$

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3.2 Transient (Path-Based) Ownership Lifting

$\boxed{\text{Transient Language}}$ extends Ownership Evaluation Language

p = countable set of heap locations

v = $i \mid n \mid p \mid (v)^\ell$

w = $\lambda x. e \mid \lambda(x:\tau). e \mid \langle v, v \rangle$

e = $\dots \mid p \mid \text{check } \tau? e p \mid (e)^\ell$

E = $\dots \mid \text{check } \tau? E p \mid (E)^\ell$

\mathcal{H} = $\mathcal{P}((p \mapsto w))$

\mathcal{B} = $\mathcal{P}((p \mapsto b^*))$

\mathcal{O} = $\mathcal{P}((p \mapsto \bar{\ell}))$

\mathcal{T} = $\cdot \mid (p:\tau?), \mathcal{T}$

$\boxed{\cdot[\cdot \mapsto \cdot] : O \times v \times \ell^* \longrightarrow O}$ ownership map replace

$$O_0[v_0 \mapsto \ell_0^*] = \begin{cases} \{v_0 \mapsto \ell_0^*\} \cup (O_0 \setminus (v_0 \mapsto \ell_1^*)) & \text{if } v_0 \in p \text{ and } (v_0 \mapsto \ell_1^*) \in O_0 \\ O_0[v_1 \mapsto \ell_0^*] & \\ \text{if } v_0 = ((v_1))^{\bar{\ell}_0} & \\ O_0 & \text{otherwise} \end{cases}$$

$\boxed{\cdot[\cdot \cup \cdot] : O \times v \longrightarrow \ell^*}$ ownership map update

$$O_0[v_0 \cup \ell_0^*] = O_0[v_0 \mapsto \ell_0^* \cup O_0(v_0)]$$

$\boxed{\mathcal{T}; \Gamma \vdash_{\mathcal{T}} e : K}$ extends $\vdash_{\mathcal{T}}$

$$\frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_0 : K_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} (e_0)^{\ell_0} : K_0}$$

$\boxed{\mathcal{T}; \Gamma \vdash_{\mathcal{T}} e : \mathcal{U}}$ extends $\vdash_{\mathcal{T}}$

$$\frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} e_0 : \mathcal{U}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathcal{T}} (e_0)^{\ell_0} : \mathcal{U}}$$

$\boxed{O; L; \ell \Vdash_{\mathcal{T}} e}$ lifts and extends \Vdash and enforces ownership consistency

$$\frac{O_0; L_0; \ell_0 \Vdash_{\mathcal{T}} e_0}{O_0; L_0; \ell_0 \Vdash_{\mathcal{T}} (e_0)^{\ell_0}} \qquad \frac{O_0; L_0; \ell_0 \Vdash_{\mathcal{T}} e_0 \quad p_0 \mapsto \ell_0 \in O_0}{O_0; L_0; \ell_0 \Vdash_{\mathcal{T}} \text{check } \tau? e_0 p_0}$$

$\boxed{\longrightarrow_{\mathcal{T}}^*}$ reflexive-transitive closure of $\longrightarrow_{\mathcal{T}}$

989	$e; \mathcal{H}; \mathcal{B} \longrightarrow_{\top} e; \mathcal{H}; \mathcal{B}$	
990		
991	$E[e_0]^{\ell_0}; \mathcal{H}_0; \mathcal{B}_0 \longrightarrow_{\top} E[e_1]^{\ell_0}; \mathcal{H}_1; \mathcal{B}_1$	
992	if $(e_0)^{\ell_0}; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} (e_1)^{\ell_0}; \mathcal{H}_1; \mathcal{B}_1$	
993		
994	$E[\text{Err}]; \mathcal{H}_0; \mathcal{B}_0 \longrightarrow_{\top} \text{Err}; \mathcal{H}_0; \mathcal{B}_0$	
995	$(e)^{\ell}; \mathcal{H}; \mathcal{B} \triangleright_{\top} (e)^{\ell}; \mathcal{H}; \mathcal{B}$	
996		
997	$(w_0)^{\ell_0}; \mathcal{H}_0; \mathcal{B}_0$	$\triangleright_{\top} (p_0)^{\ell_0}; (\{p_0 \mapsto w_0\} \cup \mathcal{H}_0); (\{p_0 \mapsto \emptyset\} \cup \mathcal{B}_0)$
998	where p_0 fresh in \mathcal{H}_0 and \mathcal{B}_0	
999	$(\text{unop}\{\tau_0\} ((v_0))^{\bar{\ell}_0}{}^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0$	$\triangleright_{\top} (\text{TagErr } \circ)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0$
1000	if $\delta(\text{unop}, \mathcal{H}_0(v_0))$ is undefined	
1001	$(\text{unop}\{\mathcal{U}\} ((v_0))^{\bar{\ell}_0}{}^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0$	$\triangleright_{\top} (\text{TagErr } \bullet)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0$
1002	if $\delta(\text{unop}, \mathcal{H}_0(v_0))$ is undefined	
1003	$(\text{unop}\{\tau?\} ((p_0))^{\bar{\ell}_0}{}^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0$	$\triangleright_{\top} (\text{check } \tau? \delta(\text{unop}, \mathcal{H}_0(p_0)) p_0)^{\bar{\ell}_0 \ell_1}; \mathcal{H}_0; \mathcal{B}_0$
1004	if $\delta(\text{unop}, \mathcal{H}_0(p_0))$ is defined	
1005	$(\text{binop}\{\tau_0\} ((v_0))^{\bar{\ell}_0} ((v_1))^{\bar{\ell}_1}{}^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0$	$\triangleright_{\top} (\text{TagErr } \circ)^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0$
1006	if $\delta(\text{binop}, v_0, v_1)$ is undefined	
1007	$(\text{binop}\{\mathcal{U}\} ((v_0))^{\bar{\ell}_0} ((v_1))^{\bar{\ell}_1}{}^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0$	$\triangleright_{\top} (\text{TagErr } \bullet)^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0$
1008	if $\delta(\text{binop}, v_0, v_1)$ is undefined	
1009	$(\text{binop}\{\tau?\} ((i_0))^{\bar{\ell}_0} ((i_1))^{\bar{\ell}_1}{}^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0$	$\triangleright_{\top} (\delta(\text{binop}, i_0, i_1))^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0$
1010	if $\delta(\text{binop}, i_0, i_1)$ is defined	
1011	$(\text{app}\{\tau_0\} ((v_0))^{\bar{\ell}_0} v_1)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0$	$\triangleright_{\top} (\text{TagErr } \circ)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0$
1012	if $\mathcal{H}_0(v_0) \notin (\lambda x. e) \cup (\lambda(x:\tau). e)$	
1013	$(\text{app}\{\mathcal{U}\} ((v_0))^{\bar{\ell}_0} v_1)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0$	$\triangleright_{\top} (\text{TagErr } \bullet)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0$
1014	if $\mathcal{H}_0(v_0) \notin (\lambda x. e) \cup (\lambda(x:\tau). e)$	
1015	$(\text{app}\{\tau?\} ((p_0))^{\bar{\ell}_0} v_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0$	$\triangleright_{\top} (\text{check } \tau? e_0[x_0 \leftarrow ((v_0))^{\bar{\ell}_1 \text{rev}(\bar{\ell}_0)}] p_0)^{\bar{\ell}_0 \ell_1}; \mathcal{H}_0; \mathcal{B}_0[v_0 \cup \text{rev}(\mathcal{B}_0(p_0))]$
1016	if $\mathcal{H}_0(p_0) = \lambda(x_0:\tau_0). e_0$ and $\text{tag-match}(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$	
1017	$(\text{app}\{\tau?\} ((p_0))^{\bar{\ell}_0} v_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0$	$\triangleright_{\top} (\text{BndryErr}(((\mathcal{B}_0(p_0))^{\bar{\ell}_0}, v_0))^{\ell_1}); \mathcal{H}_0; \mathcal{B}_0$
1018	if $\mathcal{H}_0(p_0) = \lambda(x_0:\tau_0). e_0$ and $\neg \text{tag-match}(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$	
1019	$(\text{app}\{\tau_0\} ((p_0))^{\bar{\ell}_0} v_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0$	$\triangleright_{\top} (\text{check } \tau_0 e_0[x_0 \leftarrow ((v_0))^{\bar{\ell}_1 \text{rev}(\bar{\ell}_0)}] p_0)^{\bar{\ell}_0 \ell_1}; \mathcal{H}_0; \mathcal{B}_0[v_0 \cup \text{rev}(\mathcal{B}_0(p_0))]$
1020	if $\mathcal{H}_0(p_0) = \lambda x_0. e_0$	
1021	$(\text{app}\{\mathcal{U}\} ((p_0))^{\bar{\ell}_0} v_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0$	$\triangleright_{\top} (e_0[x_0 \leftarrow ((v_0))^{\bar{\ell}_1 \text{rev}(\bar{\ell}_0)}])^{\bar{\ell}_0 \ell_1}; \mathcal{H}_0; \mathcal{B}_0$
1022	if $\mathcal{H}_0(p_0) = \lambda x_0. e_0$	
1023	$(\text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) ((v_0))^{\bar{\ell}_0}{}^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0$	$\triangleright_{\top} (v_0)^{\bar{\ell}_0 \ell_2}; \mathcal{H}_0; (\mathcal{B}_0[v_0 \cup \{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\}])$
1024	if $\text{tag-match}(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$	
1025	$(\text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) ((v_0))^{\bar{\ell}_0}{}^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0$	$\triangleright_{\top} (\text{BndryErr}(\{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\}, ((v_0))^{\bar{\ell}_0})^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0$
1026	if $\neg \text{tag-match}(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$	
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1041 $(\text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) ((v_0))^{\bar{\ell}_0}{}^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\bar{\Gamma}} (v_0)^{\bar{\ell}_0 \ell_2}; \mathcal{H}_0; (\mathcal{B}_0[v_0 \cup \{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\}])$
 1042 if $\text{tag-match}(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$
 1043
 1044 $(\text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) ((v_0))^{\bar{\ell}_0}{}^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\bar{\Gamma}} (\text{TagErr } \circ)^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0$
 1045 if $\neg \text{tag-match}(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$
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 1047 $(\text{check } \mathcal{U}((v_0))^{\bar{\ell}_0} p_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\bar{\Gamma}} (v_0)^{\bar{\ell}_0 \ell_1}; \mathcal{H}_0; \mathcal{B}_0$
 1048 $(\text{check } \tau_0((v_0))^{\bar{\ell}_0} p_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\bar{\Gamma}} ((v_0))^{\bar{\ell}_0 \ell_1}; \mathcal{H}_0; (\mathcal{B}_0[v_0 \cup \mathcal{B}_0(p_0)])$
 1049 if $\text{tag-match}(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$
 1050
 1051 $(\text{check } \tau_0((v_0))^{\bar{\ell}_0} p_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\bar{\Gamma}} (\text{BdryErr } (\mathcal{B}_0(v_0) \cup \mathcal{B}_0(p_0), ((v_0))^{\bar{\ell}_0}))^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0$
 1052 if $\neg \text{tag-match}(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$
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3.3 Transient Heap-Based Ownership Lifting

1059 $\boxed{\text{Transient Language}}$ extends Transient Ownership Evaluation Language

1060 $O = \mathcal{P}((p \mapsto \ell^*))$

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1069 $\boxed{O \Vdash_{\bar{\Gamma}} \mathcal{H}}$ enforces ownership consistency for the heap

$$\frac{\forall (p \mapsto v_0) \in \mathcal{H}_0 . O_0; \cdot; O(p_0) \Vdash_{\bar{\Gamma}} v_0}{O_0 \Vdash_{\bar{\Gamma}} \mathcal{H}_0}$$

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1082 $\boxed{\longrightarrow_{\bar{\Gamma}_2}^*}$ reflexive-transitive closure of $\longrightarrow_{\bar{\Gamma}_2}$

1083 $\boxed{e; \mathcal{H}; \mathcal{B}; O \longrightarrow_{\bar{\Gamma}_2} e; \mathcal{H}; \mathcal{B}; O}$

1084
1085 $E[e_0]^{\ell_0}; \mathcal{H}_0; \mathcal{B}_0; O_0 \longrightarrow_{\bar{\Gamma}_2} E[e_1]^{\ell_0}; \mathcal{H}_1; \mathcal{B}_1; O_0$

1086
1087 if $(e_0)^{\ell_0}; \mathcal{H}_0; \mathcal{B}_0; O_0 \triangleright_{\bar{\Gamma}_2} (e_1)^{\ell_0}; \mathcal{H}_1; \mathcal{B}_1; O_0$

1088
1089 $E[\text{Err}]; \mathcal{H}_0; \mathcal{B}_0; O_0 \longrightarrow_{\bar{\Gamma}_2} \text{Err}; \mathcal{H}_0; \mathcal{B}_0; O_0$

1093	$(e)^\ell; \mathcal{H}; \mathcal{B}; O \triangleright_{\overline{T}_2} (e)^\ell; \mathcal{H}; \mathcal{B}; O$	
1094		
1095	$(w_0)^{\ell_0}; \mathcal{H}_0; \mathcal{B}_0; O_0$	$\triangleright_{\overline{T}_2} (p_0)^{\ell_0}; (\{p_0 \mapsto w_0\} \cup \mathcal{H}_0); (\{p_0 \mapsto \emptyset\} \cup \mathcal{B}_0); (\{p_0 \mapsto \ell_0\} \cup O_0)$
1096	where p_0 fresh in \mathcal{H}_0 and \mathcal{B}_0 and O_0	
1097	$(unop\{\tau_0\} ((v_0))^{\overline{\ell}_0}{}^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0$	$\triangleright_{\overline{T}_2} (\text{TagErr } \circ)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0$
1098	if $\delta(unop, \mathcal{H}_0(v_0))$ is undefined	
1099	$(unop\{\mathcal{U}\} ((v_0))^{\overline{\ell}_0}{}^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0$	$\triangleright_{\overline{T}_2} (\text{TagErr } \bullet)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0$
1100	if $\delta(unop, \mathcal{H}_0(v_0))$ is undefined	
1101	$(unop\{\tau?\} ((p_0))^{\overline{\ell}_0}{}^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0$	$\triangleright_{\overline{T}_2} (\text{check } \tau? \delta(unop, \mathcal{H}_0(p_0)) p_0)^{\overline{\ell}_0 \ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0$
1102	if $\delta(unop, \mathcal{H}_0(p_0))$ is defined	
1103	$(binop\{\tau_0\} ((v_0))^{\overline{\ell}_0} ((v_1))^{\overline{\ell}_1}{}^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0; O_0$	$\triangleright_{\overline{T}_2} (\text{TagErr } \circ)^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0; O_0$
1104	if $\delta(binop, v_0, v_1)$ is undefined	
1105	$(binop\{\mathcal{U}\} ((v_0))^{\overline{\ell}_0} ((v_1))^{\overline{\ell}_1}{}^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0; O_0$	$\triangleright_{\overline{T}_2} (\text{TagErr } \bullet)^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0; O_0$
1106	if $\delta(binop, v_0, v_1)$ is undefined	
1107	$(binop\{\tau?\} ((i_0))^{\overline{\ell}_0} ((i_1))^{\overline{\ell}_1}{}^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0; O_0$	$\triangleright_{\overline{T}_2} (\delta(binop, i_0, i_1))^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0; O_0$
1108	if $\delta(binop, i_0, i_1)$ is defined	
1109	$(app\{\tau_0\} ((v_0))^{\overline{\ell}_0} v_1)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0$	$\triangleright_{\overline{T}_2} (\text{TagErr } \circ)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0$
1110	if $\mathcal{H}_0(v_0) \notin (\lambda x. e) \cup (\lambda(x:\tau). e)$	
1111	$(app\{\mathcal{U}\} ((v_0))^{\overline{\ell}_0} v_1)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0$	$\triangleright_{\overline{T}_2} (\text{TagErr } \bullet)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0$
1112	if $\mathcal{H}_0(v_0) \notin (\lambda x. e) \cup (\lambda(x:\tau). e)$	
1113	$(app\{\tau?\} ((p_0))^{\overline{\ell}_0} v_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0$	$\triangleright_{\overline{T}_2}$
1114	$(\text{check } \tau? e_0[x_0 \leftarrow ((v_0))^{\ell_1 rev(\overline{\ell}_0)}] p_0)^{\overline{\ell}_0 \ell_1}; \mathcal{H}_0; \mathcal{B}_0[v_0 \cup rev(\mathcal{B}_0(p_0))]; O_0[v_0 \cup O_0(p_0) \cup \{\ell_1\}]$	
1115	if $\mathcal{H}_0(p_0) = \lambda(x_0:\tau_0). e_0$ and $tag-match(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$	
1116	$(app\{\tau?\} ((p_0))^{\overline{\ell}_0} v_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0$	$\triangleright_{\overline{T}_2}$
1117	$(\text{BndryErr}(((\mathcal{B}_0(p_0))^{\overline{\ell}_0}, ((v_0))^{\ell_1 rev(\overline{\ell}_0)}))^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0[v_0 \cup O_0(p_0) \cup \ell_1]$	
1118	if $\mathcal{H}_0(p_0) = \lambda(x_0:\tau_0). e_0$ and $\neg tag-match(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$	
1119	$(app\{\tau_0\} ((p_0))^{\overline{\ell}_0} v_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0$	$\triangleright_{\overline{T}_2}$
1120	$(\text{check } \tau_0 e_0[x_0 \leftarrow ((v_0))^{\ell_1 rev(\overline{\ell}_0)}] p_0)^{\overline{\ell}_0 \ell_1}; \mathcal{H}_0; \mathcal{B}_0[v_0 \cup rev(\mathcal{B}_0(p_0))]; O_0[v_0 \cup O_0(p_0) \cup \{\ell_0\}]$	
1121	if $\mathcal{H}_0(p_0) = \lambda x_0. e_0$	
1122	$(app\{\mathcal{U}\} ((p_0))^{\overline{\ell}_0} v_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0$	$\triangleright_{\overline{T}_2} (e_0[x_0 \leftarrow ((v_0))^{\ell_1 rev(\overline{\ell}_0)}])^{\overline{\ell}_0 \ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0[v_0 \cup O_0(p_0) \cup \{\ell_0\}]$
1123	if $\mathcal{H}_0(p_0) = \lambda x_0. e_0$	
1124	$(dyn(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) ((v_0))^{\overline{\ell}_0}{}^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0; O_0$	$\triangleright_{\overline{T}_2} (v_0)^{\overline{\ell}_0 \ell_2}; \mathcal{H}_0; (\mathcal{B}_0[v_0 \cup \{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\}]); O_0[v_0 \cup \ell_2]$
1125	if $tag-match(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$	
1126	$(dyn(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) ((v_0))^{\overline{\ell}_0}{}^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0; O_0$	$\triangleright_{\overline{T}_2} (\text{BndryErr}(\{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\}, ((v_0))^{\overline{\ell}_0})^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0$
1127	if $\neg tag-match(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$	
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1145 $(\text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) ((v_0))^{\bar{\tau}_0} \ell_2; \mathcal{H}_0; \mathcal{B}_0; O_0 \triangleright_{\bar{\tau}_2} (v_0)^{\bar{\tau}_0 \ell_2}; \mathcal{H}_0; (\mathcal{B}_0[v_0 \cup \{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\}]); O_0[v_0 \cup \{\ell_2\}])$
1146 if *tag-match*($\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0)$)
1147
1148 $(\text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) ((v_0))^{\bar{\tau}_0} \ell_2; \mathcal{H}_0; \mathcal{B}_0; O_0 \triangleright_{\bar{\tau}_2} (\text{TagErr } \circ)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0)$
1149 if \neg *tag-match*($\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0)$)
1150
1151 $(\text{check } \mathcal{U}((v_0))^{\bar{\tau}_0} p_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0 \triangleright_{\bar{\tau}_2} (v_0)^{\bar{\tau}_0 \ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0[v_0 \cup O_0(p_0)]$
1152 $(\text{check } \tau_0((v_0))^{\bar{\tau}_0} p_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0 \triangleright_{\bar{\tau}_2} (v_0)^{\bar{\tau}_0 \ell_1}; \mathcal{H}_0; (\mathcal{B}_0[v_0 \cup \mathcal{B}_0(p_0)]); O_0[v_0 \cup O_0(p_0)]$
1153 if *tag-match*($\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0)$)
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1155 $(\text{check } \tau_0((v_0))^{\bar{\tau}_0} p_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0 \triangleright_{\bar{\tau}_2} (\text{BndryErr } (\mathcal{B}_0(v_0) \cup \mathcal{B}_0(p_0), ((v_0))^{\bar{\tau}_0}))^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0[v_0 \cup O_0(p_0)]$
1156 if \neg *tag-match*($\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0)$)
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1197 **4 Amnesic**

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1200 **4.1 Normal Amnesic**

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1202 $\boxed{\text{Amnesic Language}}$ extends Base Evaluation Language

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1204 $v = \dots \mid \text{trace}_v \bar{b} v \mid \text{mon}(\ell \blacktriangleleft \tau \Rightarrow \tau \blacktriangleleft \ell)(v)^\ell \mid \text{mon}(\ell \blacktriangleleft \tau \times \tau \blacktriangleleft \ell)(v)^\ell$

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1206 $e = \dots \mid \text{trace} \bar{b} e$

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1208 $E = \dots \mid \text{trace} \bar{b} E$

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1210 $\boxed{\Gamma \vdash_A e : \tau}$ extends $\Gamma \vdash e : \tau$

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1218 $\boxed{\Gamma \vdash_A e : \mathcal{U}}$ extends $\Gamma \vdash e : \mathcal{U}$

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$$\frac{\Gamma \vdash_A v_0 : \mathcal{U}}{\Gamma \vdash_A \text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 : \tau_0} \qquad \frac{}{\Gamma \vdash_A \text{Err} : \tau_0}$$

$$\frac{\Gamma \vdash_A v_0 : \tau_0}{\Gamma \vdash_A \text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 : \mathcal{U}} \qquad \frac{\Gamma \vdash_A e_0 : \mathcal{U}}{\Gamma \vdash_A \text{trace} \bar{b}_0 e_0 : \mathcal{U}} \qquad \frac{\Gamma \vdash_A e_0 : \mathcal{U}}{\Gamma \vdash_A \text{trace}_v \bar{b}_0 e_0 : \mathcal{U}} \qquad \frac{}{\Gamma \vdash_A \text{Err} : \mathcal{U}}$$

1216 $\boxed{\text{trace}_v^? \bar{b}_0 v_0}$ short for a v_1 such that $\text{get-trace}(v_1) = \bar{b}_0$ and $\text{rem-trace}(v_1) = v_0$

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1249 $e \triangleright_A e$

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1251 $unop\{\tau_0\} v_0 \quad \triangleright_A \text{TagErr} \circ$

1252 if $v_0 \notin (\text{mon}(\ell \blacktriangleleft (\tau \times \tau) \blacktriangleleft \ell) v)$ and $\delta(unop, v_0)$ is undefined

1253 $unop\{\tau_0\} v_0 \quad \triangleright_A \delta(unop, v_0)$

1254 if $\delta(unop, v_0)$ is defined

1255 $fst\{\tau_0\} (\text{mon}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_0) \quad \triangleright_A \text{dyn } b_0 (\text{fst}\{\mathcal{U}\} v_0)$

1256 where $\tau_2 = \text{fst}(\tau_1)$ and $b_0 = (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)$

1258 $snd\{\tau_0\} (\text{mon}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_0) \quad \triangleright_A \text{dyn } b_0 (\text{snd}\{\mathcal{U}\} v_0)$

1259 where $\tau_2 = \text{snd}(\tau_1)$ and $b_0 = (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)$

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1261 $binop\{\tau_0\} v_0 v_1 \quad \triangleright_A \text{TagErr} \circ$

1262 if $\delta(binop, v_0, v_1)$ is undefined

1263 $binop\{\tau_0\} v_0 v_1 \quad \triangleright_A \delta(binop, v_0, v_1)$

1264 if $\delta(binop, v_0, v_1)$ is defined

1265

1266 $app\{\tau_0\} v_0 v_1 \quad \triangleright_A \text{TagErr} \circ$

1267 if $v_0 \notin (\lambda(x:\tau). e) \cup (\text{mon}(\ell \blacktriangleleft (\tau \Rightarrow \tau) \blacktriangleleft \ell) v)$

1268 $app\{\tau_0\} (\lambda(x_0:\tau_1). e_0) v_1 \quad \triangleright_A e_0[x_0 \leftarrow v_1]$

1269 $app\{\tau_0\} (\text{mon}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_0) v_1 \quad \triangleright_A \text{dyn } b_0 (\text{app}\{\mathcal{U}\} v_0 (\text{stat } b_1 v_1))$

1270 where $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_1 \blacktriangleleft \text{dom}(\tau_1) \blacktriangleleft \ell_0)$

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1272 $\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \quad \triangleright_A \text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0$

1273 if $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$ and $v_0 \in (\text{trace}_v^? \bar{b}(\lambda(x:\tau). e)) \cup (\text{trace}_v^? \bar{b}(\text{mon}(\ell \blacktriangleleft (\tau \Rightarrow \tau) \blacktriangleleft \ell) v))$

1274 $\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{trace}_v^? \bar{b}_0 i_0) \quad \triangleright_A i_0$

1275 if $\text{tag-match}(\lfloor \tau_0 \rfloor, i_0)$

1276

1277 $\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \quad \triangleright_A \text{BdryErr}((\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \bar{b}_0, v_0)$

1278 if $\neg \text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$ and $\bar{b}_0 = \text{get-trace}(v_0)$

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1301	$e \triangleright_{\mathbb{A}} e$	
1302		
1303	$unop\{\mathcal{U}\} v_0$	$\triangleright_{\mathbb{A}} \text{TagErr} \bullet$
1304	if $v_1 = \text{rem-trace}(v_0)$ and $v_1 \notin \text{mon}(\ell \blacktriangleleft (\tau \times \tau) \blacktriangleleft \ell) v$ and $\delta(unop, v_1)$ is undefined	
1305	$unop\{\mathcal{U}\} v_0$	$\triangleright_{\mathbb{A}} \text{add-trace}(get\text{-trace}(v_0), \delta(unop, v_1))$
1306	if $v_1 = \text{rem-trace}(v_0)$ and $\delta(unop, v_1)$ is defined	
1307	$\text{fst}\{\mathcal{U}\}(\text{trace}_{\checkmark}^2 \bar{b}_0(\text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0))$	$\triangleright_{\mathbb{A}} \text{trace} \bar{b}_0(\text{stat } b_0(\text{fst}\{\tau_1\} v_0))$
1308	where $\tau_1 = \text{fst}(\tau_0)$ and $b_0 = (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)$	
1310	$\text{snd}\{\mathcal{U}\}(\text{trace}_{\checkmark}^2 \bar{b}_0(\text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0))$	$\triangleright_{\mathbb{A}} \text{trace} \bar{b}_0(\text{stat } b_0(\text{snd}\{\tau_1\} v_0))$
1311	where $\tau_1 = \text{snd}(\tau_0)$ and $b_0 = (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)$	
1312		
1313	$\text{binop}\{\mathcal{U}\} v_0 v_1$	$\triangleright_{\mathbb{A}} \text{TagErr} \bullet$
1314	if $v_2 = \text{rem-trace}(v_0)$ and $v_3 = \text{rem-trace}(v_1)$ and $\delta(\text{binop}, v_2, v_3)$ is undefined	
1315	$\text{binop}\{\mathcal{U}\} v_0 v_1$	$\triangleright_{\mathbb{A}} \delta(\text{binop}, v_2, v_3)$
1316	if $v_2 = \text{rem-trace}(v_0)$ and $v_3 = \text{rem-trace}(v_1)$ and $\delta(\text{binop}, v_2, v_3)$ is defined	
1317	$\text{app}\{\mathcal{U}\} v_0 v_1$	$\triangleright_{\mathbb{A}} \text{TagErr} \bullet$
1318	if $v_0 \notin (\text{trace}_{\checkmark}^2 \bar{b}(\lambda x. e)) \cup (\text{trace}_{\checkmark}^2 \bar{b}(\text{mon}(\ell \blacktriangleleft (\tau \Rightarrow \tau) \blacktriangleleft \ell) v))$	
1320	$\text{app}\{\mathcal{U}\}(\text{trace}_{\checkmark}^2 \bar{b}_0(\lambda x_0. e_0)) v_0$	$\triangleright_{\mathbb{A}} \text{trace} \bar{b}_0(e_0[x_0 \leftarrow v_1])$
1321	where $v_1 = \text{add-trace}(\text{rev}(\bar{b}_0), v_0)$	
1322	$\text{app}\{\mathcal{U}\}(\text{trace}_{\checkmark}^2 \bar{b}_0(\text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0)) v_1$	$\triangleright_{\mathbb{A}} \text{trace} \bar{b}_0(\text{stat } b_0(\text{app}\{\tau_2\} v_0(\text{dyn } b_1 v_2)))$
1323	where $\tau_2 = \text{cod}(\tau_0)$ and $b_0 = (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_1 \blacktriangleleft \text{dom}(\tau_0) \blacktriangleleft \ell_0)$	
1324	and $v_2 = \text{add-trace}(\text{rev}(\bar{b}_0), v_1)$	
1325		
1326	$\text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0$	$\triangleright_{\mathbb{A}} \text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0$
1327	if $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$ and $v_0 \in (\lambda(x:\tau). e) \cup \langle v, v \rangle$	
1328	$\text{stat } b_0(\text{mon } b_1(\text{trace}_{\checkmark}^2 \bar{b}_0 v_0))$	$\triangleright_{\mathbb{A}} \text{trace}(b_0 b_1 \bar{b}_0) v_0$
1329	if $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ and $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$ and $v_0 \in (\lambda x. e) \cup \langle v, v \rangle \cup (\text{mon } b(\lambda(x:\tau). e)) \cup (\text{mon } b \langle v, v \rangle)$	
1331	$\text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) i_0$	$\triangleright_{\mathbb{A}} i_0$
1332	if $\text{tag-match}(\lfloor \tau_0 \rfloor i_0)$ and $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$	
1333	$\text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0$	$\triangleright_{\mathbb{A}} \text{TagErr} \circ$
1334	if $\neg \text{tag-match}(\lfloor \tau_0 \rfloor v_0)$	
1335	$\text{trace} \bar{b}_0 v_0$	$\triangleright_{\mathbb{A}} v_1$
1336	where $v_1 = \text{add-trace}(\bar{b}_0, v_0)$	
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4.2 Amnesic Ownership Lifting

Amnesic Ownership Language extends Ownership Evaluation Language

$v = \dots \mid \text{trace}_{\checkmark} \bar{b} v \mid \text{mon}(\ell \blacktriangleleft \tau \Rightarrow \tau \blacktriangleleft \ell) v \mid \text{mon}(\ell \blacktriangleleft \tau \times \tau \blacktriangleleft \ell) v$

$e = \dots \mid \text{trace} \bar{b} e$

$E = \dots \mid \text{trace} \bar{b} E$

$L; \ell \Vdash_{\mathbb{A}} e$ extends $L; \ell \Vdash e$ enforces sound and complete blame

$$\frac{L_0; \ell_1 \Vdash_{\mathbb{A}} v_0}{L_0; \ell_0 \Vdash_{\mathbb{A}} \text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)(v_0)^{\ell_1}} \quad \frac{\bar{b}_0 \simeq (\ell_n \cdots \ell_0) \quad L_0; \ell_n \Vdash_{\mathbb{A}} e_0}{L_0; \ell_0 \Vdash_{\mathbb{A}} \text{trace} \bar{b}_0((e_0))^{(\ell_n \cdots \ell_0)}} \quad \frac{\bar{b}_0 \simeq (\ell_n \cdots \ell_0) \quad L_0; \ell_n \Vdash_{\mathbb{A}} e_0}{L_0; \ell_0 \Vdash_{\mathbb{A}} \text{trace}_{\checkmark} \bar{b}_0((e_0))^{(\ell_n \cdots \ell_0)}}$$

1352

1353	$\xrightarrow[\overline{A}]{*}$	reflexive-transitive closure of $\xrightarrow{\overline{A}}$
1354		
1355	$e \xrightarrow[\overline{A}]{} e$	reflexive-transitive closure of $\triangleright_{\overline{N}} \cup \blacktriangleright_{\overline{N}}$
1356		
1357	$(e)^\ell \triangleright_{\overline{A}} (e)^\ell$	
1358		
1359	$(unop\{\tau_0\} v_0)^{\ell_0}$	$\triangleright_{\overline{A}} (\text{TagErr} \circ)^{\ell_0}$
1360	if $v_0 \notin (\text{mon}(\ell \blacktriangleleft (\tau \times \tau) \blacktriangleleft \ell) v)$ and $\delta(unop, v_0)$ is undefined	
1361	$(unop\{\tau_0\} v_0)^{\ell_0}$	$\triangleright_{\overline{A}} (\delta(unop, v_0))^{\ell_0}$
1362	if $\delta(unop, v_0)$ is defined	
1363		
1364	$(fst\{\tau_0\} ((\text{mon}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (v_0)^{\ell_2}))^{\ell_3^*} \ell_4)$	$\triangleright_{\overline{A}} ((\text{dyn } b_0 (fst\{\mathcal{U}\} v_0)^{\ell_2}))^{\ell_3^* \ell_4}$
1365	where $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$	
1366		
1367	$(snd\{\tau_0\} ((\text{mon}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (v_0)^{\ell_2}))^{\ell_3^*} \ell_4)$	$\triangleright_{\overline{A}} ((\text{dyn } b_0 (snd\{\mathcal{U}\} v_0)^{\ell_2}))^{\ell_3^* \ell_4}$
1368	where $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$	
1369		
1370	$(binop\{\tau_0\} v_0 v_1)^{\ell_0}$	$\triangleright_{\overline{A}} (\text{TagErr} \circ)^{\ell_0}$
1371	if $\delta(binop, v_0, v_1)$ is undefined	
1372	$(binop\{\tau_0\} v_0 v_1)^{\ell_0}$	$\triangleright_{\overline{A}} (\delta(binop, v_0, v_1))^{\ell_0}$
1373	if $\delta(binop, v_0, v_1)$ is defined	
1374		
1375	$(app\{\tau_0\} v_0 v_1)^{\ell_0}$	$\triangleright_{\overline{A}} (\text{TagErr} \circ)^{\ell_0}$
1376	if $v_0 \notin ((\lambda(x:\tau). e))^{\ell^*} \cup ((\text{mon}(\ell \blacktriangleleft (\tau \Rightarrow \tau) \blacktriangleleft \ell) v))^{\ell^*}$	
1377	$(app\{\tau_0\} ((\lambda(x_0:\tau_1). e_0))^{\ell_0^*} v_1)^{\ell_1}$	$\triangleright_{\overline{A}} ((e_0[x_0 \leftarrow (v_1)]^{\ell_1 \text{rev}(\ell_0^*)}))^{\ell_0^* \ell_1}$
1378		
1379	$(app\{\tau_0\} ((\text{mon}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (v_0)^{\ell_2}))^{\ell_3^*} v_1)^{\ell_4}$	$\triangleright_{\overline{A}} ((\text{dyn } b_0 (app\{\mathcal{U}\} v_0 (\text{stat } b_1 ((v_1)^{\ell_4 \text{rev}(\ell_3^*) \ell_2}))^{\ell_2}))^{\ell_3^* \ell_4}$
1380	where $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_1 \blacktriangleleft \text{dom}(\tau_1) \blacktriangleleft \ell_0)$	
1381	$(\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) ((v_0))^{\ell_2^*} \ell_3)$	$\triangleright_{\overline{A}} (\text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) ((v_0))^{\ell_2^*} \ell_3)$
1382	if $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$ and $\text{rem-trace}(v_0) \in ((\lambda x. e))^{\ell^*} \cup ((\langle v, v \rangle))^{\ell^*} \cup ((\text{mon } b v))^{\ell^*}$	
1383		
1384	$(\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) ((\text{trace}_v^? \bar{b}_0 ((i_0))^{\ell_2^*} \ell_3))^{\ell_4}$	$\triangleright_{\overline{A}} (i_0)^{\ell_4}$
1385	if $\text{tag-match}(\lfloor \tau_0 \rfloor, i_0)$	
1386		
1387	$(\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) ((v_0))^{\ell_2^*} \ell_3)$	$\triangleright_{\overline{A}} (\text{BndryErr}((\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \bar{b}_0, (v_0))^{\ell_2^*} \ell_3)$
1388	if $\neg \text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$ and $\bar{b}_0 = \text{get-trace}(v_0)$	
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1405	$(e)^\ell \blacktriangleright_A (e)^\ell$	
1406		
1407	$(unop\{\mathcal{U}\} v_0)^{\ell_0}$	$\blacktriangleright_A (\text{TagErr } \bullet)^{\ell_0}$
1408	if $v_1 = \text{rem-trace}(v_0)$ and $v_1 \notin \text{mon}(\ell \blacktriangleleft (\tau \times \tau) \blacktriangleleft \ell) v$ and $\delta(unop, v_1)$ is undefined	
1409	$(unop\{\mathcal{U}\} v_0)^{\ell_0}$	$\blacktriangleright_A (\text{add-trace}(\text{get-trace}(v_0), \delta(unop, v_1)))^{\ell_0}$
1410	if $v_1 = \text{rem-trace}(v_0)$ and $\delta(unop, v_1)$ is defined	
1411		
1412	$(fst\{\mathcal{U}\} (\text{trace}_v^? \bar{b}_0 ((\text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (v_1)^{\ell_2})^{\ell_3})^{\ell_4} v_1)^{\ell_5})$	$\blacktriangleright_A (\text{trace } \bar{b}_0 ((\text{stat } b_1 (fst\{\tau_1\} v_1)^{\ell_2})^{\ell_3})^{\ell_4} v_1)^{\ell_5}$
1413	where $\tau_1 = fst(\tau_0)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)$	
1414		
1415	$(snd\{\mathcal{U}\} (\text{trace}_v^? \bar{b}_0 ((\text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (v_1)^{\ell_2})^{\ell_3})^{\ell_4} v_1)^{\ell_5})$	$\blacktriangleright_A (\text{trace } \bar{b}_0 ((\text{stat } b_1 (snd\{\tau_1\} v_1)^{\ell_2})^{\ell_3})^{\ell_4} v_1)^{\ell_5}$
1416	where $\tau_1 = snd(\tau_0)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)$	
1417		
1418	$(binop\{\mathcal{U}\} v_0 v_1)^{\ell_0}$	$\blacktriangleright_A (\text{TagErr } \bullet)^{\ell_0}$
1419	if $v_2 = \text{rem-trace}(v_0)$ and $v_3 = \text{rem-trace}(v_1)$ and $\delta(binop, v_2, v_3)$ is undefined	
1420	$(binop\{\mathcal{U}\} v_0 v_1)^{\ell_0}$	$\blacktriangleright_A (\delta(binop, v_2, v_3))^{\ell_0}$
1421	if $v_2 = \text{rem-trace}(v_0)$ and $v_3 = \text{rem-trace}(v_1)$ and $\delta(binop, v_2, v_3)$ is defined	
1422		
1423	$(app\{\mathcal{U}\} (\text{trace}_v^? \bar{b}_0 ((v_0)^{\ell_0})^{\ell_1} v_1)^{\ell_2})$	$\blacktriangleright_A (\text{TagErr } \bullet)^{\ell_0}$
1424	if $v_0 \notin (\lambda x. e) \cup (\text{mon}(\ell \blacktriangleleft (\tau \Rightarrow \tau) \blacktriangleleft \ell) v)$	
1425		
1426	$(app\{\mathcal{U}\} (\text{trace}_v^? \bar{b}_0 ((\lambda x_0. e_0)^{\ell_0})^{\ell_1} v_1)^{\ell_2})$	$\blacktriangleright_A (\text{trace } \bar{b}_0 ((e_0[x_0 \leftarrow v_2])^{\ell_0})^{\ell_1} v_1)^{\ell_2}$
1427	where $v_2 = \text{add-trace}(\text{rev}(\bar{b}_0), (v_1)^{\ell_2 \text{rev}(\ell_1^* \text{rev}(\ell_0^*))})$	
1428		
1429	$(app\{\mathcal{U}\} (\text{trace}_v^? \bar{b}_0 ((\text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (v_0)^{\ell_2})^{\ell_3})^{\ell_4} v_1)^{\ell_5})$	$\blacktriangleright_A ((\text{trace } \bar{b}_0 ((\text{stat } b_1 (app\{\tau_3\} v_0 (\text{dyn } b_2 v_2))^{\ell_2})^{\ell_3})^{\ell_4} v_1)^{\ell_5})$
1430	where $\tau_2 = \text{cod}(\tau_0)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)$ and $b_2 = (\ell_1 \blacktriangleleft \text{dom}(\tau_0) \blacktriangleleft \ell_0)$ and $\tau_3 = \text{forget}(\tau_2)$	
1431	and $v_2 = (\text{add-trace}(\text{rev}(\bar{b}_0), (v_1)^{\ell_5 \text{rev}(\ell_3^* \ell_4^*)}))^{\ell_2}$	
1432		
1433	$(\text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0)^{\ell_2}$	$\blacktriangleright_A (\text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0)^{\ell_2}$
1434	if $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$ and $v_0 \in ((\lambda(x:\tau). e))^{\ell^*} \cup ((\langle v, v \rangle))^{\ell^*}$	
1435		
1436	$(\text{stat } b_0 ((\text{mon } b_1 ((\text{trace}_v^? \bar{b}_2 v_0)^{\ell_0})^{\ell_1} v_1)^{\ell_2}))$	$\blacktriangleright_A (\text{trace}(b_0 b_1 \bar{b}_2) ((v_0)^{\ell_0 \ell_1 \ell_2})^{\ell_2})$
1437	if $b_0 = (\ell_3 \blacktriangleleft \tau_0 \blacktriangleleft \ell_4)$ and $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$	
1438	and $v_0 \in ((\lambda x. e))^{\ell^*} \cup ((\langle v, v \rangle))^{\ell^*} \cup ((\text{mon } b (\lambda(x:\tau). e))^{\ell^*} \cup ((\text{mon } b \langle v, v \rangle))^{\ell^*})$	
1439		
1440	$(\text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) ((i_0))^{\ell_2})^{\ell_3}$	$\blacktriangleright_A (i_0)^{\ell_3}$
1441	if $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$	
1442	$(\text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) ((v_0))^{\ell_2})^{\ell_3}$	$\blacktriangleright_A (\text{TagErr } \circ)^{\ell_3}$
1443	if $\neg \text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$	
1444		
1445	$(\text{trace } \bar{b}_0 v_0)^{\ell_0}$	$\blacktriangleright_A (v_1)^{\ell_0}$
1446	where $v_1 = \text{add-trace}(\bar{b}_0, v_0)$	
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1457 **5 Natural THEOREMS, LEMMAS, AND PROOFS**

1458 **5.1 Natural Theorems**

1459 **THEOREM 5.1 (TYPE SOUNDNESS).** *If $e_0 : \tau_0$ wf then one of the following holds:*

- 1460 • $e_0 \rightarrow_N^* v_0$ and $\cdot \vdash_N v_0 : \tau_0$
- 1461 • e_0 diverges
- 1462 • $e_0 \rightarrow_N^* E_0[\text{dyn } b_1 E[e_1]]$ and $e_1 \blacktriangleright_N \text{TagErr} \bullet$
- 1463 • $e_0 \rightarrow_N^* \text{DivErr}$
- 1464 • $e_0 \rightarrow_N^* \text{BndryErr}(\bar{b}_1, v_1)$

1465 **PROOF.** By progress and preservation lemmas (lemma 5.6 & lemma 5.7).

□

1466 **THEOREM 5.2 (DYNAMIC SOUNDNESS).** *If $e_0 : \mathcal{U}$ wf then one of the following holds:*

- 1467 • $e_0 \rightarrow_N^* v_0$ and $\cdot \vdash_N v_0 : \mathcal{U}$
- 1468 • e_0 diverges
- 1469 • $e_0 \rightarrow_N^* E_0[e_1]$ and $e_1 \blacktriangleright_N \text{TagErr} \bullet$
- 1470 • $e_0 \rightarrow_N^* \text{DivErr}$
- 1471 • $e_0 \rightarrow_N^* \text{BndryErr}(\bar{b}_1, v_1)$

1472 **PROOF.** By progress and preservation lemmas (lemma 5.6 & lemma 5.7).

□

1473 **THEOREM 5.3.** *If $e_0 : \tau? \overline{\text{wf}}$ then $\text{forget}(e_0) : \tau? \text{wf}$ and $e_0 \rightarrow_N e_1$ iff $\text{forget}(e_0) \rightarrow_N \text{forget}(e_1)$*

1474 **PROOF.** By the definition of \rightarrow_N .

□

1475 **THEOREM 5.4 (COMPLETE MONITORING).** *If $e_0 : \tau? \overline{\text{wf}}$ and $e_0 \rightarrow_N^* e_1$ then $\cdot ; \ell \Vdash e_1$.*

1476 **PROOF.** By lemma 5.20 and lemma 5.21.

□

1477 **THEOREM 5.5 (CORRECT BLAME).** *If $e_0 : \tau? \overline{\text{wf}}$ and $e_0 \rightarrow_N^* \text{BndryErr}(\bar{b}_0, v_0)$ then:*

- 1478 • $\text{senders}(\bar{b}_0) = \text{owners}(v_0)$
- 1479 • $\bar{b}_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
- 1480 • *either has-boundary* $((\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1), e_0)$ *or has-boundary* $((\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0), e_0)$

1481 **PROOF.**

1482 1. Either $e_0 \rightarrow_N^* v_0$ where $v_0 : \tau? \overline{\text{wf}}$ or e_0 diverges, or $e_0 \rightarrow_N^* \text{Err}$

1483 by lemma 5.20 and lemma 5.21

1484 2. ASSUME $e_0 \rightarrow_N^* \text{BndryErr}(\bar{b}_0, v_0)$

1485 2.1. $\exists \ell_0, \tau_0, \ell_1. \bar{b}_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$

1486 2.1.1. QED

1509 by definition of $\xrightarrow{*}_{\mathbb{N}}$
 1510 2.2. $\text{owners}(v_0) = \ell_1$
 1511 2.2.1. QED
 1512 by lemma 5.21
 1513 2.3. $\exists v_1, E_0$ such that $e_0 \xrightarrow{*}_{\mathbb{N}} E_0[\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_1]$
 1514 by lemma 5.24
 1515 2.4. $(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \in e_0$ or $(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0) \in e_0$
 1516 by lemma 5.35
 1517 3. QED

□

1523 5.2 Natural Lemmas

1524 LEMMA 5.6 ($\vdash_{\mathbb{N}}$ PROGRESS). *If $\cdot \vdash_{\mathbb{N}} e_0 : \tau$? then one of the following holds:*

- 1525 • $e_0 \in v$
- 1526 • $e_0 \in \text{Err}$
- 1527 • $\exists e_1$ such that $e_0 \rightarrow_{\mathbb{N}} e_1$

1528 PROOF. By case analysis of e_0 .

1529 By lemma 5.10 it suffices to consider the following cases.

- 1530 1. CASE $e_0 \in v$
 - 1531 1.1. QED
- 1532 2. CASE $e_0 = E_0[\text{Err}]$
 - 1533 2.1. QED
- 1534 3. CASE $e_0 = E_0[\text{app}\{\tau_1\} v_0 v_1]$
 - 1535 3.1. $v_0 \in (\lambda(x:\tau). e) \cup (\text{mon } b v)$
 1536 by inversion $\vdash_{\mathbb{N}}$
 - 1537 3.2. SCASE $v_0 = \lambda(x_2:\tau_2). e_2$
 - 1538 3.2.1. QED
 1539 $e_0 \triangleright_{\mathbb{N}} E_0[e_2[x_2 \leftarrow v_1]]$
 - 1540 3.3. SCASE $v_0 = \text{mon}(\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2) \blacktriangleleft \ell_1) v_2$
 - 1541 3.3.1. QED
 1542 $e_0 \triangleright_{\mathbb{N}} E_0[\text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{app}\{\mathcal{U}\} v_2 (\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1))]$
- 1543 4. CASE $e_0 = E_0[\text{app}\{\mathcal{U}\} v_0 v_1]$
 - 1544 4.1. SCASE $v_0 = \lambda x_2. e_2$
 - 1545 4.1.1. QED
 1546 $e_0 \triangleright_{\mathbb{N}} E_0[e_2[x_2 \leftarrow v_1]]$
 - 1547 4.2. SCASE $v_0 = \text{mon}(\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2) \blacktriangleleft \ell_1) v_2$
 - 1548 4.2.1. QED
 1549 $e_0 \triangleright_{\mathbb{N}} E_0[\text{stat}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{app}\{\tau_2\} v_2 (\text{dyn}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1))]$
 - 1550 4.3. SCASE $v_0 \notin (\lambda(x:\tau). e) \cup (\text{mon } b v)$
 - 1551 4.3.1. QED

1560

1561 $e_0 \triangleright_N E_0[\text{TagErr} \bullet]$

1562 5. CASE $e_0 = E_0[\text{unop}\{\tau?\} v_0]$

1563 5.1. QED

1564 by lemma 5.11 and lemma 5.13

1565 6. CASE $e_0 = E_0[\text{binop}\{\tau?\} v_0 v_1]$

1566 6.1. QED

1567 by lemma 5.11 and lemma 5.13

1568 7. CASE $e_0 = E_0[\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0]$

1569 7.1. QED

1570 by lemma 5.11 and lemma 5.15

1571 8. CASE $e_0 = E_0[\text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0]$

1572 8.1. QED

1573 by lemma 5.11 and lemma 5.16

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LEMMA 5.7 (\vdash_N STATIC PRESERVATION). *If $\cdot \vdash_N e_0 : \tau?$ and $e_0 \rightarrow_N e_1$ then $\cdot \vdash_N e_1 : \tau?$.*

1581

1582

PROOF. By lemma 5.8 and lemma 5.9.

1583

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LEMMA 5.8 (\triangleright_N PRESERVATION). *If $\cdot \vdash_N e_0 : \tau_0$ and $e_0 \triangleright_N e_1$ then $\cdot \vdash_N e_1 : \tau_0$.*

1587

PROOF. By case analysis of \triangleright_N .

1588

1589

1. CASE $\text{unop}\{\tau_0\} v_0 \triangleright_N \delta_N(\text{unop}, v_0)$

1590

1.1. QED

1591

by lemma 5.14

1592

2. CASE $\text{binop}\{\tau_0\} v_0 v_1 \triangleright_N \delta_N(\text{binop}, v_0, v_1)$

1593

2.1. QED

1594

by lemma 5.14

1595

3. CASE $\text{app}\{\tau_0\}(\lambda(x_1 : \tau_1). e_1) v_2 \triangleright_N e_1[x_1 \leftarrow v_2]$

1596

3.1. QED

1597

by lemma 5.19

1598

4. CASE $\text{app}\{\tau_0\}(\text{mon}(\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_0) \blacktriangleleft \ell_1) v_0) v_1$

1599

$\triangleright_N \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)(\text{app}\{\mathcal{U}\} v_0 (\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1))$

1600

4.1. QED

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$$\frac{\frac{\text{by inversion } \vdash_N}{\cdot \vdash_N v_0 : \mathcal{U}} \quad \frac{\text{by inversion } \vdash_N}{\cdot \vdash_N v_1 : \tau_1}}{\cdot \vdash_N \text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1 : \mathcal{U}}}{\cdot \vdash_N \text{app}\{\mathcal{U}\} v_0 (\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1) : \mathcal{U}}}{\cdot \vdash_N \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)(\text{app}\{\mathcal{U}\} v_0 (\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1)) : \tau_0}$$

1613 5. CASE $\text{dyn } b_0 v_0 \triangleright_N v_1$

1614 5.1. QED

1615 by lemma 5.17

1617

1618

1619 LEMMA 5.9 (\triangleright_N PRESERVATION). *If $\cdot \vdash_N e_0 : \mathcal{U}$ and $e_0 \triangleright_N e_1$ then $\cdot \vdash_N e_1 : \mathcal{U}$.*

1620

1621 PROOF. By case analysis of \triangleright_N .

1622 1. CASE $\text{unop}\{\mathcal{U}\} v_0 \triangleright_N \text{TagErr} \bullet$

1623 1.1. QED

1624 $\cdot \vdash_N \text{TagErr} \bullet : \mathcal{U}$

1626 2. CASE $\delta_N(\text{unop}, v_0)$ is defined and $\text{unop}\{\mathcal{U}\} v_0 \triangleright_N \delta_N(\text{unop}, v_0)$

1627 2.1. QED

1628 by lemma 5.14

1629 3. CASE $\text{binop}\{\mathcal{U}\} v_0 v_1 \triangleright_N \text{TagErr} \bullet$

1630 3.1. QED

1632 $\cdot \vdash_N \text{TagErr} \bullet : \mathcal{U}$

1633 4. CASE $\delta_N(\text{binop}, v_0, v_1)$ is defined and $\text{binop}\{\mathcal{U}\} v_0 v_1 \triangleright_N \delta_N(\text{binop}, v_0, v_1)$

1634 4.1. QED

1636 by lemma 5.14

1637 5. CASE $\text{app}\{\mathcal{U}\} (\lambda x_1. e_1) v_2 \triangleright_N e_1[x_1 \leftarrow v_2]$

1638 5.1. QED

1639 by lemma 5.19

1641 6. CASE $\text{app}\{\mathcal{U}\} (\text{mon } (\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_0) \blacktriangleleft \ell_1) v_0) v_1$

1642 $\triangleright_N \text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{app}\{\tau_0\} v_0 (\text{dyn } (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1))$

1643 6.1. QED

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7. CASE $\text{stat } b_0 v_0 \triangleright_N v_1$

7.1. QED

by lemma 5.18

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1664

LEMMA 5.10 (UNIQUE DECOMPOSITION). *If $\cdot \vdash_N e_0 : \tau?$ then either:*

- $e_0 \in v$
- $e_0 = E_0[\text{app}\{\tau?\} v_0 v_1]$
- $e_0 = E_0[\text{unop}\{\tau?\} v_0]$

- 1665 • $e_0 = E_0[\text{binop}\{\tau?\} v_0 v_1]$
- 1666 • $e_0 = E_0[\text{dyn } b_1 v_1]$
- 1667 • $e_0 = E_0[\text{stat } b_1 v_1]$
- 1668 • $e_0 = E_0[\text{Err}]$

1670

PROOF. By induction on the structure of e_0 .

1671

1. CASE $e_0 = x_0$

1672

1.1. CONTRADICTION:

1673

• $\vdash_{\mathbb{N}} e_0 : \tau?$

1674

2. CASE $e_0 = v_0$

1675

2.1. QED

1676

3. CASE $e_0 = \langle e_1, e_2 \rangle$

1677

3.1. SCASE $e_1 \notin v$

1678

3.1.1. QED

1679

by the induction hypothesis

1680

3.2. SCASE $e_1 \in v$ and $e_2 \notin v$

1681

3.2.1. QED

1682

by the induction hypothesis

1683

3.3. SCASE $e_1 \in v$ and $e_2 \in v$

1684

3.3.1. QED

1685

$e_0 \in v$

1686

4. CASE $e_0 = \text{app}\{\tau?\} e_1 e_2$

1687

4.1. QED

1688

by the induction hypothesis

1689

5. CASE $e_0 = \text{unop}\{\tau?\} e_1$

1690

5.1. QED

1691

by the induction hypothesis

1692

6. CASE $e_0 = \text{binop}\{\tau?\} e_1 e_2$

1693

6.1. QED

1694

by the induction hypothesis

1695

7. CASE $e_0 = \text{dyn } b_1 e_1$

1696

7.1. QED

1697

by the induction hypothesis

1698

8. CASE $e_0 = \text{stat } b_1 e_1$

1699

8.1. QED

1700

by the induction hypothesis

1701

9. CASE $e_0 \in \text{Err}$

1702

9.1. QED

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LEMMA 5.11. *If $\cdot \vdash_{\mathbb{N}} E_0[e_0] : \tau?$ then one of the following holds:*

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□

- 1717 • $\cdot \vdash_N e_0 : \mathcal{U}$
- 1718 • $\exists \tau_0 . \cdot \vdash_N e_0 : \tau_0$
- 1719

1720 PROOF. By induction on the structure of E_0 and case analysis of \vdash_N .

- 1721 1. CASE $E_0 = []$
- 1722 1.1. QED
- 1723
- 1724 2. CASE $E_0 = \langle E_1, e_2 \rangle$
- 1725 2.1. $\cdot \vdash_N E_1[e_0] : \tau?$
- 1726 by inversion \vdash_N
- 1727 2.2. QED
- 1728 by the induction hypothesis
- 1729
- 1730 3. CASE $E_0 = \langle v_1, E_2 \rangle$
- 1731 3.1. QED
- 1732 by the induction hypothesis
- 1733
- 1734 4. CASE $E_0 = \text{app}\{\tau?\} E_1 e_2$
- 1735 4.1. QED
- 1736 by the induction hypothesis
- 1737
- 1738 5. CASE $E_0 = \text{app}\{\tau?\} v_1 E_2$
- 1739 5.1. QED
- 1740 by the induction hypothesis
- 1741
- 1742 6. CASE $E_0 = \text{unop}\{\tau?\} E_1$
- 1743 6.1. QED
- 1744 by the induction hypothesis
- 1745
- 1746 7. CASE $E_0 = \text{binop}\{\tau?\} E_1 e_2$
- 1747 7.1. QED
- 1748 by the induction hypothesis
- 1749
- 1750 8. CASE $E_0 = \text{binop}\{\tau?\} v_1 E_2$
- 1751 8.1. QED
- 1752 by the induction hypothesis
- 1753
- 1754 9. CASE $E_0 = \text{dyn } b_1 E_1$
- 1755 9.1. QED
- 1756 by the induction hypothesis
- 1757
- 1758 10. CASE $E_0 = \text{stat } b_1 E_1$
- 1759 10.1. QED
- 1760 by the induction hypothesis

□

1762 LEMMA 5.12 (\vdash_N REPLACEMENT).

- 1764 • If $\cdot \vdash_N E_0[e_0] : \tau?$ and the derivation contains a proof of $\cdot \vdash_N e_0 : \tau_0$ and $\cdot \vdash_N e_1 : \tau_0$ then $\cdot \vdash_N E_0[e_1] : \tau?$.
- 1765 • If $\cdot \vdash_N E_0[e_0] : \tau?$ and the derivation contains a proof of $\cdot \vdash_N e_0 : \mathcal{U}$ and $\cdot \vdash_N e_1 : \mathcal{U}$ then $\cdot \vdash_N E_0[e_1] : \tau?$.
- 1766

1767 PROOF. By induction on E_0 .

1768

- 1769 1. CASE $E_0 = []$
 1770 1.1. QED
 1771 1771 2. CASE $E_0 = \langle E_1, e_2 \rangle$
 1772 1772 2.1. QED
 1773 1773 by the induction hypothesis
 1774 1774 3. CASE $E_0 = \langle v_1, E_2 \rangle$
 1775 1775 3.1. QED
 1776 1776 by the induction hypothesis
 1777 1777 4. CASE $E_0 = \text{app}\{\tau?\} E_1 e_2$
 1778 1778 4.1. QED
 1779 1779 by the induction hypothesis
 1780 1780 5. CASE $E_0 = \text{app}\{\tau?\} v_1 E_2$
 1781 1781 5.1. QED
 1782 1782 by the induction hypothesis
 1783 1783 6. CASE $E_0 = \text{unop}\{\tau?\} E_1$
 1784 1784 6.1. QED
 1785 1785 by the induction hypothesis
 1786 1786 7. CASE $E_0 = \text{binop}\{\tau?\} E_1 e_2$
 1787 1787 7.1. QED
 1788 1788 by the induction hypothesis
 1789 1789 8. CASE $E_0 = \text{binop}\{\tau?\} v_1 E_2$
 1790 1790 8.1. QED
 1791 1791 by the induction hypothesis
 1792 1792 9. CASE $E_0 = \text{dyn } b_1 E_1$
 1793 1793 9.1. QED
 1794 1794 by the induction hypothesis
 1795 1795 10. CASE $E_0 = \text{stat } b_1 E_1$
 1796 1796 10.1. QED
 1797 1797 by the induction hypothesis
 1798 1798
 1799 1799
 1800 1800
 1801 1801
 1802 1802
 1803 1803
 1804 1804
 1805 1805
 1806 1806

□

LEMMA 5.13 (δ_N TYPE PROGRESS).

- 1809 • If $\cdot \vdash_N \text{unop}\{\tau_1\} v_0 : \tau_0$ then $\delta_N(\text{unop}, v_0)$ is defined.
 1810 • if $\cdot \vdash_N \text{binop}\{\tau_1\} v_0 v_1 : \tau_0$ then $\delta_N(\text{binop}, v_0, v_1)$ is defined.
 1811 • If $\cdot \vdash_N \text{unop}\{\mathcal{U}\} v_0 : \mathcal{U}$ then $\text{unop}\{\mathcal{U}\} v_0 \blacktriangleright_N e_1$.
 1812 • if $\cdot \vdash_N \text{binop}\{\mathcal{U}\} v_0 v_1 : \mathcal{U}$ then $\text{binop}\{\mathcal{U}\} v_0 v_1 \blacktriangleright_N e_1$.
 1813
 1814

PROOF. By case analysis of δ_N , \vdash_N , and \blacktriangleright_N .

- 1816 1. CASE $\cdot \vdash_N \text{fst}\{\tau_1\} v_0 : \tau_0$
 1817 1.1. $v_0 = \langle v_1, v_2 \rangle$
 1818 by \vdash_N canonical forms
 1819

1821 1.2. QED
 1822 $\delta_N(\text{unop}, v_0) = v_1$
 1823
 1824 2. CASE $\cdot \vdash_N \text{snd}\{\tau_1\} v_0 : \tau_0$
 1825 2.1. $v_0 = \langle v_1, v_2 \rangle$
 1826 by \vdash_N canonical forms
 1827 2.2. QED
 1828 $\delta_N(\text{unop}, v_0) = v_2$
 1829
 1830 3. CASE $\cdot \vdash_N \text{sum}\{\tau_1\} v_0 v_1 : \tau_0$
 1831 3.1. $v_0 \in i$ and $v_1 \in i$
 1832 by \vdash_N canonical forms
 1833 3.2. QED
 1834 $\delta_N(\text{unop}, v_0, v_1) \in i$
 1835
 1836 4. CASE $\cdot \vdash_N \text{quotient}\{\tau_1\} v_0 v_1 : \tau_0$
 1837 4.1. $v_0 \in i$ and $v_1 \in i$
 1838 by \vdash_N canonical forms
 1839 4.2. QED
 1840 $\delta_N(\text{unop}, v_0, v_1) \in i \cup \text{DivErr}$
 1841
 1842 5. CASE $\cdot \vdash_N \text{unop}\{\mathcal{U}\} v_0 : \mathcal{U}$
 1843 5.1. SCASE $\delta_N(\text{unop}, v_0)$ is defined
 1844 5.1.1. QED
 1845 $\text{unop}\{\mathcal{U}\} v_0 \blacktriangleright_N \delta_N(\text{unop}, v_0)$
 1846 5.2. SCASE $\delta_N(\text{unop}, v_0)$ is not defined
 1847 5.2.1. QED
 1848 $\text{unop}\{\mathcal{U}\} v_0 \blacktriangleright_N \text{TagErr} \bullet$
 1849
 1850 6. CASE $\cdot \vdash_N \text{binop}\{\mathcal{U}\} v_0 v_1 : \mathcal{U}$
 1851 6.1. SCASE $\delta_N(\text{binop}, v_0, v_1)$ is defined
 1852 6.1.1. QED
 1853 $\text{binop}\{\mathcal{U}\} v_0 v_1 \blacktriangleright_N \delta_N(\text{binop}, v_0, v_1)$
 1854 6.2. SCASE $\delta_N(\text{binop}, v_0, v_1)$ is not defined
 1855 6.2.1. QED
 1856 $\text{binop}\{\mathcal{U}\} v_0 v_1 \blacktriangleright_N \text{TagErr} \bullet$
 1857
 1858
 1859

□

1860
 1861
 1862 LEMMA 5.14 (δ_N TYPE PRESERVATION).
 1863

- 1864 • If $\cdot \vdash_N \text{unop}\{\tau_1\} v_0 : \tau_0$ then $\cdot \vdash_N \delta_N(\text{unop}, v_0) : \tau_0$.
- 1865 • If $\cdot \vdash_N \text{binop}\{\tau_1\} v_0 v_1 : \tau_0$ then $\cdot \vdash_N \delta_N(\text{binop}, v_0, v_1) : \tau_0$.
- 1866 • If $\cdot \vdash_N \text{unop}\{\mathcal{U}\} v_0 : \mathcal{U}$ and $\delta_N(\text{unop}, v_0)$ is defined then $\cdot \vdash_N \delta_N(\text{unop}, v_0) : \mathcal{U}$.
- 1867 • If $\cdot \vdash_N \text{binop}\{\mathcal{U}\} v_0 v_1 : \mathcal{U}$ and $\delta_N(\text{binop}, v_0, v_1)$ is defined then $\cdot \vdash_N \delta_N(\text{binop}, v_0, v_1) : \mathcal{U}$.

1869 PROOF. By case analysis of δ_N and \vdash_N .
 1870

1871 1. CASE $\cdot \vdash_N \text{fst}\{\tau_0\} v_0 : \tau_0$
 1872

- 1873 1.1. $v_0 = \langle v_1, v_2 \rangle$ and $\cdot \vdash_N v_1 : \tau_0$
 1874 by inversion \vdash_N
 1875 1.2. QED
 1876
 1877 2. CASE $\cdot \vdash_N \text{snd}\{\tau_0\} v_0 : \tau_0$
 1878 2.1. $v_0 = \langle v_1, v_2 \rangle$ and $\cdot \vdash_N v_2 : \tau_0$
 1879 by \vdash_N canonical forms and \vdash_N inversion
 1880 2.2. QED
 1881
 1882 3. CASE $\cdot \vdash_N \text{sum}\{\tau_1\} v_0 v_1 : \tau_0$
 1883 3.1. $\tau_0 \in \text{Int} \cup \text{Nat}$
 1884 by inversion \vdash_N
 1885 3.2. SCASE $\tau_0 = \text{Int}$
 1886 3.2.1. $\cdot \vdash_N v_0 : \text{Int}$ and $\cdot \vdash_N v_1 : \text{Int}$
 1887 by inversion \vdash_N
 1888 3.2.2. $v_0 \in i$ and $v_1 \in i$
 1889 by \vdash_N canonical forms
 1890 3.2.3. QED
 1891 $\delta_N(\text{binop}, v_0, v_1) \in i$
 1892
 1893 3.3. SCASE $\tau_0 = \text{Nat}$
 1894 3.3.1. $\cdot \vdash_N v_0 : \text{Nat}$ and $\cdot \vdash_N v_1 : \text{Nat}$
 1895 by inversion \vdash_N
 1896 3.3.2. $v_0 \in n$ and $v_1 \in n$
 1897 by \vdash_N canonical forms
 1898 3.3.3. QED
 1899 $\delta_N(\text{binop}, v_0, v_1) \in n$
 1900
 1901 4. CASE $\cdot \vdash_N \text{quotient}\{\tau_1\} v_0 v_1 : \tau_0$
 1902 4.1. $\tau_0 \in \text{Int} \cup \text{Nat}$
 1903 by inversion \vdash_N
 1904 4.2. SCASE $\tau_0 = \text{Int}$
 1905 4.2.1. $\cdot \vdash_N v_0 : \text{Int}$ and $\cdot \vdash_N v_1 : \text{Int}$
 1906 by inversion \vdash_N
 1907 4.2.2. $v_0 \in i$ and $v_1 \in i$
 1908 by \vdash_N canonical forms
 1909 4.2.3. QED
 1910 $\delta_N(\text{binop}, v_0, v_1) \in i \cup \text{DivErr}$
 1911
 1912 4.3. SCASE $\tau_0 = \text{Nat}$
 1913 4.3.1. $\cdot \vdash_N v_0 : \text{Nat}$ and $\cdot \vdash_N v_1 : \text{Nat}$
 1914 by inversion \vdash_N
 1915 4.3.2. $v_0 \in n$ and $v_1 \in n$
 1916 by \vdash_N canonical forms
 1917 4.3.3. QED
 1918 $\delta_N(\text{binop}, v_0, v_1) \in n \cup \text{DivErr}$
 1919
 1920
 1921
 1922
 1923
 1924

- 1925 5. CASE $\cdot \vdash_N \text{fst}\{\mathcal{U}\} v_0 : \mathcal{U}$
 1926 5.1. $v_0 = \langle v_1, v_2 \rangle$
 1927 because $\delta_N(\text{unop}, v_0)$ is defined
 1928 5.2. $\cdot \vdash_N v_1 : \mathcal{U}$
 1929 by inversion \vdash_N
 1930 5.3. QED
 1931 $\delta_N(\text{unop}, v_0) = v_1$
 1932
 1933 6. CASE $\cdot \vdash_N \text{snd}\{\mathcal{U}\} v_0 : \mathcal{U}$
 1934 6.1. $v_0 = \langle v_1, v_2 \rangle$
 1935 because $\delta_N(\text{unop}, v_0)$ is defined
 1936 6.2. $\cdot \vdash_N v_2 : \mathcal{U}$
 1937 by inversion \vdash_N
 1938 6.3. QED
 1939 $\delta_N(\text{unop}, v_0) = v_2$
 1940
 1941 7. CASE $\cdot \vdash_N \text{sum}\{\mathcal{U}\} v_0 v_1 : \mathcal{U}$
 1942 7.1. $\delta_N(\text{binop}, v_0, v_1) \in i$
 1943 by definition δ_N
 1944 7.2. QED
 1945 $\cdot \vdash_N \delta_N(\text{binop}, v_0, v_1) : \mathcal{U}$
 1946
 1947 8. CASE $\cdot \vdash_N \text{quotient}\{\mathcal{U}\} v_0 v_1 : \mathcal{U}$
 1948 8.1. $\delta_N(\text{binop}, v_0, v_1) \in i \cup \text{DivErr}$
 1949 by definition δ_N
 1950 8.2. QED
 1951 $\cdot \vdash_N \delta_N(\text{binop}, v_0, v_1) : \mathcal{U}$
 1952
 1953
 1954
 1955

□

1956
 1957 LEMMA 5.15. *If $\cdot \vdash_N \text{dyn } b_0 v_0 : \tau_0$ and $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ then $\exists e_1$ such that $\text{dyn } b_0 v_0 \triangleright_N e_1$.*

1958 PROOF. By case analysis of $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$.

- 1959 1. CASE $\text{tag-match}(\lfloor \tau_0 \rfloor, \lambda x_1. e_1)$
 1960 1.1. QED
 1961 $\text{dyn } b_0 v_0 \triangleright_N \text{mon } b_0 v_0$
 1962
 1963 2. CASE $\text{tag-match}(\lfloor \tau_0 \rfloor, \lambda(x_1 : \tau_1). e_1)$
 1964 2.1. CONTRADICTION:
 1965 $\cdot \vdash_N \text{dyn } b_0 v_0 : \tau_0$
 1966
 1967 3. CASE $\text{tag-match}(\lfloor \tau_0 \rfloor, \text{mon } b_1 v_1)$
 1968 3.1. QED
 1969 $\text{dyn } b_0 v_0 \triangleright_N \text{mon } b_0 v_0$
 1970
 1971 4. CASE $\text{tag-match}(\lfloor (\tau_1 \times \tau_2) \rfloor, \langle v_1, v_2 \rangle)$
 1972 4.1. QED
 1973 $\text{dyn } b_0 v_0 \triangleright_N \langle \text{dyn } (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_1, \text{dyn } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rangle$
 1974
 1975 5. CASE $\text{tag-match}(\lfloor \text{Int} \rfloor, v_0)$
 1976

1977 5.1. QED

1978 $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} v_0$

1979 6. CASE $\text{tag-match}([\text{Nat}], v_0)$

1980 6.1. QED

1981 $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} v_0$

1982 7. CASE $\neg \text{tag-match}([\tau_0], v_0)$

1983 7.1. QED

1984 $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \text{BndryErr}(b_0, v_0)$

□

1985 LEMMA 5.16. *If $\cdot \vdash_{\mathbb{N}} \text{stat } b_0 v_0 : \mathcal{U}$ and $b_0 = (\ell_0 \triangleleft \tau_0 \triangleleft \ell_1)$ then $\exists e_1$ such that $\text{stat } b_0 v_0 \blacktriangleright_{\mathbb{N}} e_1$.*

1986 PROOF. By case analysis on v_0 .

1987 1. CASE $v_0 \in \lambda x. e$

1988 1.1. CONTRADICTION:

1989 $\cdot \vdash_{\mathbb{N}} \text{stat } b_0 v_0 : \mathcal{U}$

1990 2. CASE $v_0 \in \lambda(x:\tau). e$

1991 2.1. QED

1992 $\text{stat } b_0 v_0 \blacktriangleright_{\mathbb{N}} \text{mon } b_0 v_0$

1993 3. CASE $v_0 \in \text{mon } b e$

1994 3.1. QED

1995 $\text{stat } b_0 v_0 \blacktriangleright_{\mathbb{N}} \text{mon } b_0 v_0$

1996 4. CASE $v_0 = \langle v_1, v_2 \rangle$

1997 4.1. $\tau_0 = \tau_1 \times \tau_2$

1998 by inversion $\vdash_{\mathbb{N}}$

1999 4.2. QED

2000 $\text{stat } b_0 v_0 \blacktriangleright_{\mathbb{N}} \langle \text{stat } (\ell_0 \triangleleft \tau_1 \triangleleft \ell_1) v_1, \text{stat } (\ell_0 \triangleleft \tau_2 \triangleleft \ell_1) v_2 \rangle$

2001 5. CASE $v_0 \in i$

2002 5.1. QED

2003 $\text{stat } b_0 v_0 \blacktriangleright_{\mathbb{N}} v_0$

□

2004 LEMMA 5.17 (N-dyn PRESERVATION). *If $\cdot \vdash_{\mathbb{N}} \text{dyn } b_0 v_0 : \tau_0$ and $b_0 = (\ell_0 \triangleleft \tau_0 \triangleleft \ell_1)$ and $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} e_1$ then $\cdot \vdash_{\mathbb{N}} e_1 : \tau_0$.*

2005 PROOF. By case analysis of $\triangleright_{\mathbb{N}}$.

2006 1. CASE $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \text{mon } b_0 v_0$

2007 and $v_0 \in (\lambda x. e) \cup (\text{mon } b v)$

2008 1.1. QED

2009 by inversion $\vdash_{\mathbb{N}}$

2010 $\cdot \vdash_{\mathbb{N}} v_0 : \mathcal{U}$

2011 $\cdot \vdash_{\mathbb{N}} \text{mon } b_0 v_0 : \tau_0$

2029 2. CASE $\tau_0 = \tau_1 \times \tau_2$
 2030 and $\text{dyn}(\ell_0 \blacktriangleleft (\tau_1 \times \tau_2) \blacktriangleleft \ell_1) \langle v_1, v_2 \rangle \triangleright_N \langle \text{dyn}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_1, \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rangle$
 2031

2032 2.1. QED

$$\begin{array}{c}
 \text{by inversion } \vdash_N \qquad \qquad \qquad \text{by inversion } \vdash_N \\
 \hline
 \cdot \vdash_N v_1 : \mathcal{U} \qquad \qquad \qquad \cdot \vdash_N v_2 : \mathcal{U} \\
 \hline
 \cdot \vdash_N \text{dyn}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_1 : \tau_1 \qquad \cdot \vdash_N \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 : \tau_2 \\
 \hline
 \cdot \vdash_N \langle \text{dyn}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_1, \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rangle : \tau_1 \times \tau_2
 \end{array}$$

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 2039 3. CASE $\text{dyn } b_0 i_0 \triangleright_N i_0$
 2040 and $\text{tag-match}(\lfloor \tau_0 \rfloor, i_0)$

2041 3.1. QED

2042 by case analysis of $\text{tag-match}(\lfloor \tau_0 \rfloor, i_0)$

2043 4. CASE $\text{dyn } b_0 v_0 \triangleright_N \text{BndryErr}(b_0, v_0)$

2044 4.1. QED

2045 $\cdot \vdash_N \text{BndryErr}(b_0, v_0) : \tau_0$

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LEMMA 5.18 (N-stat PRESERVATION). *If $\cdot \vdash_N \text{stat } b_0 v_0 : \mathcal{U}$ and $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ and $\text{stat } b_0 v_0 \blacktriangleright_N e_1$ then $\cdot \vdash_N e_1$.*

PROOF. By case analysis of \blacktriangleright_N .

1. CASE $v_0 \in (\lambda(x:\tau). e) \cup (\text{mon } b v)$

and $\text{stat } b_0 v_0 \blacktriangleright_N \text{mon } b_0 v_0$

1.1. QED

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$$\begin{array}{c}
 \text{by inversion } \vdash_N \\
 \hline
 \cdot \vdash_N v_0 : \tau_0 \\
 \hline
 \cdot \vdash_N \text{mon } b_0 v_0 : \mathcal{U}
 \end{array}$$

2. CASE $\tau_0 = \tau_1 \times \tau_2$
 and $\text{stat}(\ell_0 \blacktriangleleft (\tau_1 \times \tau_2) \blacktriangleleft \ell_1) \langle v_1, v_2 \rangle \blacktriangleright_N \langle \text{stat}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_1, \text{stat}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rangle$

2.1. QED

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$$\begin{array}{c}
 \text{by inversion } \vdash_N \qquad \qquad \qquad \text{by inversion } \vdash_N \\
 \hline
 \cdot \vdash_N v_1 : \tau_1 \qquad \qquad \qquad \cdot \vdash_N v_2 : \tau_2 \\
 \hline
 \cdot \vdash_N \text{stat}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_1 : \mathcal{U} \qquad \cdot \vdash_N \text{stat}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 : \mathcal{U} \\
 \hline
 \cdot \vdash_N \langle \text{stat}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_1, \text{stat}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rangle : \mathcal{U}
 \end{array}$$

3. CASE $\text{stat } b_0 i_0 \blacktriangleright_N i_0$

3.1. QED

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4. CASE $\text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \blacktriangleright_N \text{TagErr} \circ$

4.1. CONTRADICTION:

2081 $\cdot \vdash_N v_0 : \tau_0$

2082

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□

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LEMMA 5.19.

2086

• *If $(x_0 : \tau_0), \Gamma_0 \vdash_N e_1 : \tau?$ and $\cdot \vdash_N v_0 : \tau_0$ then $\Gamma_0 \vdash_N e_1[x_0 \leftarrow v_0] : \tau?$*

2087

• *If $(x_0 : \mathcal{U}), \Gamma_0 \vdash_N e_1 : \tau?$ and $\cdot \vdash_N v_0 : \mathcal{U}$ then $\Gamma_0 \vdash_N e_1[x_0 \leftarrow v_0] : \tau?$*

2088

2089

PROOF. By induction on e_1 .

2090

1. CASE $e_1 = x_2$

2091

1.1. SCASE $x_0 = x_2$

2092

1.1.1. QED

2093

$e_1[x_0 \leftarrow v_0] = v_0$

2094

2095

1.2. SCASE $x_0 \neq x_2$

2096

1.2.1. QED

2097

$e_1[x_0 \leftarrow v_0] = e_1$

2098

2099

2. CASE $e_1 = i_0$

2100

2.1. QED

2101

$e_1[x_0 \leftarrow v_0] = e_1$

2102

2103

3. CASE $e_1 = \lambda x_2. e_2$

2104

3.1. SCASE $x_0 = x_2$

2105

3.1.1. QED

2106

by the induction hypothesis

2107

3.2. SCASE $x_0 \neq x_2$

2108

3.2.1. QED

2109

$e_1[x_0 \leftarrow v_0] = e_1$

2110

2111

4. CASE $e_1 = \lambda(x_2 : \tau_2). e_2$

2112

4.1. SCASE $x_0 = x_2$

2113

4.1.1. QED

2114

by the induction hypothesis

2115

2116

4.2. SCASE $x_0 \neq x_2$

2117

4.2.1. QED

2118

$e_1[x_0 \leftarrow v_0] = e_1$

2119

2120

5. CASE $e_1 = \langle e_2, e_3 \rangle$

2121

5.1. QED

2122

by the induction hypothesis

2123

6. CASE $e_1 = \text{app}\{\tau?\} e_2 e_3$

2124

6.1. QED

2125

by the induction hypothesis

2126

2127

7. CASE $e_1 = \text{unop}\{\tau?\} e_2$

2128

7.1. QED

2129

by the induction hypothesis

2130

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- 2133 8. CASE $e_1 = \text{binop}\{\tau?\} e_2 e_3$
 2134 8.1. QED
 2135 by the induction hypothesis
 2136
 2137 9. CASE $e_1 = \text{dyn } b_2 e_2$
 2138 9.1. QED
 2139 by the induction hypothesis
 2140
 2141 10. CASE $e_1 = \text{stat } b_2 e_2$
 2142 10.1. QED
 2143 by the induction hypothesis
 2144
 2145

□

2146 LEMMA 5.20 ($\overline{\text{P}}\text{-PROGRESS}$). *If $(e_0)^{\ell_0} : \tau? \overline{\text{wf}}$ and $;\ell_0 \overline{\text{P}} e_0$ then one of the following holds:*

- 2148 • $e_0 \in ((v))^\ell$
 2149 • $e_0 \in E[\text{Err}]^\ell$
 2150
 2151 • $\exists e_1$ such that $e_0 \xrightarrow{\text{N}} e_1$
 2152

2153 PROOF. By case analysis of e_0 .

2154 By lemma 5.25, it suffices to consider the following cases.

- 2155 1. CASE $e_0 \in ((v))^\ell$
 2156 1.1. QED
 2157
 2158 2. CASE $e_0 \in E[\text{Err}]^\ell$
 2159 2.1. QED
 2160
 2161 3. CASE $e_0 = E[\text{app}\{\tau_1\} ((v_0))^{\ell_0} ((v_1))^{\ell_0}]^{\ell_0}$
 2162 3.1. $v_0 \in (\lambda(x:\tau). e) \cup (\text{mon } b v)$
 2163 by \vdash_{N} canonical forms and \vdash_{N} inversion
 2164 3.2. SCASE $v_0 = \lambda(x_2:\tau_2). e_2$
 2165 3.2.1. QED
 2166 $e_0 \triangleright_{\text{N}} E[e_2[x_2 \leftarrow (v_1)^{\ell_0 \ell_0 \ell_0}]]^{\ell_0}$
 2167
 2168 3.3. SCASE $v_0 = \text{mon } (\ell_0 \blacktriangleleft (\tau_2 \Rightarrow \tau_3) \blacktriangleleft \ell_1) v_2$
 2169 3.3.1. LET $b_3 = (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)$
 2170 and $b_4 = (\ell_1 \blacktriangleleft \tau_2 \blacktriangleleft \ell_0)$
 2171 3.3.2. QED
 2172 $e_0 \triangleright_{\text{N}} E[\text{dyn } b_3 (\text{app}\{\mathcal{U}\} v_2 (\text{stat } b_4 ((v_1))^{\ell_0 \ell_0 \ell_0})^{\ell_2})]^{\ell_0}$
 2173
 2174 4. CASE $e_0 = E[\text{app}\{\mathcal{U}\} ((v_0))^{\ell_0} ((v_1))^{\ell_0}]^{\ell_0}$
 2175 4.1. SCASE $v_0 = \lambda x_2. e_2$
 2176 4.1.1. QED
 2177 $e_0 \triangleright_{\text{N}} E[e_2[x_2 \leftarrow (v_0)^{\ell_0 \ell_0 \ell_0}]]^{\ell_0}$
 2178
 2179 4.2. SCASE $v_0 = \text{mon } (\ell_0 \blacktriangleleft (\tau_2 \Rightarrow \tau_3) \blacktriangleleft \ell_1) v_2$
 2180 4.2.1. LET $b_3 = (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)$
 2181 and $b_4 = (\ell_1 \blacktriangleleft \tau_2 \blacktriangleleft \ell_0)$
 2182 and $\tau_4 = \text{forget}(\tau_3)$
 2183
 2184

2185 4.2.2. QED

$$2186 e_0 \triangleright_{\mathbb{N}} E[\text{stat } b_3 (\text{app}\{\tau_4\} v_2 (\text{dyn } b_4 ((v_1))^{\ell_0 \ell_0 \ell_0}))^{\ell_2}]^{\ell_0}$$

2187 4.3. SCASE $v_0 \notin (\lambda x. e) \cup (\text{mon } b v)$

2188 4.3.1. QED

$$2189 e_0 \triangleright_{\mathbb{N}} E[\text{TagErr } \bullet]^{\ell_0}$$

2190 5. CASE $e_0 = E[\text{unop}\{\tau?\} ((v_0))^{\ell_0}]^{\ell_0}$

2192 5.1. QED

2193 by lemma 5.11 and lemma 5.28

2194 6. CASE $e_0 = E[\text{binop}\{\tau?\} ((v_0))^{\ell_0} ((v_1))^{\ell_0}]^{\ell_0}$

2195 6.1. QED

2196 by lemma 5.11 and lemma 5.28

2197 7. CASE $e_0 = E[\text{dyn } b_1 ((v_1))^{\ell_0}]^{\ell_0}$

2198 7.1. QED

2199 by lemma 5.11 and lemma 5.30

2200 8. CASE $e_0 = E[\text{stat } b_1 ((v_1))^{\ell_0}]^{\ell_0}$

2201 8.1. QED

2202 by lemma 5.11 and lemma 5.31

2203 □

2204 LEMMA 5.21 ($\overline{\text{IF}}$ -PRESERVATION). *If $(e_0)^{\ell_0} : \tau? \overline{\text{wf}}$ and $e_0 \longrightarrow_{\mathbb{N}} e_1$ then $;\ell_0 \overline{\text{IF}} e_1$*

2205 PROOF. By lemma 5.22 and lemma 5.23.

2206 LEMMA 5.22. *If $(e_0)^{\ell_0} : \tau_0 \overline{\text{wf}}$ and $e_0 \triangleright_{\mathbb{N}} e_1$ then $;\ell_0 \overline{\text{IF}} e_1$*

2207 PROOF. By case analysis of $\triangleright_{\mathbb{N}}$. □

2208 1. CASE $\delta_{\mathbb{N}}(\text{unop}, ((v_0))^{\ell_0})$ is defined

$$2209 \text{ and } (\text{unop}\{\tau?\} ((v_0))^{\ell_0})^{\ell_0} \triangleright_{\mathbb{N}} (\delta_{\mathbb{N}}(\text{unop}, ((v_0))^{\ell_0}))^{\ell_0}$$

2210 1.1. QED

2211 by lemma 5.29

2212 2. CASE $\delta_{\mathbb{N}}(\text{binop}, ((v_0))^{\ell_0}, ((v_1))^{\ell_0})$ is defined

$$2213 \text{ and } (\text{binop}\{\tau?\} ((v_0))^{\ell_0} ((v_1))^{\ell_0})^{\ell_0} \triangleright_{\mathbb{N}} (\delta_{\mathbb{N}}(\text{binop}, ((v_0))^{\ell_0}, ((v_1))^{\ell_0}))^{\ell_0}$$

2214 2.1. QED

2215 by lemma 5.29

2216 3. CASE $(\text{app}\{\tau_0\} ((\lambda(x_0 : \tau_1). e_0))^{\ell_0} ((v_1))^{\ell_0})^{\ell_0} \triangleright_{\mathbb{N}} (e_0[x_0 \leftarrow ((v_1))^{\ell_0 \ell_0 \ell_0}])^{\ell_0}$

2217 3.1. QED

2218 by lemma 5.34

2219 4. CASE $(\text{app}\{\tau_0\} ((\text{mon } (\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2)^{\ell_1} \blacktriangleleft \ell_1) (v_0)^{\ell_2}))^{\ell_0} ((v_1))^{\ell_0})^{\ell_0}$

$$2220 \triangleright_{\mathbb{N}} (\text{dyn } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{app}\{\mathcal{U}\} v_0 (\text{stat } (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) ((v_1))^{\ell_0 \ell_0 \ell_0}))^{\ell_2})^{\ell_0}$$

2221 4.0.1. $\ell_1 = \ell_2$

2222 by inversion $\overline{\text{IF}}$

2223 2019-10-03 17:26. Page 43 of 1–148.

4.0.2. QED

$$\begin{array}{c}
\text{by inversion } \overline{\Gamma} \\
\frac{}{\vdash; \ell_0 \overline{\Gamma} v_1} \\
\text{by inversion } \overline{\Gamma} \quad \frac{}{\vdash; \ell_0 \overline{\Gamma} ((v_1))^{\ell_0 \ell_0 \ell_0}} \\
\frac{\vdash; \ell_1 \overline{\Gamma} v_0 \quad \vdash; \ell_1 \overline{\Gamma} \text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) ((v_1))^{\ell_0 \ell_0 \ell_0}}{\vdash; \ell_1 \overline{\Gamma} \text{app}\{\mathcal{U}\} v_0 (\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) ((v_1))^{\ell_0 \ell_0 \ell_0})} \\
\frac{\vdash; \ell_1 \overline{\Gamma} (\text{app}\{\mathcal{U}\} v_0 (\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) ((v_1))^{\ell_0 \ell_0 \ell_0}))^{\ell_2}}{\vdash; \ell_0 \overline{\Gamma} \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{app}\{\mathcal{U}\} v_0 (\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) ((v_1))^{\ell_0 \ell_0 \ell_0}))^{\ell_2}} \\
\vdash; \ell_0 \overline{\Gamma} (\text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{app}\{\mathcal{U}\} v_0 (\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) ((v_1))^{\ell_0 \ell_0 \ell_0}))^{\ell_2})^{\ell_0}
\end{array}$$

5. CASE $(\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) ((v_0))^{\ell_2})^{\ell_0} \triangleright_{\mathbb{N}} e_2$
by lemma 5.32

□

LEMMA 5.23. *If $(e_0)^{\ell_0} : \mathcal{U} \overline{\text{wf}}$ and $e_0 \triangleright_{\mathbb{N}} e_1$ then $\vdash; \ell_0 \overline{\Gamma} e_1$*

PROOF. By case analysis of $\triangleright_{\mathbb{N}}$.

1. CASE $\delta_N(\text{unop}, ((v_0))^{\ell_0})$ is not defined
and $(\text{unop}\{\tau?\} ((v_0))^{\ell_0})^{\ell_0} \triangleright_{\mathbb{N}} (\text{TagErr} \bullet)^{\ell_0}$
- 1.1. QED
2. CASE $\delta_N(\text{unop}, ((v_0))^{\ell_0})$ is defined
and $(\text{unop}\{\tau?\} ((v_0))^{\ell_0})^{\ell_0} \triangleright_{\mathbb{N}} (\delta_N(\text{unop}, ((v_0))^{\ell_0}))^{\ell_0}$
- 2.1. QED
by lemma 5.29
3. CASE $\delta_N(\text{binop}, ((v_0))^{\ell_0}, ((v_1))^{\ell_0})$ is not defined
and $(\text{binop}\{\tau?\} ((v_0))^{\ell_0} ((v_1))^{\ell_0})^{\ell_0} \triangleright_{\mathbb{N}} (\text{TagErr} \bullet)^{\ell_0}$
- 3.1. QED
4. CASE $\delta_N(\text{binop}, ((v_0))^{\ell_0}, ((v_1))^{\ell_0})$ is defined
and $(\text{binop}\{\tau?\} ((v_0))^{\ell_0} ((v_1))^{\ell_0})^{\ell_0} \triangleright_{\mathbb{N}} (\delta_N(\text{binop}, ((v_0))^{\ell_0}, ((v_1))^{\ell_0}))^{\ell_0}$
- 4.1. QED
by lemma 5.29
5. CASE $(\text{app}\{\mathcal{U}\} ((\lambda x_0. e_0))^{\ell_0} ((v_1))^{\ell_0})^{\ell_0} \triangleright_{\mathbb{N}} (e_0[x_0 \leftarrow ((v_1))^{\ell_0 \ell_0 \ell_0}])^{\ell_0}$
- 5.1. QED
by lemma 5.34
6. CASE $(\text{app}\{\mathcal{U}\} ((\text{mon}(\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1)^{\ell_1} \blacktriangleleft \ell_1) (v_0)^{\ell_2}))^{\ell_0} ((v_1))^{\ell_0})^{\ell_0} \triangleright_{\mathbb{N}}$
 $(\text{stat}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{app}\{\text{forget}(\tau_1)\} v_0 (\text{dyn}(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0) ((v_1))^{\ell_0 \ell_0 \ell_0}))^{\ell_2})^{\ell_0}$
- 6.1. $\ell_1 = \ell_2$
by inversion $\overline{\Gamma}$
- 6.2. QED

$$\begin{array}{c}
\text{by inversion } \overline{\vdash} \\
\frac{}{;\ell_0 \overline{\vdash} v_1} \\
\hline
\text{by inversion } \overline{\vdash} \quad \frac{}{;\ell_0 \overline{\vdash} ((v_1))^{\ell_0 \ell_0 \ell_0}} \\
\frac{}{;\ell_0 \overline{\vdash} v_0} \quad \frac{}{;\ell_0 \overline{\vdash} \text{dyn } (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0) ((v_1))^{\ell_0 \ell_0 \ell_0}} \\
\hline
\frac{}{;\ell_0 \overline{\vdash} \text{app}\{\text{forget}(\tau_1)\} v_0 (\text{dyn } (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0) ((v_1))^{\ell_0 \ell_0 \ell_0})} \\
\hline
\frac{}{;\ell_0 \overline{\vdash} (\text{app}\{\text{forget}(\tau_1)\} v_0 (\text{dyn } (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0) ((v_1))^{\ell_0 \ell_0 \ell_0}))^{\ell_2}} \\
\hline
\frac{}{;\ell_0 \overline{\vdash} \text{stat } (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{app}\{\text{forget}(\tau_1)\} v_0 (\text{dyn } (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0) ((v_1))^{\ell_0 \ell_0 \ell_0}))^{\ell_2}} \\
\hline
;\ell_0 \overline{\vdash} (\text{stat } (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{app}\{\text{forget}(\tau_1)\} v_0 (\text{dyn } (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0) ((v_1))^{\ell_0 \ell_0 \ell_0}))^{\ell_2})^{\ell_0}
\end{array}$$

7. CASE $(\text{stat } (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) ((v_0))^{\ell_2})^{\ell_0} \blacktriangleright_{\mathbb{N}} e_2$

7.1. QED

by lemma 5.33

□

LEMMA 5.24 (UNWIND BOUNDARY ERROR). *If $\cdot \vdash_{\mathbb{N}} e_0 : \tau?$ and $e_0 \rightarrow_{\mathbb{N}}^* \text{BndryErr}(b_0, v_1)$ then $\exists v_1, E$ such that $e_0 \rightarrow_{\mathbb{N}}^* E_0[\text{dyn } b_0 v_1] \rightarrow_{\mathbb{N}} E_0[\text{BndryErr}(b_0, v_1)]$*

PROOF. By the definitions of $\triangleright_{\mathbb{N}}$ and $\blacktriangleright_{\mathbb{N}}$, only dyn expression can step to a boundary error.

□

LEMMA 5.25. *If $\cdot \vdash_{\mathbb{N}} e_0 : \tau?$ and $;\ell_0 \overline{\vdash} e_0$ then either:*

- $e_0 \in ((v))^{\ell}$
- $e_0 = E_0[\text{app}\{\tau?\} ((v_0))^{\ell_0} ((v_1))^{\ell_0}]^{\ell_0}$
- $e_0 = E_0[\text{unop}\{\tau?\} ((v_0))^{\ell_0}]^{\ell_0}$
- $e_0 = E_0[\text{binop}\{\tau?\} ((v_0))^{\ell_0} ((v_1))^{\ell_0}]^{\ell_0}$
- $e_0 = E_0[\text{dyn } b_1 ((v_1))^{\ell_0}]^{\ell_0}$
- $e_0 = E_0[\text{stat } b_1 ((v_1))^{\ell_0}]^{\ell_0}$
- $e_0 = E_0[\text{Err}]^{\ell_0}$

PROOF. By induction on the structure of e_0 .

1. CASE $e_0 = ((x_0))^{\ell_0}$

1.1. CONTRADICTION:

$;\ell_0 \overline{\vdash} e_0$

2. CASE $e_0 \in ((v))^{\ell_0}$

2.1. QED

3. CASE $e_0 = ((\langle e_1, e_2 \rangle))^{\ell_0}$

3.1. $\cdot \vdash_{\mathbb{N}} e_1 : \tau?$ and $\cdot \vdash_{\mathbb{N}} e_2 : \tau?$

by inversion $\vdash_{\mathbb{N}}$

3.2. $;\ell_0 \overline{\vdash} e_1$ and $;\ell_0 \overline{\vdash} e_2$

by inversion $\overline{\vdash}$

2341 3.3. SCASE $e_1 \notin ((v))\bar{\ell}$
 2342 3.3.1. QED
 2343 by the induction hypothesis
 2344 3.4. SCASE $e_1 \in ((v))\bar{\ell}$ and $e_2 \notin ((v))\bar{\ell}$
 2345 3.4.1. QED
 2346 by the induction hypothesis
 2347 3.5. SCASE $e_1 \in ((v))\bar{\ell}$ and $e_2 \in ((v))\bar{\ell}$
 2348 3.5.1. QED
 2349 $e_0 \in ((v))\bar{\ell}$
 2350 4. CASE $e_0 = \text{app}\{\tau?\} e_1 e_2$
 2351 4.1. QED
 2352 by the induction hypothesis
 2353 5. CASE $e_0 = \text{unop}\{\tau?\} e_1$
 2354 5.1. QED
 2355 by the induction hypothesis
 2356 6. CASE $e_0 = \text{binop}\{\tau?\} e_1 e_2$
 2357 6.1. QED
 2358 by the induction hypothesis
 2359 7. CASE $e_0 = \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (e_1)^{\ell_1}$
 2360 7.1. $\cdot; \ell_1 \bar{\Gamma} e_1$
 2361 by inversion $\bar{\Gamma}$
 2362 7.2. QED
 2363 by the induction hypothesis
 2364 8. CASE $e_0 = \text{stat } b_1 e_1$
 2365 8.1. QED
 2366 by the induction hypothesis
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 2376 LEMMA 5.26. *If $\cdot; \ell_0 \bar{\Gamma} E_0[e_0]$ then $\exists \ell_1$ such that $\cdot; \ell_1 \bar{\Gamma} e_0$*
 2377 PROOF. By induction on the structure of E_0 .
 2378 1. $E_0 = []$
 2379 1.1. QED
 2380 2. $E_0 = \langle E_1, e_2 \rangle$
 2381 2.1. QED
 2382 by the induction hypothesis
 2383 3. $E_0 = \langle e_1, E_2 \rangle$
 2384 3.1. QED
 2385 by the induction hypothesis
 2386 4. $E_0 = \text{unop}\{\tau?\} E_1$
 2387 4.1. QED
 2388 by the induction hypothesis
 2389 5. $E_0 = \text{binop}\{\tau?\} E_1 E_2$
 2390 5.1. QED
 2391 by the induction hypothesis
 2392

□

- 2393 5. $E_0 = \text{binop}\{\tau?\} E_1 e_2$
 2394 5.1. QED
 2395 by the induction hypothesis
 2396
 2397 6. $E_0 = \text{binop}\{\tau?\} e_1 E_2$
 2398 6.1. QED
 2399 by the induction hypothesis
 2400
 2401 7. $E_0 = \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) (E_1)^{\ell_3}$
 2402 7.1. $\cdot; \ell_2 \Vdash E_1[e_0]$
 2403 7.2. QED
 2404 by the induction hypothesis
 2405
 2406 8. $E_0 = \text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) (E_1)^{\ell_3}$
 2407 8.1. QED
 2408 by the induction hypothesis
 2409
 2410 9. $E_0 = (E_1)^{\ell_0}$
 2411 9.1. QED
 2412 by the induction hypothesis

□

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 2415 LEMMA 5.27 (\Vdash REPLACEMENT). *If $\cdot; \ell_0 \Vdash E_0[e_0]$ and the derivation contains a proof of $\cdot; \ell_1 \Vdash e_0$ and $\cdot; \ell_1 \Vdash e_1$ then*
 2416 $L_0; \ell_0 \Vdash E_0[e_1]$
 2417

2418 PROOF. By induction on the structure of E_0 .

- 2419 1. $E_0 = []$
 2420 1.1. QED
 2421
 2422 2. $E_0 = \langle E_1, e_2 \rangle$
 2423 2.1. QED
 2424 by the induction hypothesis
 2425
 2426 3. $E_0 = \langle e_1, E_2 \rangle$
 2427 3.1. QED
 2428 by the induction hypothesis
 2429
 2430 4. $E_0 = \text{unop}\{\tau?\} E_1$
 2431 4.1. QED
 2432 by the induction hypothesis
 2433
 2434 5. $E_0 = \text{binop}\{\tau?\} E_1 e_2$
 2435 5.1. QED
 2436 by the induction hypothesis
 2437
 2438 6. $E_0 = \text{binop}\{\tau?\} e_1 E_2$
 2439 6.1. QED
 2440 by the induction hypothesis
 2441
 2442 7. $E_0 = \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) (E_1)^{\ell_3}$
 2443 7.1. $\cdot; \ell_2 \Vdash E_1[e_0]$
 2444 7.2. QED

2445 by the induction hypothesis

2446 8. $E_0 = \text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)(E_1)^{\ell_3}$

2447 8.1. QED

2448 by the induction hypothesis

2449 9. $E_0 = (E_1)^{\ell_0}$

2450 9.1. QED

2451 by the induction hypothesis

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LEMMA 5.28 (δ_N LABEL PROGRESS).

- If $\cdot \vdash_N \text{unop}\{\tau_1\} v_0 : \tau_0$ and $\cdot; \ell_0 \overline{\vdash} \text{unop}\{\tau_1\} v_0$ then $\delta_N(\text{unop}, v_0)$ is defined and $\text{unop}\{\tau_1\} v_0 \triangleright_{\overline{N}} \delta_N(\text{unop}, v_0)$.
- if $\cdot \vdash_N \text{binop}\{\tau_1\} v_0 v_1 : \tau_0$ and $\cdot; \ell_0 \overline{\vdash} \text{binop}\{\tau_1\} v_0 v_1$ then $\delta_N(\text{binop}, v_0, v_1)$ is defined and $\text{binop}\{\tau_1\} v_0 v_1 \triangleright_{\overline{N}} \delta_N(\text{binop}, v_0, v_1)$.
- If $\cdot \vdash_N \text{unop}\{\mathcal{U}\} v_0 : \mathcal{U}$ and $\cdot; \ell_0 \overline{\vdash} \text{unop}\{\mathcal{U}\} v_0$ then $\text{unop}\{\mathcal{U}\} v_0 \blacktriangleright_{\overline{N}} e_1$.
- if $\cdot \vdash_N \text{binop}\{\mathcal{U}\} v_0 v_1 : \mathcal{U}$ and $\cdot; \ell_0 \overline{\vdash} \text{binop}\{\mathcal{U}\} v_0 v_1$ then $\text{binop}\{\mathcal{U}\} v_0 v_1 \blacktriangleright_{\overline{N}} e_1$.

PROOF. By case analysis of δ_N , \vdash_N , $\overline{\vdash}$, and $\blacktriangleright_{\overline{N}}$.

1. CASE $\cdot \vdash_N \text{unop}\{\tau_1\} v_0 : \tau_0$

1.1. $v_0 = ((v_1))^{\ell_0}$

by inversion $\overline{\vdash}$

1.2. $v_1 = \langle v_2, v_3 \rangle$

by inversion \vdash_N

1.3. QED

by definition δ_N

2. CASE $\cdot \vdash_N \text{binop}\{\tau_1\} v_0 v_1 : \tau_0$

2.1. $v_0 = ((v_2))^{\ell_0}$ and $v_1 = ((v_3))^{\ell_0}$

by inversion $\overline{\vdash}$

2.2. $v_0 \in i$ and $v_1 \in i$

by inversion \vdash_N

2.3. QED

by definition δ_N

3. CASE $\cdot \vdash_N \text{unop}\{\mathcal{U}\} v_0 : \mathcal{U}$

3.1. QED

by inversion $\overline{\vdash}$

4. CASE $\cdot \vdash_N \text{binop}\{\mathcal{U}\} v_0 v_1 : \mathcal{U}$

4.1. QED

by inversion $\overline{\vdash}$

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LEMMA 5.29 (δ_N LABEL PRESERVATION).

- If $\cdot; \ell_0 \overline{\vdash} \text{unop}\{\tau?\} v_0$ and $(\text{unop}\{\tau?\} v_0)^{\ell_0} (\triangleright_{\overline{N}} \cup \blacktriangleright_{\overline{N}}) (e_1)^{\ell_0}$ then $\cdot; \ell_0 \overline{\vdash} e_1$.

• If $\cdot; \ell_0 \Vdash \text{binop}\{\tau?\} v_0 v_1$ and $(\text{binop}\{\tau?\} v_0 v_1)^{\ell_0} \triangleright_{\mathbb{N}} \cup \blacktriangleright_{\mathbb{N}} (e_1)^{\ell_0}$ then $\cdot; \ell_0 \Vdash e_1$.

PROOF. By case analysis of $(\triangleright_{\mathbb{N}} \cup \blacktriangleright_{\mathbb{N}})$.

1. $(\text{fst}\{\tau?\} (\langle v_1, v_2 \rangle))^{\ell_0} \triangleright_{\mathbb{N}} \cup \blacktriangleright_{\mathbb{N}} (v_1)^{\ell_0}$
 - 1.1. $\cdot; \ell_0 \Vdash v_1$
by inversion \Vdash
 - 1.2. QED
2. $(\text{snd}\{\tau?\} (\langle v_1, v_2 \rangle))^{\ell_0} \triangleright_{\mathbb{N}} \cup \blacktriangleright_{\mathbb{N}} (v_2)^{\ell_0}$
 - 2.1. $\cdot; \ell_0 \Vdash v_2$
by inversion \Vdash
 - 2.2. QED
3. $(\text{sum}\{\tau?\} (i_1)^{\ell_1} (i_2)^{\ell_2})^{\ell_0} \triangleright_{\mathbb{N}} \cup \blacktriangleright_{\mathbb{N}} (i_1 + i_2)^{\ell_0}$
 - 3.1. QED
4. $(\text{quotient}\{\tau?\} (i_1)^{\ell_1} (i_2)^{\ell_2})^{\ell_0} \triangleright_{\mathbb{N}} \cup \blacktriangleright_{\mathbb{N}} (\text{DivErr})^{\ell_0}$
 - 4.1. QED
5. $(\text{quotient}\{\tau?\} (i_1)^{\ell_1} (i_2)^{\ell_2})^{\ell_0} \triangleright_{\mathbb{N}} \cup \blacktriangleright_{\mathbb{N}} (\lfloor i_1 / i_2 \rfloor)^{\ell_0}$
 - 5.1. QED

□

LEMMA 5.30. If $\cdot \vdash_{\mathbb{N}} \text{dyn } b_0 v_0 : \tau_0$ and $\cdot; \ell_0 \Vdash \text{dyn } b_0 v_0$ then $\exists e_1$ such that $(\text{dyn } b_0 v_0)^{\ell_0} \triangleright_{\mathbb{N}} (e_1)^{\ell_0}$.

PROOF. By inversion of \Vdash and case analysis of $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$.

1. $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
and $\cdot; \ell_1 \Vdash v_0$
by inversion \Vdash
2. $v_0 = (v_1)^{\ell_1}$
by inversion \Vdash
3. CASE $v_1 = \lambda x_1. e_1$ and $\text{tag-match}(\lfloor \tau_0 \rfloor, v_1)$
 - 3.1. QED
 $(\text{dyn } b_0 v_0)^{\ell_0} \triangleright_{\mathbb{N}} (\text{mon } b_0 v_0)^{\ell_0}$
4. CASE $v_1 = \lambda(x_1 : \tau_1). e_1$ and $\text{tag-match}(\lfloor \tau_0 \rfloor, v_1)$
 - 4.1. CONTRADICTION:
 $\cdot \vdash_{\mathbb{N}} \text{dyn } b_0 v_0 : \tau_0$
5. CASE $v_1 = \text{mon } b_1 v_2$ and $\text{tag-match}(\lfloor \tau_0 \rfloor, v_1)$
 - 5.1. QED
 $(\text{dyn } b_0 v_0)^{\ell_0} \triangleright_{\mathbb{N}} (\text{mon } b_0 v_0)^{\ell_0}$
6. CASE $v_1 = \langle v_2, v_3 \rangle$ and $\text{tag-match}(\lfloor (\tau_1 \times \tau_2)^{\ell_1} \rfloor, v_1)$
 - 6.1. QED
 $(\text{dyn } b_0 v_0)^{\ell_0} \triangleright_{\mathbb{N}} ((\text{dyn } (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_2, \text{dyn } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_3))^{\ell_0}$
7. CASE $v_1 \in i$ and $\text{tag-match}(\lfloor \text{Int} \rfloor, v_1)$
 - 7.1. QED
 $(\text{dyn } b_0 v_0)^{\ell_0} \triangleright_{\mathbb{N}} (v_1)^{\ell_0}$

2549 8. CASE $v_1 \in n$ and $\text{tag-match}(\lfloor \text{Nat} \rfloor, v_1)$

2550 8.1. QED

2551 $(\text{dyn } b_0 v_0)^{\ell_0} \triangleright_{\mathbb{N}} (v_1)^{\ell_0}$

2552 9. CASE $\neg \text{tag-match}(\lfloor \tau_0 \rfloor, v_1)$

2553 9.1. QED

2554 $(\text{dyn } b_0 v_0)^{\ell_0} \triangleright_{\mathbb{N}} (\text{BndryErr}(b_0, v_0))^{\ell_0}$

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2559 LEMMA 5.31. *If $\cdot \vdash_{\mathbb{N}} \text{stat } b_0 v_0 : \mathcal{U}$ and $;\ell_0 \Vdash \text{stat } b_0 v_0$ then $\exists e_1$ such that $(\text{stat } b_0 v_0)^{\ell_0} \blacktriangleright_{\mathbb{N}} (e_1)^{\ell_0}$.*

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2561 PROOF. By case analysis on v_0 .

2562 1. $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$

2563 and $;\ell_1 \Vdash v_0$

2564 by inversion \Vdash

2565 2. $v_0 = ((v_1))^{\ell_1}$

2566 by inversion \Vdash

2567 3. CASE $v_1 \in \lambda x. e$

2568 3.1. CONTRADICTION:

2569 $\cdot \vdash_{\mathbb{N}} \text{stat } b_0 v_0 : \mathcal{U}$

2570 4. CASE $v_1 \in \lambda(x:\tau). e$

2571 4.1. QED

2572 $(\text{stat } b_0 v_0)^{\ell_0} \blacktriangleright_{\mathbb{N}} (\text{mon } b_0 v_0)^{\ell_0}$

2573 5. CASE $v_1 \in \text{mon } b e$

2574 5.1. QED

2575 $(\text{stat } b_0 v_0)^{\ell_0} \blacktriangleright_{\mathbb{N}} (\text{mon } b_0 v_0)^{\ell_0}$

2576 6. CASE $v_1 = \langle v_2, v_3 \rangle$

2577 6.1. $\tau_0 = (\tau_1 \times \tau_2)^{\ell_1}$

2578 by inversion $\vdash_{\mathbb{N}}$

2579 6.2. QED

2580 $(\text{stat } b_0 v_0)^{\ell_0} \blacktriangleright_{\mathbb{N}} (\langle \text{stat } (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_2, \text{stat } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_3 \rangle)^{\ell_0}$

2581 7. CASE $v_1 \in i$

2582 7.1. QED

2583 $(\text{stat } b_0 v_0)^{\ell_0} \blacktriangleright_{\mathbb{N}} (v_1)^{\ell_0}$

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2591 LEMMA 5.32. *If $\cdot \vdash_{\mathbb{N}} \text{dyn } b_0 v_0 : \tau_0$ and $;\ell_0 \Vdash \text{dyn } b_0 v_0$ and $(\text{dyn } b_0 v_0)^{\ell_0} \triangleright_{\mathbb{N}} (e_1)^{\ell_0}$ then $;\ell_0 \Vdash e_1$.*

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2593 PROOF. By case analysis of $\triangleright_{\mathbb{N}}$.

2594 1. $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$

2595 and $;\ell_1 \Vdash v_0$

2596 by inversion \Vdash

2597 2. $v_0 = ((v_1))^{\ell_1}$

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2601 by inversion $\overline{\Gamma}$

2602 3. CASE $v_1 \in (\lambda x. e) \cup (\text{mon } b \ v)$

2603 and $(\text{dyn } b_0 \ v_0)^{\ell_0} \triangleright_{\mathbb{N}} (\text{mon } b_0 \ v_0)^{\ell_0}$

2604 3.1. QED

$$\frac{\text{by inversion } \overline{\Gamma}}{\frac{; \ell_1 \overline{\Gamma} \ v_0}{; \ell_0 \overline{\Gamma} \ \text{mon } b_0 \ v_0}}$$

2611 4. CASE $v_1 = \langle v_2, v_3 \rangle$

2612 and $b_0 = (\ell_0 \blacktriangleleft (\tau_2 \times \tau_3)^{\ell_1} \blacktriangleleft \ell_1)$

2613 and $(\text{dyn } b_0 \ (\langle v_0, v_1 \rangle))^{\ell_2} \triangleright_{\mathbb{N}} (\langle \text{dyn } b_3 \ (\langle v_2 \rangle)^{\ell_1}, \text{dyn } b_4 \ (\langle v_3 \rangle)^{\ell_1} \rangle)^{\ell_0}$

2614 4.1. $b_3 = (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)$

2615 and $b_4 = (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)$

2616 by definition $\triangleright_{\mathbb{N}}$

2617 4.2. QED

$$\frac{\frac{\text{by inversion } \overline{\Gamma}}{; \ell_1 \overline{\Gamma} \ (\langle v_2 \rangle)^{\ell_1}}}{; \ell_0 \overline{\Gamma} \ \text{dyn } b_3 \ (\langle v_2 \rangle)^{\ell_1}} \quad \frac{\text{by inversion } \overline{\Gamma}}{; \ell_1 \overline{\Gamma} \ (\langle v_3 \rangle)^{\ell_1}}}{; \ell_0 \overline{\Gamma} \ \text{dyn } b_4 \ (\langle v_3 \rangle)^{\ell_1}}$$

$$\frac{}{; \ell_0 \overline{\Gamma} \ \langle \text{dyn } b_3 \ (\langle v_2 \rangle)^{\ell_1}, \text{dyn } b_4 \ (\langle v_3 \rangle)^{\ell_1} \rangle}$$

2621 5. CASE $v_1 \in i$

2622 and $(\text{dyn } b_0 \ v_0)^{\ell_0} \triangleright_{\mathbb{N}} (v_1)^{\ell_0}$

2623 5.1. QED

2624 6. CASE $(\text{dyn } b_0 \ v_0)^{\ell_0} \triangleright_{\mathbb{N}} (\text{BndryErr}(b_0, v_0))^{\ell_0}$

2625 6.1. QED

2626 \square

2627 LEMMA 5.33 (N-stat PRESERVATION). *If $\cdot \vdash_{\mathbb{N}} \text{stat } b_0 \ v_0 : \mathcal{U}$ and $; \ell_0 \overline{\Gamma} \ \text{stat } b_0 \ v_0$ and $(\text{stat } b_0 \ v_0)^{\ell_0} \blacktriangleright_{\mathbb{N}} (e_1)^{\ell_0}$ then*

2628 $; \ell_0 \overline{\Gamma} \ e_1$.

2629 PROOF. By case analysis of $\blacktriangleright_{\mathbb{N}}$.

2630 1. $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$

2631 and $; \ell_1 \overline{\Gamma} \ v_0$

2632 and $v_0 = (\langle v_1 \rangle)^{\ell_1}$

2633 by inversion $\overline{\Gamma}$

2634 2. CASE $v_1 \in (\lambda(x:\tau). e) \cup (\text{mon } b \ v)$

2635 and $(\text{stat } b_0 \ v_0)^{\ell_0} \blacktriangleright_{\mathbb{N}} (\text{mon } b_0 \ v_0)^{\ell_0}$

2636 2.1. QED

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$$\frac{\text{by inversion } \overline{\Gamma}}{\cdot; \ell_1 \overline{\Gamma} v_0}}{\cdot; \ell_0 \overline{\Gamma} \text{mon } b_0 v_0}$$

3. CASE $v_1 = \langle v_2, v_3 \rangle$
 and $\tau_0 = (\tau_2 \times \tau_3)^{\ell_1}$
 and $(\text{stat } b_0 v_0)^{\ell_0} \triangleright_{\mathbb{N}} ((\text{stat } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{stat } (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3))^{\ell_0}$

3.1. QED

$$\frac{\frac{\text{by inversion } \overline{\Gamma}}{\cdot; \ell_1 \overline{\Gamma} v_2}}{\cdot; \ell_0 \overline{\Gamma} \text{stat } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2} \quad \frac{\text{by inversion } \overline{\Gamma}}{\cdot; \ell_1 \overline{\Gamma} v_3}}{\cdot; \ell_0 \overline{\Gamma} \text{stat } (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3}}{\cdot; \ell_0 \overline{\Gamma} \langle \text{stat } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{stat } (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle}$$

4. CASE $v_1 \in i$
 and $(\text{stat } b_0 v_0)^{\ell_0} \triangleright_{\mathbb{N}} (v_1)^{\ell_0}$
 qedstep

□

LEMMA 5.34. *If $(x_0 : \tau?_0), \Gamma_0 \vdash_{\mathbb{N}} e_1 : \tau?$ and $(x_0 : \ell_0), L_0; \ell_1 \overline{\Gamma} e_1$ and $\cdot \vdash_{\mathbb{N}} v_0 : \tau?_0$ and $\cdot; \ell_0 \overline{\Gamma} v_0$ then $\Gamma_0 \vdash_{\mathbb{N}} e_1[x_0 \leftarrow v_0] : \tau?$ and $L_0 \overline{\Gamma} e_1[x_0 \leftarrow v_0]$.*

PROOF. By induction on the structure of e_0 .

1. $e_0 = x_2$
 1.1. SCASE $x_0 = x_2$
 1.1.1. QED
 1.2. SCASE $x_0 \neq x_2$
 1.2.1. QED
 $e_1[x_0 \leftarrow v_0] = e_1$
 2. CASE $e_0 \in i$
 2.1. QED
 $e_1[x_0 \leftarrow v_0] = e_1$
 3. CASE $e_0 = \lambda x_2. e_2$
 ore $e_0 = \lambda(x_2 : \tau_2). e_2$
 3.1. SCASE $x_0 = x_2$
 3.1.1. QED
 $e_1[x_0 \leftarrow v_0] = e_1$
 3.2. SCASE $x_0 \neq x_2$
 3.2.1. QED
 by the induction hypothesis
 4. CASE $e_0 = \langle e_1, e_2 \rangle$
 4.1. QED

2705 by the induction hypothesis

2706 5. CASE $e_0 = \text{app}\{\tau?\} e_1 e_2$

2707 5.1. QED

2708 by the induction hypothesis

2709 6. CASE $e_0 = \text{unop}\{\tau?\} e_1$

2710 6.1. QED

2711 by the induction hypothesis

2712 7. CASE $e_0 = \text{binop}\{\tau?\} e_1 e_2$

2713 7.1. QED

2714 by the induction hypothesis

2715 8. CASE $e_0 = \text{dyn } b_1 e_1$

2716 8.1. QED

2717 by the induction hypothesis

2718 9. CASE $e_0 = \text{stat } b_1 e_1$

2719 9.1. QED

2720 by the induction hypothesis

2721 10. CASE $e_0 = (e_1)^{\ell_1}$

2722 10.1. $\ell_0 = \ell_1$

2723 by inversion $\overline{\text{IF}}$

2724 10.2. QED

2725 by the induction hypothesis

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LEMMA 5.35 (BOUNDARY PRESERVATION). *If $e_0 : \tau?$ wf and $e_0 \rightarrow_N^* E[\text{dyn } b_0 v_1]$ then either $\text{has-boundary}(e_0, b_0)$ or $\text{has-boundary}(e_0, \text{flip}(b_0))$.*

PROOF. By case analysis of \triangleright_N and \blacktriangleright_N , evaluation does not create new labels and only creates a new boundary by flipping an existing boundary.

LEMMA 5.36. *If $e_0 : \tau?$ wf and $e_0 \rightarrow_N^* E[\text{mon}(\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1)^{\ell_1} \blacktriangleleft \ell_2) v_0]$ then $\text{tag-match}([\tau_0 \Rightarrow \tau_1], v_0)$*

PROOF. Surface expressions do not contain monitors, and the only ways to create one via \blacktriangleright_N and \triangleright_N require tag-match as a precondition.

2757 **6 Transient THEOREMS, LEMMAS, AND PROOFS**

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6.1 Transient Theorems

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THEOREM 6.1 (INCOMPLETE MONITORING). *There exists $e_0 : \tau? \overline{\mathbf{wf}}$ such that $e_0; \emptyset; \emptyset \xrightarrow{*}_{\overline{\mathbf{T}}} e_1; \mathcal{H}_1; \mathcal{B}_1$ and for all O_1 such that $O_1 \Vdash_{\overline{\mathbf{T}}} \mathcal{H}_1$, we have $O_1; \cdot; \ell \not\vdash_{\overline{\mathbf{T}}} e_1$.*

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PROOF. Let

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$$e_f = \text{stat } (\ell_0 \blacktriangleleft (\text{Int} \Rightarrow \text{Int}) \blacktriangleleft \ell_1) (\text{dyn } (\ell_1 \blacktriangleleft (\text{Int} \Rightarrow \text{Int}) \blacktriangleleft \ell_2) (\lambda x_0. (\text{sum}\{x_0\} 1)))^{\ell_2})^{\ell_1}$$

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$$e_0 = (\text{app}\{\mathcal{U}\} e_0 (\lambda x_0. (\text{sum}\{x_0\} 1)))^{\ell_0}$$

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$$v_f = ((p_0))^{\ell_2 \ell_1 \ell_0}$$

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$$e_1 = (\text{app}\{\mathcal{U}\} v_f (\lambda x_1. 0))^{\ell_0}$$

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By the reduction rules:

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$$e_0; \emptyset; \emptyset \xrightarrow{*}_{\overline{\mathbf{T}}} e_1; \mathcal{H}_0; \mathcal{B}_0$$

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$$\text{where } \mathcal{H}_0 = \{p_0 \mapsto \lambda x_0. (\text{sum}\{x_0\} 1)\}$$

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$$\text{and } \mathcal{B}_0 = \{p_0 \mapsto \{(\ell_0 \blacktriangleleft (\text{Int} \Rightarrow \text{Int}) \blacktriangleleft \ell_1), (\ell_1 \blacktriangleleft (\text{Int} \Rightarrow \text{Int}) \blacktriangleleft \ell_2)\}\}$$

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In the result, $\overline{\mathbf{T}}$ fails because v_f has multiple owners.

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□

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THEOREM 6.2 (UN SOUND BLAME). *There exists $e_0 : \tau? \overline{\mathbf{wf}}$ such that $e_0; \emptyset; \emptyset \xrightarrow{*}_{\overline{\mathbf{T}}} \text{BndryErr}((\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1), v_1); \mathcal{H}_1; \mathcal{B}_1$ and senders $(\overline{b}_1) \not\subseteq \text{owners}(v_1)$.*

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PROOF. Consider the following example e_0 where the let-expressions are sugar for untyped function applications:

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$$e_0 = (\text{let } f_0 = (\lambda x_0. \langle x_0, x_0 \rangle) \text{ in}$$

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$$\text{let } f_1 = (\text{stat } (\ell_0 \blacktriangleleft (\text{Int} \Rightarrow \text{Int}) \blacktriangleleft \ell_1) (\text{dyn } (\ell_1 \blacktriangleleft (\text{Int} \Rightarrow \text{Int}) \blacktriangleleft \ell_0) (f_0)^{\ell_0})^{\ell_1}) \text{ in}$$

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$$\text{stat } (\ell_0 \blacktriangleleft \text{Int} \blacktriangleleft \ell_2) (\text{app}\{\text{Int}\} (\text{dyn } (\ell_2 \blacktriangleleft (\text{Int} \Rightarrow \text{Int}) \blacktriangleleft \ell_0) (f_0)^{\ell_0}) 5)^{\ell_2})^{\ell_0}$$

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By the reduction rules:

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$$e_0; \emptyset; \emptyset$$

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$$\xrightarrow{*}_{\overline{\mathbf{T}}} (\text{stat } (\ell_0 \blacktriangleleft \text{Int} \blacktriangleleft \ell_2) (\text{app}\{\text{Int}\} ((p_0))^{\ell_0 \ell_2} 5)^{\ell_2})^{\ell_0}; \mathcal{H}_0; \mathcal{B}_0$$

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$$\xrightarrow{*}_{\overline{\mathbf{T}}} (\text{stat } (\ell_0 \blacktriangleleft \text{Int} \blacktriangleleft \ell_2) (\text{check Int } ((p_1))^{\ell_0 \ell_2} p_0)^{\ell_2})^{\ell_0}; \mathcal{H}_1; \mathcal{B}_1$$

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$$\xrightarrow{*}_{\overline{\mathbf{T}}} \text{BndryErr}(\mathcal{B}_0(p_1) \cup \mathcal{B}_0(p_0), ((p_1))^{\ell_0 \ell_2}); \mathcal{H}_1; \mathcal{B}_0$$

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$$\text{where } \mathcal{H}_1 = \{p_0 \mapsto \lambda x_0. \langle x_0, x_0 \rangle, p_1 \mapsto \langle 5, 5 \rangle\}$$

2795

$$\text{and } \mathcal{B}_1 = \{p_0 \mapsto (\ell_1 \blacktriangleleft (\text{Int} \Rightarrow \text{Int}) \blacktriangleleft \ell_0), (\ell_0 \blacktriangleleft (\text{Int} \Rightarrow \text{Int}) \blacktriangleleft \ell_1), (\ell_2 \blacktriangleleft (\text{Int} \Rightarrow \text{Int}) \blacktriangleleft \ell_0), p_1 \mapsto \emptyset\}$$

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Thus senders $(\mathcal{B}_0(p_1) \cup \mathcal{B}_0(p_0)) = \{\ell_0, \ell_1\} \not\subseteq \{\ell_0, \ell_2\} = \text{owners}(((p_1))^{\ell_0 \ell_2})$.

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THEOREM 6.3 (INCOMPLETE BLAME). *There exists $e_0 : \tau? \overline{\mathbf{wf}}$ such that $e_0; \emptyset; \emptyset \xrightarrow{*}_{\overline{\mathbf{T}}} \text{BndryErr}((\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1), v_1); \mathcal{H}_1; \mathcal{B}_1$ and senders $(\overline{b}_1) \not\subseteq \text{owners}(v_1)$.*

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PROOF. Consider the following example e_0 where the let-expressions are sugar for untyped function applications:

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2809 (let $f_0 = \text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{dyn } (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) (\lambda x_0. x_0))$ in
 2810 let $f_1 = \text{stat } (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_3) (\text{dyn } (\ell_3 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_4) (\lambda x_1. x_1))$ in
 2811 $\text{stat } (\ell_0 \blacktriangleleft (\text{Int} \times \text{Int}) \blacktriangleleft \ell_5)$
 2812 $(\text{app}\{\text{Int} \times \text{Int}\} (\text{dyn } (\ell_5 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) (\text{app}\{\mathcal{U}\} f_1 f_0)^{\ell_0} 42)^{\ell_5})^{\ell_0}; \emptyset; \emptyset$
 2813 $\longrightarrow_{\overline{\tau}}^*$ $(\text{stat } (\ell_0 \blacktriangleleft (\text{Int} \times \text{Int}) \blacktriangleleft \ell_5)$
 2814 $(\text{app}\{\text{Int} \times \text{Int}\} (\text{dyn } (\ell_5 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) (\text{app}\{\mathcal{U}\} ((p_1)^{\ell_4 \ell_3 \ell_0} ((p_0)^{\ell_2 \ell_1 \ell_0})^{\ell_0} 42)^{\ell_5})^{\ell_0})^{\ell_5})^{\ell_0}; \mathcal{H}_0; \mathcal{B}_0$
 2815 $\longrightarrow_{\overline{\tau}}^*$ $(\text{stat } (\ell_0 \blacktriangleleft (\text{Int} \times \text{Int}) \blacktriangleleft \ell_5) (\text{app}\{\text{Int} \times \text{Int}\} ((p_0)^{\overline{\ell}_0} 42)^{\ell_5})^{\ell_0}; \mathcal{H}_1; \mathcal{B}_1$
 2816 $\longrightarrow_{\overline{\tau}}^*$ $(\text{stat } (\ell_0 \blacktriangleleft (\text{Int} \times \text{Int}) \blacktriangleleft \ell_5) (\text{check } (\text{Int} \times \text{Int}) ((42)^{\overline{\ell}_1} p_0)^{\ell_5})^{\ell_0}; \mathcal{H}_2; \mathcal{B}_2$
 2817 $\longrightarrow_{\overline{\tau}}^*$ $\text{BndryErr } (\mathcal{B}_2(p_0), ((42)^{\overline{\ell}_1}); \mathcal{H}_2; \mathcal{B}_2$
 2818 where $\tau_0 = (\text{Int} \Rightarrow \text{Int})$
 2819 and $\tau_1 = (\text{Int} \Rightarrow \text{Int} \times \text{Int})$
 2820 and $\overline{\ell}_0 = \ell_2 \ell_1 \ell_0 \ell_3 \ell_4 \ell_3 \ell_0 \ell_5$
 2821 and $\overline{\ell}_1 = \ell_5 \overline{\ell}_0 (\text{rev}(\overline{\ell}_0))$
 2822 and $\mathcal{H}_2 = \{(p_0 \mapsto \lambda x_0. x_0), (p_1 \mapsto \lambda x_1. x_1)\}$
 2823 and $\mathcal{B}_2 = \{(p_0 \mapsto \{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1), (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2), (\ell_5 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\}),$
 2824 $(p_1 \mapsto \{(\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_3), (\ell_3 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_4)\})\}$

2825 Hence, $\text{senders}(\mathcal{B}_2(p_0)) = \{\ell_0, \ell_1, \ell_2\} \not\supseteq \{\ell_0, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5\} = \text{owners}(((42)^{\overline{\ell}_1})$. The crucial labels ℓ_3 and ℓ_4 are nowhere
 2826 to be found.

□

2827 **THEOREM 6.4 (ALL-PATH SOUND BLAME).** *If $e_0 : \tau ? \overline{\mathbf{wf}}$:*

- 2828 • $e_0 \longrightarrow_{\overline{\tau}}^* E_0[\text{dyn } (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2) v_1]; \mathcal{H}_0; \mathcal{B}_0; O_0 \triangleright_{\overline{\tau}} \text{BndryErr}((\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2), v_1); \mathcal{H}_1; \mathcal{B}_1; O_1$ then
 2829 (1) *has-boundary*($e_0, (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)$)
 2830 (2) $v_1 = ((v))^{n \dots \ell_2}$
 2831 • $e_0 \longrightarrow_{\overline{\tau}}^* E_0[\text{app}\{\tau?\} v_0 v_1]; \mathcal{H}_0; \mathcal{B}_0; O_0 \triangleright_{\overline{\tau}} \text{BndryErr}(\overline{b}_1, v_1); \mathcal{H}_1; \mathcal{B}_1; O_1$ then
 2832 (1) $\forall b_1 \in \overline{b}_1$, either *has-boundary*(e_0, b_1) or *has-boundary*($e_0, \text{flip}(b_1)$)
 2833 (2) $v_1 = ((p_1))^{\overline{\ell}_0}$, $\exists \overline{\ell}_1. \overline{\ell}_1 \simeq \overline{b}_1$ and $\overline{\ell}_0 \cup O_1(p_1) \subseteq \overline{\ell}_1$.
 2834 • $e_0 \longrightarrow_{\overline{\tau}}^* E_0[\text{check}(\tau_0, v_0, p_1)]; \mathcal{H}_0; \mathcal{B}_0; O_0 \triangleright_{\overline{\tau}} \text{BndryErr}(\overline{b}_1, v_0); \mathcal{H}_0; \mathcal{B}_0; O_0$ then
 2835 (1) $\forall b_1 \in \overline{b}_1$, either *has-boundary*(e_0, b_1) or *has-boundary*($e_0, \text{flip}(b_1)$)
 2836 (2) $v_0 = ((v_1))^{\overline{\ell}_0}$, $\exists \overline{\ell}_1. \overline{\ell}_1 \simeq \overline{b}_1$, and $\overline{\ell}_0 \cup O_1(v_1) \cup O_1(p_1) \subseteq \text{senders}(\overline{b}_1)$.

2837 **PROOF SKETCH.** The proof relies on a straight-forward subject reduction argument showing that the evaluation trace
 2838 of any well-formed term e_0 satisfies two invariants:

- 2839 • for all $\text{dyn } (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2) v_1$ that occur in all configurations $e; \mathcal{H}; \mathcal{B}; O$ in the trace, *has-boundary*($e_0, (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)$)
 2840 and $v_1 = ((v))^{n \dots \ell_2}$;
 2841 • for all configurations $e; \mathcal{H}; \mathcal{B}; O$ in the trace, (i) if $e \neq \text{BndryErr}(\overline{b}, v)$ then for all p in the domain of O , $O(p) \subseteq \ell^*$
 2842 where $\ell^* \simeq \mathcal{B}(p_0)$ and (ii) $\forall b_1 \in \mathcal{B}(p_0)$, either *has-boundary*(e_0, b_1) or *has-boundary*($e_0, \text{flip}(b_1)$).

2861 We employ the first invariant to establish the first case of the theorem and the second for the others. \square

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2863 THEOREM 6.5 (ALL-PATH INCOMPLETE BLAME). *There exists e_0 such that $\cdot; \ell_\bullet \Vdash e_0$,*

2864 • $e_0 \xrightarrow{*}_{\top} E_0[\text{check}(\tau_0, \nu_0, p_1)]; \mathcal{H}_0; \mathcal{B}_0; O_0 \triangleright_{\top} \text{BndryErr}(\bar{b}_1, \nu_0); \mathcal{H}_0; \mathcal{B}_0; O_0$ and

2865 (1) $\forall b_1 \in \bar{b}_1$, *either has-boundary(e_0, b_1) or has-boundary($e_0, \text{flip}(b_1)$)*

2866 (2) $\nu_0 = ((\nu_1))^{\ell^*}$, $\nu \neq p$, $\ell_1 \dots \ell_n \simeq \bar{b}_1$ and $\ell_1 \dots \ell_n \not\supseteq O_1(p_1) \cup \ell^*$.

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2869 PROOF. Consider the following example e_0 where the let-expressions are sugar for untyped function applications:

2870 $e_0 = (\text{let } f_0 = \text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{dyn}(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) (\lambda x_0. x_0)) \text{ in}$
 2871 $\text{let } f_1 = \text{stat}(\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_3) (\text{dyn}(\ell_3 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_4) (\lambda x_1. x_1)) \text{ in}$
 2872 $\text{stat}(\ell_0 \blacktriangleleft (\text{Int} \times \text{Int}) \blacktriangleleft \ell_5)$
 2873 $(\text{app}\{\text{Int} \times \text{Int}\} (\text{dyn}(\ell_5 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) (\text{app}\{\mathcal{U}\} f_1 f_0)^{\ell_0} 42)^{\ell_5})^{\ell_0}$

2874 With a straight-forward application of the reduction rules we obtain:

2875 $e_0; \emptyset; \emptyset; \emptyset$
 2876 $\xrightarrow{*}_{\top} \text{stat}(\ell_0 \blacktriangleleft (\text{Int} \times \text{Int}) \blacktriangleleft \ell_5)$
 2877 $(\text{app}\{\text{Int} \times \text{Int}\} (\text{dyn}(\ell_5 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) (\text{app}\{\mathcal{U}\} ((p_1))^{\ell_4 \ell_3 \ell_0} ((p_0))^{\ell_2 \ell_1 \ell_0} 42)^{\ell_5})^{\ell_0}; \mathcal{H}_0; \mathcal{B}_0; O_0$
 2878 $\xrightarrow{*}_{\top} (\text{stat}(\ell_0 \blacktriangleleft \text{Int} \times \text{Int} \blacktriangleleft \ell_5) (\text{app}\{\text{Int} \times \text{Int}\} ((p_0))^{\bar{\ell}_0} 42)^{\ell_5})^{\ell_0}; \mathcal{H}_1; \mathcal{B}_1; O_1$
 2879 $\xrightarrow{*}_{\top} (\text{stat}(\ell_0 \blacktriangleleft \text{Int} \times \text{Int} \blacktriangleleft \ell_5) (\text{check Int} \times \text{Int} ((42))^{\bar{\ell}_1} p_0)^{\ell_5})^{\ell_0}; \mathcal{H}_2; \mathcal{B}_2; O_2$
 2880 $\triangleright_{\top} \text{BndryErr}(\mathcal{B}_2(p_0), ((42))^{\bar{\ell}_1}); \mathcal{H}_2; \mathcal{B}_2; O_2$
 2881 where $\tau_0 = (\text{Int} \Rightarrow \text{Int})$ and $\tau_1 = (\text{Int} \Rightarrow \text{Int} \times \text{Int})$
 2882 and $\bar{\ell}_0 = \ell_2 \ell_1 \ell_0 \ell_3 \ell_4 \ell_3 \ell_0 \ell_5$ and $\bar{\ell}_1 = \ell_5 \bar{\ell}_0 (\text{rev}(\bar{\ell}_0))$
 2883 and $\mathcal{H}_2 = \{(p_0 \mapsto \lambda x_0. x_0), (p_1 \mapsto \lambda x_1. x_1)\}$
 2884 and $\mathcal{B}_2 = \{(p_0 \mapsto \{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1), (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2), (\ell_5 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\}),$
 2885 $(p_1 \mapsto \{(\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_3), (\ell_3 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_4)\})\}$

2886 Thus $\{\ell_0, \ell_1, \ell_2, \ell_5\} \simeq \mathcal{B}_2(p_0)$ and $\{\ell_0, \ell_1, \ell_2, \ell_5\} \not\supseteq O_2(p_0) \cup \{\ell_0, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5\}$ independently of the details of O_2 .

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2889 THEOREM 6.6 (TYPE SOUNDNESS). *If $e_0 : \tau?$ wf then one of the following holds:*

- 2890 • $e_0; \emptyset; \emptyset \xrightarrow{*}_{\top} \nu_0; \mathcal{H}_0; \mathcal{B}_0$ and $\exists \mathcal{T}_0. \mathcal{T}_0; \cdot \vdash_{\top} \nu_0 : \lfloor \tau? \rfloor$
- 2891 • $e_0; \emptyset; \emptyset$ diverges
- 2892 • $e_0; \emptyset; \emptyset \xrightarrow{*}_{\top} E_0[\text{dyn } b_1 E[e_1]]; \mathcal{H}_0; \mathcal{B}_0$ and $e_1; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{TagErr} \bullet; \mathcal{H}_1; \mathcal{B}_1$
- 2893 • $e_0; \emptyset; \emptyset \xrightarrow{*}_{\top} \text{DivErr}; \mathcal{H}_0; \mathcal{B}_0$
- 2894 • $e_0; \emptyset; \emptyset \xrightarrow{*}_{\top} \text{BndryErr}(\bar{b}_1, \nu_1); \mathcal{H}_0; \mathcal{B}_0$

2895 PROOF SKETCH. By progress and preservation lemmas; preservation constructs a heap typing (\mathcal{T}) at each step. \square

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2913 7 Amnesic THEOREMS, LEMMAS, AND PROOFS

2914 7.1 Amnesic Theorems

2916 THEOREM 7.1. *If $e_0 : \tau? \overline{\mathbf{wf}}$ then $\mathit{forget}(e_0) : \tau? \mathbf{wf}$ and $e_0 \longrightarrow_{\overline{\mathbf{A}}} e_1$ iff $\mathit{forget}(e_0) \rightarrow_{\mathbf{A}} \mathit{forget}(e_1)$*

2918 PROOF. By the definition of $\longrightarrow_{\overline{\mathbf{A}}}$.

2919 □

2921 THEOREM 7.2 (TYPE SOUNDNESS). *If $e_0 : \tau_0 \mathbf{wf}$ then one of the following holds:*

- 2922 • $e_0 \rightarrow_{\mathbf{A}}^* v_0$ and $\cdot \vdash_{\mathbf{A}} v_0 : \tau_0$
- 2923 • e_0 diverges
- 2924 • $e_0 \rightarrow_{\mathbf{A}}^* E_0[\mathit{dyn} \ b_1 \ E[e_1]]$ and $e_1 \blacktriangleright_{\mathbf{A}} \mathit{TagErr} \bullet$
- 2925 • $e_0 \rightarrow_{\mathbf{A}}^* \mathit{DivErr}$
- 2926 • $e_0 \rightarrow_{\mathbf{A}}^* \mathit{BndryErr}(\overline{b}_1, v_1)$
- 2927 • $e_0 \rightarrow_{\mathbf{A}}^* \mathit{BndryErr}(\overline{b}_1, v_1)$

2929 PROOF. By progress and preservation lemmas (lemma 7.11 and lemma 7.12).

2930 □

2931 THEOREM 7.3 (DYNAMIC SOUNDNESS). *If $e_0 : \mathcal{U} \mathbf{wf}$ then one of the following holds:*

- 2932 • $e_0 \rightarrow_{\mathbf{A}}^* v_0$ and $\cdot \vdash_{\mathbf{A}} v_0 : \mathcal{U}$
- 2933 • e_0 diverges
- 2934 • $e_0 \rightarrow_{\mathbf{A}}^* E_0[e_1]$ and $e_1 \blacktriangleright_{\mathbf{A}} \mathit{TagErr} \bullet$
- 2935 • $e_0 \rightarrow_{\mathbf{A}}^* \mathit{DivErr}$
- 2936 • $e_0 \rightarrow_{\mathbf{A}}^* \mathit{BndryErr}(\overline{b}_1, v_1)$

2937 PROOF. By progress and preservation lemmas (lemma 7.11 & lemma 7.12).

2938 □

2939 THEOREM 7.4 (INCOMPLETE MONITORING). *There exist $e_0, e_1, \ell_0, \tau?$ such that $(e_0)^{\ell_0} : \tau? \overline{\mathbf{wf}}$ and $e_0 \longrightarrow_{\overline{\mathbf{A}}}^* e_1$ and $\cdot; \ell \Vdash e_1$.*

2940 PROOF. Let

$$\begin{aligned}
 2941 e_f &= \mathit{stat}(\ell_0 \blacktriangleleft (\mathit{Int} \Rightarrow \mathit{Int}) \blacktriangleleft \ell_1) (\mathit{dyn}(\ell_1 \blacktriangleleft (\mathit{Int} \Rightarrow \mathit{Int}) \blacktriangleleft \ell_2) (\lambda x_0. (\mathit{sum}\{\mathit{Int}\} \ x_0 \ 1))^{\ell_2})^{\ell_1} \\
 2942 e_0 &= (\mathit{app}\{\mathcal{U}\} e_0 (\lambda x_0. (\mathit{sum}\{\mathit{Int}\} \ x_0 \ 1)))^{\ell_0} \\
 2943 v_f &= (\mathit{trace}_v \{(\ell_0 \blacktriangleleft (\mathit{Int} \Rightarrow \mathit{Int}) \blacktriangleleft \ell_1), (\ell_1 \blacktriangleleft (\mathit{Int} \Rightarrow \mathit{Int}) \blacktriangleleft \ell_2)\} ((\lambda x_0. (\mathit{sum}\{\mathit{Int}\} \ x_0 \ 1))^{\ell_2} \ell_1))^{\ell_0} \\
 2944 e_1 &= (\mathit{app}\{\mathcal{U}\} v_0 (\lambda x_1. 0))^{\ell_0}
 \end{aligned}$$

2945 With a straight-forward application of the reduction rules we obtain:

$$\begin{aligned}
 2946 (e_f)^{\ell_0} &\longrightarrow_{\overline{\mathbf{A}}}^* (\mathit{stat}(\ell_0 \blacktriangleleft (\mathit{Int} \Rightarrow \mathit{Int}) \blacktriangleleft \ell_1) (\mathit{mon}(\ell_1 \blacktriangleleft (\mathit{Int} \Rightarrow \mathit{Int}) \blacktriangleleft \ell_2) (\lambda x_0. (\mathit{sum}\{\mathit{Int}\} \ x_0 \ 1))^{\ell_2})^{\ell_1})^{\ell_0} \\
 2947 &\longrightarrow_{\overline{\mathbf{A}}}^* (\mathit{trace}_v \{(\ell_0 \blacktriangleleft (\mathit{Int} \Rightarrow \mathit{Int}) \blacktriangleleft \ell_1), (\ell_1 \blacktriangleleft (\mathit{Int} \Rightarrow \mathit{Int}) \blacktriangleleft \ell_2)\} ((\lambda x_0. (\mathit{sum}\{\mathit{Int}\} \ x_0 \ 1))^{\ell_2} \ell_1))^{\ell_0} \\
 2948 &= v_f
 \end{aligned}$$

2949 therefore

$$2950 e_0 \longrightarrow_{\overline{\mathbf{A}}}^* e_1$$

2951 □

2952 THEOREM 7.5 (SOUND AND COMPLETE BLAME). *If $\cdot; \ell \bullet \Vdash e_0$ and $e_0 \longrightarrow_{\overline{\mathbf{A}}}^* \mathit{BndryErr}((\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1), v_1)$ then*

- 2965 • either *has-boundary* $((\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1), e_0)$ or *has-boundary* $((\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0), e_0)$, and
 2966 • *senders* $(\bar{b}_1) = \text{owners}(v_1)$
 2967

2968 THEOREM 7.6 (BLAME CORRECTNESS). *If $(e_0)^{\ell_0} : \tau ? \overline{\mathbf{wf}}$ then one of the following holds:*

- 2969 • $e_0 \xrightarrow{\bar{A}}^* v_0$ and $\cdot ; \ell_0 \Vdash_A v_0$
 2970 • e_0 *diverges*
 2971 • $e_0 \xrightarrow{\bar{A}}^* \text{TagErr}$ •
 2972 • $e_0 \xrightarrow{\bar{A}}^* \text{DivErr}$
 2973 • $e_0 \xrightarrow{\bar{A}}^* E_0[\text{dyn}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2) v_1] \xrightarrow{\bar{A}} \text{BndryErr}(\bar{b}_1, v_1)$ and furthermore:
 2974 (1) $\forall b_1 \in \bar{b}_1$, either *has-boundary* (e_0, b_1) or *has-boundary* $(e_0, \text{flip}(b_1))$
 2975 (2) one of the following holds:
 2976 (a) $v_1 \notin (\text{trace}_v \bar{b}((v))^{\bar{\ell}})$ and $\cdot ; \ell_2 \Vdash_A v_1$
 2977 (b) $v_1 = (\text{trace}_v \bar{b}_2((v_2))^{\bar{\ell}_2 \ell_2})$ and $\bar{b}_2 \simeq \ell_2 \bar{\ell}_2 \ell_2$ and $\cdot ; \text{last}(\ell_3) \Vdash_A v_2$
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2982 PROOF.

- 2983 1. SUFFICES ASSUME $e_0 \xrightarrow{\bar{A}}^* E_0[\text{dyn}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2) v_1] \xrightarrow{\bar{A}} \text{BndryErr}(\bar{b}_1, v_1)$
 2984 PROVE (1) $\forall b_1 \in \bar{b}_1$, either *has-boundary* (e_0, b_1) or *has-boundary* $(e_0, \text{flip}(b_1))$
 2985 (2) one of the following holds:
 2986 (a) $v_1 \notin (\text{trace}_v \bar{b}((v))^{\bar{\ell}})$ and $\cdot ; \ell_2 \Vdash_A v_1$
 2987 (b) $v_1 = (\text{trace}_v \bar{b}_2((v_2))^{\bar{\ell}_2 \ell_2})$ and $\bar{b}_2 \simeq \ell_2 \bar{\ell}_2 \ell_2$ and $\cdot ; \text{last}(\ell_3) \Vdash_A v_2$
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2989 by lemma 7.26 and lemma 7.27

- 2990 2. $(\ell_1 \blacktriangleleft \ell_2 \blacktriangleleft \Vdash_A) \tau_1$ and $\ell_2 \Vdash_A v_1$
 2991 by lemma 7.27
 2992 3. $\forall b_1 \in \bar{b}_1$ either $b_1 \in e_0$ or $\text{flip}(b_1) \in e_0$
 2993 by lemma 7.41
 2994 4. either $\bar{b}_1 = (\ell_1 \blacktriangleleft \ell_2 \blacktriangleleft)$
 2995 or $v_1 = (\text{trace}_v \bar{b}_0 v_0)$ and $\bar{b}_1 = (\ell_1 \blacktriangleleft \ell_2 \blacktriangleleft) \bar{b}_0$
 2996 by the definition of $\xrightarrow{\bar{A}}$
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3001 COROLLARY 7.7 (MINIMAL BLAME INFO). *If $e_0 : \tau ? \mathbf{wf}$ and $e_0 \xrightarrow{\bar{A}}^* \text{BndryErr}(\bar{b}_1, v_1)$ then $\bar{b}_1 \neq \cdot$.*

3002 PROOF. by THEOREM 7.6

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COROLLARY 7.8 (BLAME/OWNERSHIP MATCH). *If $L_0; \ell_0 \Vdash_A e_0$ then for all subterms $(\text{trace}_v \bar{b}_1 e_1)$ there exists $\bar{\ell}_1$ such that $e_1 = ((e_2))^{\bar{\ell}_1}$ and $\bar{b}_1 \simeq \bar{\ell}_1$*

PROOF. by definition of \Vdash_A

THEOREM 7.9. *If $e_0 : \tau_0 \mathbf{wf}$ and $e_0 \xrightarrow{\bar{A}}^* v_1$ then $\text{mon-depth}(v_1) \leq 2$*

PROOF. 1. SUFFICES if $\text{dyn } b_0 v_2 \triangleright_A v_3$ then $\text{mon-depth}(v_3) \leq 2$

3017 because the only way to increase the *mon-depth* of a value is by crossing a boundary

3018 2. QED

3019 by lemma 7.10 and the definition of \triangleright_A

3021 \square

3022 THEOREM 7.10. *If $e_0 : \mathcal{U}$ wf and $e_0 \rightarrow_A^* v_1$ then $\text{mon-depth}(v_1) \leq 1$*

3023 PROOF. 1. SUFFICES if $\text{stat } b_0 v_2 \triangleright_A v_3$ then $\text{mon-depth}(v_3) \leq 1$

3024 because the only way to increase the *mon-depth* of a value is by crossing a boundary

3025 2. QED

3026 by definition of \triangleright_A

3027 \square

3032 7.2 Amnesic Lemmas

3033 LEMMA 7.11 (\vdash_A PROGRESS). *If $\cdot \vdash_A e_0 : \tau$? then one of the following holds:*

- 3034 • $e_0 \in v$
- 3035 • $e_0 \in \text{Err}$
- 3036 • $\exists e_1$ such that $e_0 \rightarrow_A e_1$

3037 PROOF. By case analysis of e_0 .

3038 By lemma 7.15 it suffices to consider the following cases.

3039 1. CASE $e_0 \in v$

3040 1.1. QED

3041 2. CASE $e_0 = E_0[\text{Err}]$

3042 2.1. QED

3043 3. CASE $e_0 = E_0[\text{app}\{\tau_1\} v_0 v_1]$

3044 3.1. $v_0 \in (\lambda(x:\tau). e) \cup (\text{mon } b v)$

3045 by lemma 7.16 and inversion \vdash_A

3046 3.2. SCASE $v_0 = \lambda(x_2:\tau_2). e_2$

3047 3.2.1. QED

$$3048 e_0 \triangleright_A E_0[e_2[x_2 \leftarrow v_1]]$$

3049 3.3. SCASE $v_0 = \text{mon}(\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2) \blacktriangleleft \ell_1) v_2$

3050 3.3.1. QED

$$3051 e_0 \triangleright_A E_0[\text{dyn}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)(\text{app}\{\mathcal{U}\} v_2 (\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1))]$$

3052 4. CASE $e_0 = E_0[\text{app}\{\tau_1\} v_0 v_1]$

3053 4.1. SCASE $v_0 = \text{trace}_v^? \bar{b}_0(\lambda(x_2:\tau_2). e_2)$

3054 4.1.1. QED

$$3055 e_0 \triangleright_A E_0[\text{trace } \bar{b}_0(e_2[x_2 \leftarrow v_1])]$$

3056 4.2. SCASE $v_0 = \text{trace}_v^? \bar{b}_0(\text{mon}(\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2) \blacktriangleleft \ell_1) v_2)$

3057 4.2.1. QED

$$3058 e_0 \triangleright_A E_0[\text{trace } \bar{b}_0(\text{stat}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)(\text{app}\{\text{forget}(\tau_2)\} v_2 (\text{dyn}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1)))]$$

3059 4.3. SCASE $v_0 \notin (\text{trace}_v^? \bar{b}(\lambda(x:\tau). e) \cup (\text{trace}_v^? \bar{b}(\text{mon } b v)))$

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3069 4.3.1. QED

3070 $e_0 \triangleright_{\mathbb{A}} E_0[\text{TagErr } \bullet]$

3071 5. CASE $e_0 = E_0[\text{unop}\{\tau?\} v_0]$

3072 5.1. QED

3073 by lemma 7.16 and lemma 7.18

3074 6. CASE $e_0 = E_0[\text{binop}\{\tau?\} v_0 v_1]$

3075 6.1. QED

3076 by lemma 7.16 and lemma 7.18

3077 7. CASE $e_0 = E_0[\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0]$

3078 7.1. QED

3079 by lemma 7.16 and lemma 7.20

3080 8. CASE $e_0 = E_0[\text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0]$

3081 8.1. QED

3082 by lemma 7.16 and lemma 7.21

3083 9. CASE $e_0 = E_0[\text{trace } \bar{b}_0 v_0]$

3084 9.1. QED

3085 $e_0 \triangleright_{\mathbb{A}} E_0[\text{add-trace}(\bar{b}_0, v_0)]$

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LEMMA 7.12 ($\vdash_{\mathbb{A}}$ TYPE PRESERVATION). *If $\cdot \vdash_{\mathbb{A}} e_0 : \tau?$ and $e_0 \rightarrow_{\mathbb{A}} e_1$ then $\cdot \vdash_{\mathbb{A}} e_1 : \tau?$.*

PROOF. By lemma 7.13 and lemma 7.14.

LEMMA 7.13 ($\triangleright_{\mathbb{A}}$ PRESERVATION). *If $\cdot \vdash_{\mathbb{A}} e_0 : \tau_0$ and $e_0 \triangleright_{\mathbb{A}} e_1$ then $\cdot \vdash_{\mathbb{A}} e_1 : \tau_0$.*

PROOF. By case analysis of $\triangleright_{\mathbb{A}}$.

1. CASE $\delta_{\mathbb{A}}(\text{unop}, v_0)$ is defined

and $\text{unop}\{\tau?\} v_0 \triangleright_{\mathbb{A}} \delta_{\mathbb{A}}(\text{unop}, v_0)$

1.1. QED

by lemma 7.19

2. CASE $\text{fst}\{\tau_0\}(\text{mon}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_0) \triangleright_{\mathbb{A}} \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)(\text{fst}\{\mathcal{U}\} v_0)$

2.1. QED

by inversion $\vdash_{\mathbb{A}}$

$\cdot v_0 : \mathcal{U}$

$\cdot \vdash_{\mathbb{A}} \text{fst}\{\mathcal{U}\} v_0 : \mathcal{U}$

$\cdot \vdash_{\mathbb{A}} \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)(\text{fst}\{\mathcal{U}\} v_0) : \tau_0$

3. CASE $\text{snd}\{\tau_0\}(\text{mon}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_0) \triangleright_{\mathbb{A}} \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)(\text{snd}\{\mathcal{U}\} v_0)$

3.1. QED

$$\begin{array}{c}
\text{3121} \\
\text{3122} \\
\text{3123} \\
\text{3124} \\
\text{3125} \\
\text{3126} \\
\text{3127}
\end{array}
\frac{\frac{\text{by inversion } \vdash_A}{\cdot v_0 : \mathcal{U}}}{\cdot \vdash_A \text{snd}\{\mathcal{U}\} v_0 : \mathcal{U}}}{\cdot \vdash_A \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)(\text{snd}\{\mathcal{U}\} v_0) : \tau_0}$$

4. CASE $\text{binop}\{\tau?\} v_0 v_1 \triangleright_A \delta_A(\text{binop}, v_0, v_1)$

4.1. QED

by lemma 7.19

5. CASE $\text{app}\{\tau_0\}(\lambda(x_1 : \tau_1). e_1) v_2 \triangleright_A e_1[x_1 \leftarrow v_2]$

5.1. QED

by lemma 7.25

6. CASE $\text{app}\{\tau_0\}(\text{mon}(\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2) \blacktriangleleft \ell_1) v_0) v_1$

$\triangleright_A \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)(\text{app}\{\mathcal{U}\} v_0(\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1))$

6.1. QED

$$\begin{array}{c}
\text{3140} \\
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\text{3144} \\
\text{3145} \\
\text{3146} \\
\text{3147}
\end{array}
\frac{\frac{\text{by inversion } \vdash_A}{\cdot \vdash_A v_0 : \mathcal{U}} \quad \frac{\text{by inversion } \vdash_A}{\cdot \vdash_A v_1 : \tau_1}}{\cdot \vdash_A \text{app}\{\mathcal{U}\} v_0(\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1) : \mathcal{U}}}{\cdot \vdash_A \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)(\text{app}\{\mathcal{U}\} v_0(\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1)) : \tau_0}$$

7. CASE $\text{dyn } b_0 v_0 \triangleright_A v_1$

7.1. QED

by lemma 7.22

□

LEMMA 7.14 (\blacktriangleright_A PRESERVATION). *If $\cdot \vdash_A e_0 : \mathcal{U}$ and $e_0 \blacktriangleright_A e_1$ then $\cdot \vdash_A e_1 : \mathcal{U}$.*

PROOF. By case analysis of \blacktriangleright_A .

1. CASE $\text{unop}\{\tau?\} v_0 \blacktriangleright_A \text{TagErr} \bullet$

1.1. QED

$\cdot \vdash_A \text{TagErr} \bullet : \mathcal{U}$

2. CASE $\text{unop}\{\tau?\} v_0 \blacktriangleright_A \delta_A(\text{unop}, v_0)$

2.1. QED

by lemma 7.19

3. CASE $\text{fst}\{\mathcal{U}\}(\text{trace}_v^? \bar{b}_0(\text{mon}(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) v_1)) \blacktriangleright_A \text{trace } \bar{b}_0(\text{stat } b_7(\text{fst}\{\text{fst}(\tau_0)\} v_1))$

where $b_7 = (\ell_1 \blacktriangleleft \text{fst}(\tau_0) \blacktriangleleft \ell_2)$

3.1. QED

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- by inversion \vdash_A
- $$\frac{\frac{\frac{\cdot \vdash_A v_1 : \tau_0}{\cdot \vdash_A \text{fst}\{\text{fst}(\tau_0)\} v_1 : \text{fst}(\tau_0)}}{\cdot \vdash_A \text{stat } b_7 (\text{fst}\{\text{fst}(\tau_0)\} v_1) : \mathcal{U}}}{\cdot \vdash_A \text{trace } \bar{b}_0 (\text{stat } b_7 (\text{fst}\{\text{fst}(\tau_0)\} v_1)) : \mathcal{U}}$$
4. CASE $\text{snd}\{\mathcal{U}\} (\text{trace}_v^2 \bar{b}_0 (\text{mon} (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) v_1)) \blacktriangleright_A \text{trace } \bar{b}_0 (\text{stat } b_7 (\text{snd}\{\text{snd}(\tau_0)\} v_1))$
 where $b_7 = (\ell_1 \blacktriangleleft \text{snd}(\tau_0) \blacktriangleleft \ell_2)$
- 4.1. QED
- by inversion \vdash_A
- $$\frac{\frac{\frac{\cdot \vdash_A v_1 : \tau_0}{\cdot \vdash_A \text{snd}\{\text{snd}(\tau_0)\} v_1 : \text{snd}(\tau_0)}}{\cdot \vdash_A \text{stat } b_7 (\text{snd}\{\text{snd}(\tau_0)\} v_1) : \mathcal{U}}}{\cdot \vdash_A \text{trace } \bar{b}_0 (\text{stat } b_7 (\text{snd}\{\text{snd}(\tau_0)\} v_1)) : \mathcal{U}}$$
5. CASE $\text{binop}\{\tau?\} v_0 v_1 \blacktriangleright_A \text{TagErr} \bullet$
- 5.1. QED
- $\cdot \vdash_A \text{TagErr} \bullet : \mathcal{U}$
6. CASE $\text{binop}\{\tau?\} v_0 v_1 \blacktriangleright_A \delta_A(\text{binop}, v_0, v_1)$
- 6.1. QED
- by lemma 7.19
7. CASE $\text{app}\{\mathcal{U}\} (\text{trace}_v^2 \bar{b}_0 (\lambda x_1. e_1)) v_2$
 $\blacktriangleright_A \text{trace } \bar{b}_0 (e_1[x_1 \leftarrow v_2])$
- 7.1. QED
- by lemma 7.25
8. CASE $\text{app}\{\mathcal{U}\} (\text{trace}_v^2 \bar{b}_0 (\text{mon} (\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_0) \blacktriangleleft \ell_1) v_0)) v_1$
 $\blacktriangleright_A \text{trace } \bar{b}_0 (\text{stat} (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{app}\{\tau_0\} v_0 (\text{dyn} (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1)))$
- 8.1. QED
- by inversion \vdash_A
- $$\frac{\frac{\frac{\text{by inversion } \vdash_A}{\cdot \vdash_A v_0 : \tau_1 \Rightarrow \tau_0} \quad \frac{\text{by inversion } \vdash_A}{\cdot \vdash_A v_1 : \mathcal{U}}}{\cdot \vdash_A \text{app}\{\tau_0\} v_0 (\text{dyn} (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1) : \tau_0}}{\cdot \vdash_A \text{trace } \bar{b}_0 (\text{stat} (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{app}\{\tau_0\} v_0 (\text{dyn} (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1))) : \mathcal{U}}$$
9. CASE $\text{stat } b_0 v_0 \blacktriangleright_A v_1$
- 9.1. QED
- by lemma 7.23
10. CASE $\text{trace } \bar{b}_0 v_0 \blacktriangleright_A \text{add-trace}(\bar{b}_0, v_0)$

3225 10.1. QED

3226 by lemma 7.24

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3230 LEMMA 7.15 (UNIQUE DECOMPOSITION). *If $\cdot \vdash_A e_0 : \tau?$ then either:*

- 3231 • $e_0 \in v$
- 3232 • $e_0 = E_0[\text{app}\{\tau?\} v_0 v_1]$
- 3233 • $e_0 = E_0[\text{unop}\{\tau?\} v_0]$
- 3234 • $e_0 = E_0[\text{binop}\{\tau?\} v_0 v_1]$
- 3235 • $e_0 = E_0[\text{dyn } b_1 v_1]$
- 3236 • $e_0 = E_0[\text{stat } b_1 v_1]$
- 3237 • $e_0 = E_0[\text{trace } \bar{b}_1 v_1]$
- 3238 • $e_0 = E_0[\text{Err}]$

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3242 PROOF. By induction on the structure of e_0 .

3243 1. CASE $e_0 = x_0$

3244 1.1. CONTRADICTION:

3245 $\cdot \vdash_A e_0 : \tau?$

3246 2. CASE $e_0 = v_0$

3247 2.1. QED

3248 3. CASE $e_0 = \langle e_1, e_2 \rangle$

3249 3.1. SCASE $e_1 \notin v$

3250 3.1.1. QED

3251 by the induction hypothesis

3252 3.2. SCASE $e_1 \in v$ and $e_2 \notin v$

3253 3.2.1. QED

3254 by the induction hypothesis

3255 3.3. SCASE $e_1 \in v$ and $e_2 \in v$

3256 3.3.1. QED

3257 $e_0 \in v$

3258 4. CASE $e_0 = \text{app}\{\tau?\} e_1 e_2$

3259 4.1. QED

3260 by the induction hypothesis

3261 5. CASE $e_0 = \text{unop}\{\tau?\} e_1$

3262 5.1. QED

3263 by the induction hypothesis

3264 6. CASE $e_0 = \text{binop}\{\tau?\} e_1 e_2$

3265 6.1. QED

3266 by the induction hypothesis

3267 7. CASE $e_0 = \text{dyn } b_1 e_1$

3268 7.1. QED

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□

3277 by the induction hypothesis

3278 8. CASE $e_0 = \text{stat } b_1 e_1$

3279 8.1. QED

3280 by the induction hypothesis

3281 9. CASE $e_0 = \text{trace } \bar{b}_1 e_1$

3282 9.1. QED

3283 by the induction hypothesis

3284 10. CASE $e_0 \in \text{Err}$

3285 10.1. QED

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LEMMA 7.16. *If $\cdot \vdash_A E_0[e_0] : \tau?$ then one of the following holds:*

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PROOF. By induction on the structure of E_0 and case analysis of \vdash_A .

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1. CASE $E_0 = []$

1.1. QED

2. CASE $E_0 = \langle E_1, e_2 \rangle$

2.1. $\cdot \vdash_A E_1[e_0] : \tau?$

by inversion \vdash_A

2.2. QED

by the induction hypothesis

3. CASE $E_0 = \langle v_1, E_2 \rangle$

3.1. QED

by the induction hypothesis

4. CASE $E_0 = \text{app}\{\tau?\} E_1 e_2$

4.1. QED

by the induction hypothesis

5. CASE $E_0 = \text{app}\{\tau?\} v_1 E_2$

5.1. QED

by the induction hypothesis

6. CASE $E_0 = \text{unop}\{\tau?\} E_1$

6.1. QED

by the induction hypothesis

7. CASE $E_0 = \text{binop}\{\tau?\} E_1 e_2$

7.1. QED

by the induction hypothesis

8. CASE $E_0 = \text{binop}\{\tau?\} v_1 E_2$

8.1. QED

by the induction hypothesis

□

- 3329 9. CASE $E_0 = \text{dyn } b_1 E_1$
 3330 9.1. QED
 3331 by the induction hypothesis
 3332
 3333 10. CASE $E_0 = \text{stat } b_1 E_1$
 3334 10.1. QED
 3335 by the induction hypothesis
 3336
 3337 11. CASE $E_0 = \text{trace } \bar{b}_1 E_1$
 3338 11.1. QED
 3339 by the induction hypothesis
 3340

□

LEMMA 7.17 (\vdash_A REPLACEMENT).

- If $\vdash_A E_0[e_0] : \tau?$ and the derivation contains a proof of $\vdash_A e_0 : \tau_0$ and $\vdash_A e_1 : \tau_0$ then $\vdash_A E_0[e_1] : \tau?$.
- If $\vdash_A E_0[e_0] : \tau?$ and the derivation contains a proof of $\vdash_A e_0 : \mathcal{U}$ and $\vdash_A e_1 : \mathcal{U}$ then $\vdash_A E_0[e_1] : \tau?$.

PROOF. By induction on E_0 .

- 3347
 3348 1. CASE $E_0 = []$
 3349 1.1. QED
 3350
 3351 2. CASE $E_0 = \langle E_1, e_2 \rangle$
 3352 2.1. QED
 3353 by the induction hypothesis
 3354
 3355 3. CASE $E_0 = \langle v_1, E_2 \rangle$
 3356 3.1. QED
 3357 by the induction hypothesis
 3358
 3359 4. CASE $E_0 = \text{app}\{\tau?\} E_1 e_2$
 3360 4.1. QED
 3361 by the induction hypothesis
 3362
 3363 5. CASE $E_0 = \text{app}\{\tau?\} v_1 E_2$
 3364 5.1. QED
 3365 by the induction hypothesis
 3366
 3367 6. CASE $E_0 = \text{unop}\{\tau?\} E_1$
 3368 6.1. QED
 3369 by the induction hypothesis
 3370
 3371 7. CASE $E_0 = \text{binop}\{\tau?\} E_1 e_2$
 3372 7.1. QED
 3373 by the induction hypothesis
 3374
 3375 8. CASE $E_0 = \text{binop}\{\tau?\} v_1 E_2$
 3376 8.1. QED
 3377 by the induction hypothesis
 3378
 3379 9. CASE $E_0 = \text{dyn } b_1 E_1$
 3380 9.1. QED

3381 by the induction hypothesis

3382 10. CASE $E_0 = \text{stat } b_1 E_1$

3383 10.1. QED

3384 by the induction hypothesis

3386 11. CASE $E_0 = \text{trace } \bar{b}_1 E_1$

3387 11.1. QED

3388 by the induction hypothesis

3390

□

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3392 LEMMA 7.18.

3393

3394 • If $\cdot \vdash_A \text{unop}\{\tau_1\} v_0 : \tau_0$ then $\text{unop}\{\tau_1\} v_0 \triangleright_A e_1$.

3395 • if $\cdot \vdash_A \text{binop}\{\tau_1\} v_0 v_1 : \tau_0$ then $\text{binop}\{\tau_1\} v_0 v_1 \triangleright_A e_1$.

3396 • If $\cdot \vdash_A \text{unop}\{\mathcal{U}\} v_0 : \mathcal{U}$ then $\text{unop}\{\mathcal{U}\} v_0 \blacktriangleright_A e_1$.

3397 • if $\cdot \vdash_A \text{binop}\{\mathcal{U}\} v_0 v_1 : \mathcal{U}$ then $\text{binop}\{\mathcal{U}\} v_0 v_1 \blacktriangleright_A e_1$.

3399

3400 PROOF. By case analysis of δ_A , \vdash_A , and \blacktriangleright_A .

3401 1. CASE $\cdot \vdash_A \text{fst}\{\tau_0\} v_0$

3402 1.1. $v_0 \in \langle v, v \rangle \cup \text{mon}(\ell \blacktriangleleft \tau \times \tau \blacktriangleleft \ell) v$

3403 by \vdash_A canonical forms

3404 1.2. SCASE $v_0 = \langle v_1, v_2 \rangle$

3405 1.2.1. QED

3406 $\text{fst}\{\tau_0\} v_0 \triangleright_A v_1$

3407 1.3. SCASE $v_0 = \text{mon}(\ell_0 \blacktriangleleft \tau_1 \times \tau_2 \blacktriangleleft \ell_1) v_1$

3408 1.3.1. QED

3409 $\text{fst}\{\tau_0\} v_0 \triangleright_A \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{fst}\{\mathcal{U}\} v_0)$

3410 2. CASE $\cdot \vdash_A \text{snd}\{\tau_0\} v_0$

3411 2.1. QED

3412 similar to the fst case

3413 3. CASE $\cdot \vdash_A \text{fst}\{\mathcal{U}\} v_0$

3414 3.1. SCASE $v_0 = \langle v_1, v_2 \rangle$

3415 3.1.1. QED

3416 $\text{fst}\{\mathcal{U}\} v_0 \blacktriangleright_A v_1$

3417 3.2. SCASE $v_0 = \text{trace}_v^? \bar{b}_0 (\text{mon}(\ell_1 \blacktriangleleft \tau_1 \times \tau_2 \blacktriangleleft \ell_2) v_1)$

3418 3.2.1. QED

3419 $\text{fst}\{\mathcal{U}\} v_0 \blacktriangleright_A \text{trace } \bar{b}_0 (\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2) (\text{fst}\{\tau_1\} v_1))$

3420 3.3. SCASE $v_0 \notin \langle v, v \rangle \cup (\text{mon}(\ell \blacktriangleleft \tau \times \tau \blacktriangleleft \ell) v)$

3421 3.3.1. QED

3422 $\text{fst}\{\mathcal{U}\} v_0 \blacktriangleright_A \text{TagErr} \bullet$

3423 4. CASE $\cdot \vdash_A \text{snd}\{\mathcal{U}\} v_0$

3424 4.1. QED

3425 similar to the fst case

3426

3433 5. CASE $\cdot \vdash_A \text{binop}\{\tau_1\} v_0 v_1 : \tau_0$

3434 5.1. $v_0 \in i$ and $v_1 \in i$

3435 by \vdash_A canonical forms

3436 5.2. QED

3438 $\text{binop}\{\tau_1\} v_0 v_1 \triangleright_A \delta_A(\text{binop}, v_0, v_1)$

3439 6. CASE $\cdot \vdash_A \text{binop}\{\mathcal{U}\} v_0 v_1 : \mathcal{U}$

3440 6.1. SCASE $v_0 \in i$ and $v_1 \in i$

3441 6.1.1. QED

3442 $\text{binop}\{\mathcal{U}\} v_0 v_1 \blacktriangleright_A \delta_A(\text{binop}, v_0, v_1)$

3443 6.2. SCASE $v_0 \notin i$ or $v_1 \notin i$

3444 6.2.1. QED

3445 $\text{binop}\{\mathcal{U}\} v_0 v_1 \blacktriangleright_A \text{TagErr} \bullet$

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LEMMA 7.19.

- If $\cdot \vdash_A \text{unop}\{\tau_1\} v_0 : \tau_0$ and $\text{unop}\{\tau_1\} v_0 \triangleright_A e_1$ then $\cdot \vdash_A e_1 : \tau_0$.
- If $\cdot \vdash_A \text{binop}\{\tau_1\} v_0 v_1 : \tau_0$ and $\text{binop}\{\tau_1\} v_0 v_1 \triangleright_A e_2$ then $\cdot \vdash_A e_2 : \tau_0$.
- If $\cdot \vdash_A \text{unop}\{\mathcal{U}\} v_0 : \mathcal{U}$ and $\text{unop}\{\mathcal{U}\} v_0 \blacktriangleright_A e_1$ then $\cdot \vdash_A e_1 : \mathcal{U}$.
- If $\cdot \vdash_A \text{binop}\{\mathcal{U}\} v_0 v_1 : \mathcal{U}$ and $\text{binop}\{\mathcal{U}\} v_0 v_1 \blacktriangleright_A e_2$ then $\cdot \vdash_A e_2 : \mathcal{U}$.

PROOF. By case analysis of δ_A and \vdash_A .

1. CASE $\cdot \vdash_A \text{fst}\{\tau_0\} v_0 : \tau_0$

1.1. SCASE $\text{fst}\{\tau_0\} \langle v_1, v_2 \rangle \triangleright_A v_1$

1.1.1. QED

by inversion \vdash_A

1.2. SCASE $\text{fst}\{\tau_0\} (\text{mon} (\ell_0 \blacktriangleleft \tau_1 \times \tau_2 \blacktriangleleft \ell_1) v_1)$

$\triangleright_A \text{dyn} (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{fst}\{\mathcal{U}\} v_1)$

1.2.1. QED

by inversion \vdash_A

$\cdot \vdash_A v_1 : \mathcal{U}$

$\cdot \vdash_A \text{fst}\{\mathcal{U}\} v_1 : \mathcal{U}$

by inversion \vdash_A

$\cdot \vdash_A \text{dyn} (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{fst}\{\mathcal{U}\} v_1) : \tau_1$ $\tau_1 \leq \tau_0$

$\cdot \vdash_A \text{dyn} (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{fst}\{\mathcal{U}\} v_1) : \tau_0$

2. CASE $\cdot \vdash_A \text{snd}\{\tau_0\} v_0 : \tau_0$

2.1. QED

similar to fst

3. CASE $\cdot \vdash_A \text{sum}\{\tau_1\} v_0 v_1 : \tau_0$

3.1. $\text{sum}\{\tau_1\} v_0 v_1 \triangleright_A \delta_A(\text{sum}, v_0, v_1)$

3.2. $\tau_0 \in \text{Int} \cup \text{Nat}$

by inversion \vdash_A

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□

3485 3.3. SCASE $\tau_0 = \text{Int}$

3486 3.3.1. $\cdot \vdash_A v_0 : \text{Int}$ and $\cdot \vdash_A v_1 : \text{Int}$

3487 by inversion \vdash_A

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3489 3.3.2. $v_0 \in i$ and $v_1 \in i$

3490 by \vdash_A canonical forms

3491

3492 3.3.3. QED

3493 $\cdot \vdash_A \delta_A(\text{binop}, v_0, v_1) : \text{Int}$

3494 3.4. SCASE $\tau_0 = \text{Nat}$

3495 3.4.1. $\cdot \vdash_A v_0 : \text{Nat}$ and $\cdot \vdash_A v_1 : \text{Nat}$

3496 by inversion \vdash_A

3497

3498 3.4.2. $v_0 \in n$ and $v_1 \in n$

3499 by \vdash_A canonical forms

3500

3501 3.4.3. QED

3502 $\cdot \vdash_A \delta_A(\text{binop}, v_0, v_1) : \text{Nat}$

3503 4. CASE $\cdot \vdash_A \text{quotient}\{\tau_1\} v_0 v_1 : \tau_0$

3504 4.1. $\text{quotient}\{\tau_1\} v_0 v_1 \triangleright_A \delta_A(\text{quotient}, v_0, v_1)$

3505

3506 4.2. $\tau_0 \in \text{Int} \cup \text{Nat}$

3507 by inversion \vdash_A

3508

3509 4.3. SCASE $\tau_0 = \text{Int}$

3510 4.3.1. $\cdot \vdash_A v_0 : \text{Int}$ and $\cdot \vdash_A v_1 : \text{Int}$

3511 by inversion \vdash_A

3512

3513 4.3.2. $v_0 \in i$ and $v_1 \in i$

3514 by \vdash_A canonical forms

3515

3516 4.3.3. QED

3517 $\delta_A(\text{binop}, v_0, v_1) \in i \cup \text{DivErr}$

3518 4.4. SCASE $\tau_0 = \text{Nat}$

3519 4.4.1. $\cdot \vdash_A v_0 : \text{Nat}$ and $\cdot \vdash_A v_1 : \text{Nat}$

3520 by inversion \vdash_A

3521

3522 4.4.2. $v_0 \in n$ and $v_1 \in n$

3523 by \vdash_A canonical forms

3524

3525 4.4.3. QED

3526 $\delta_A(\text{binop}, v_0, v_1) \in n \cup \text{DivErr}$

3527 5. CASE $\cdot \vdash_A \text{fst}\{\mathcal{U}\} v_0 : \mathcal{U}$

3528 5.1. SCASE $v_0 = \text{trace}_v^? \bar{b}_0 \langle v_1, v_2 \rangle$

3529 and $\text{fst}\{\mathcal{U}\} v_0 \blacktriangleright_A \text{add-trace}(\bar{b}_0, v_1)$

3530 5.1.1. $\cdot \vdash_A v_1 : \mathcal{U}$

3531 by inversion \vdash_A

3532

3533 5.1.2. QED

3534 by lemma 7.24

3535 5.2. SCASE $v_0 = \text{trace}_v^? \bar{b}_0 (\text{mon}(\ell_1 \blacktriangleleft \tau_1 \times \tau_2 \blacktriangleleft \ell_2) v_1)$

3536 and $\text{fst}\{\mathcal{U}\} v_0 \blacktriangleright_A \text{trace} \bar{b}_0 (\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2) (\text{fst}\{\tau_1\} v_1))$

5.2.1. QED

$$\frac{\frac{\text{by inversion } \vdash_A}{\cdot \vdash_A v_1 : \mathcal{U}}}{\cdot \vdash_A \text{fst}\{\tau_1\} v_1 : \mathcal{U}}}{\cdot \vdash_A \text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)(\text{fst}\{\tau_1\} v_1) : \mathcal{U}}}{\cdot \vdash_A \text{trace } \bar{b}_0(\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)(\text{fst}\{\tau_1\} v_1)) : \mathcal{U}}$$

5.3. SCASE $v_0 \notin (\text{trace}_v^? \bar{b} \langle v, v \rangle) \cup (\text{trace}_v^? \bar{b}(\text{mon } b \ v))$
and $\text{fst}\{\mathcal{U}\} v_0 \blacktriangleright_A \text{TagErr} \bullet$

5.3.1. QED

$\cdot \vdash_A \text{TagErr} \bullet : \mathcal{U}$

6. CASE $\cdot \vdash_A \text{snd}\{\mathcal{U}\} v_0 : \mathcal{U}$

6.1. QED

similar to fst

7. CASE $\cdot \vdash_A \text{sum}\{\mathcal{U}\} v_0 v_1 : \mathcal{U}$

7.1. $\text{sum}\{\mathcal{U}\} v_0 v_1 \blacktriangleright_A \delta_A(\text{binop}, v_0, v_1)$

7.2. $\delta_A(\text{binop}, v_0, v_1) \in i$

by definition δ_A

7.3. QED

$\cdot \vdash_A \delta_A(\text{binop}, v_0, v_1) : \mathcal{U}$

8. CASE $\cdot \vdash_A \text{quotient}\{\mathcal{U}\} v_0 v_1 : \mathcal{U}$

8.1. $\text{quotient}\{\mathcal{U}\} v_0 v_1 \blacktriangleright_A \delta_A(\text{binop}, v_0, v_1)$

8.2. $\delta_A(\text{binop}, v_0, v_1) \in i \cup \text{DivErr}$

by definition δ_A

8.3. QED

$\cdot \vdash_A \delta_A(\text{binop}, v_0, v_1) : \mathcal{U}$

□

LEMMA 7.20. If $\cdot \vdash_A \text{dyn } b_0 \ v_0 : \tau_0$ and $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ then $\exists e_1$ such that $\text{dyn } b_0 \ v_0 \triangleright_A e_1$.

PROOF. By case analysis of $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$.

1. CASE $\text{tag-match}(\lfloor \tau_0 \rfloor, \text{trace}_v^? \bar{b}_2(\lambda x_1. e_1))$

1.1. QED

$\text{dyn } b_0 \ v_0 \triangleright_A \text{mon } b_0 \ v_0$

2. CASE $\text{tag-match}(\lfloor \tau_0 \rfloor, \lambda(x_1 : \tau_1). e_1)$

2.1. CONTRADICTION:

$\cdot \vdash_A \text{dyn } b_0 \ v_0 : \tau_0$

3. CASE $\text{tag-match}(\lfloor \tau_0 \rfloor, \text{trace}_v^? \bar{b}_2(\text{mon } b_1 \ v_1))$

3.1. QED

$\text{dyn } b_0 \ v_0 \triangleright_A \text{mon } b_0 \ v_0$

4. CASE $\text{tag-match}(\lfloor (\tau_1 \times \tau_2) \rfloor, \text{trace}_v^? \bar{b}_2 \langle v_1, v_2 \rangle)$

- 3589 4.1. QED
 3590 $\text{dyn } b_0 v_0 \triangleright_A \text{mon } b_0 v_0$
 3591 5. CASE $\text{tag-match}([\text{Int}], \text{trace}_v^? \bar{b}_1 v_1)$
 3592 5.1. QED
 3593 $\text{dyn } b_0 v_0 \triangleright_A v_1$
 3594 6. CASE $\text{tag-match}([\text{Nat}], \text{trace}_v^? \bar{b}_1 v_1)$
 3595 6.1. QED
 3596 $\text{dyn } b_0 v_0 \triangleright_A v_1$
 3597 7. CASE $\neg \text{tag-match}([\tau_0], v_0)$
 3598 7.1. QED
 3599 $\text{dyn } b_0 v_0 \triangleright_A \text{BndryErr}(b_0, v_0)$

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LEMMA 7.21. *If $\cdot \vdash_A \text{stat } b_0 v_0 : \mathcal{U}$ and $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ then $\exists e_1$ such that $\text{stat } b_0 v_0 \blacktriangleright_A e_1$.*

PROOF. By case analysis on v_0 .

1. CASE $v_0 \in \text{trace}_v^? \bar{b}_2 (\lambda x. e)$
 1.1. CONTRADICTION:
 $\cdot \vdash_A \text{stat } b_0 v_0 : \mathcal{U}$
 2. CASE $v_0 \in \lambda(x:\tau). e$
 2.1. QED
 $\text{stat } b_0 v_0 \blacktriangleright_A \text{mon } b_0 v_0$
 3. CASE $v_0 = \text{mon } b_1 v_1$
 3.1. SCASE $v_1 = \text{trace}_v^? \bar{b}_2 v_2$
 and $v_2 \in (\lambda x. e) \cup \langle v, v \rangle$
 3.1.1. QED
 $\text{stat } b_0 v_0 \blacktriangleright_A \text{trace } b_0 b_1 \bar{b}_2 v_0$
 3.2. SCASE $v_1 = \text{trace}_v^? \bar{b}_2 (\text{mon } b_3 v_2)$
 and $v_2 \in (\lambda(x:\tau). v) \cup \langle v, v \rangle$
 $\text{stat } b_0 v_0 \blacktriangleright_A \text{trace } b_0 b_1 \bar{b}_2 (\text{mon } b_3 v_2)$
 4. CASE $v_0 = \langle v_1, v_2 \rangle$
 4.1. $\tau_0 = \tau_1 \times \tau_2$
 by inversion \vdash_A
 4.2. QED
 $\text{stat } b_0 v_0 \blacktriangleright_A \text{mon } b_0 v_0$
 5. CASE $v_0 \in i$
 5.1. QED
 $\text{stat } b_0 v_0 \blacktriangleright_A v_0$

LEMMA 7.22. *If $\cdot \vdash_A \text{dyn } b_0 v_0 : \tau_0$ and $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ and $\text{dyn } b_0 v_0 \triangleright_A e_1$ then $\cdot \vdash_A e_1 : \tau_0$.*

PROOF. By case analysis of \triangleright_A .

3641 1. CASE $\text{dyn } b_0 v_0 \triangleright_A \text{mon } b_0 v_0$

3642 1.1. QED

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3649 2. CASE $\text{dyn } b_0 \text{trace}_v^? \bar{b}_1 i_0 \triangleright_A i_0$

3650

2.1. QED

3651

by case analysis of $\text{tag-match}(\lfloor \tau_0 \rfloor, i_0)$

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3653 3. CASE $\text{dyn } b_0 v_0 \triangleright_A \text{BndryErr}(b_0, v_0)$

3654

3.1. QED

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$\cdot \vdash_A \text{BndryErr}(b_0, v_0) : \tau_0$

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LEMMA 7.23. If $\cdot \vdash_A \text{stat } b_0 v_0 : \mathcal{U}$ and $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ and $\text{stat } b_0 v_0 \blacktriangleright_A e_1$ then $\cdot \vdash_A e_1$.

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PROOF. By case analysis of \blacktriangleright_A .

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3662 1. CASE $v_0 \in (\lambda(x:\tau). e) \cup (\langle v, v \rangle)$

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and $\text{stat } b_0 v_0 \blacktriangleright_A \text{mon } b_0 v_0$

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1.1. QED

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by inversion \vdash_A

$\cdot \vdash_A v_0 : \tau_0$

$\cdot \vdash_A \text{mon } b_0 v_0 : \mathcal{U}$

3671

2. CASE $v_0 \in (\lambda x. e) \cup (\langle v, v \rangle)$

3672

and $\text{stat } b_0 (\text{mon } b_1 (\text{trace}_v^? \bar{b}_2 v_0)) \blacktriangleright_A \text{trace } b_0 b_1 \bar{b}_2 v_0$

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2.1. QED

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by inversion \vdash_A

$\cdot \vdash_A v_0 : \mathcal{U}$

$\cdot \vdash_A \text{trace } b_0 b_1 \bar{b}_2 v_0 : \mathcal{U}$

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3. CASE $v_0 \in (\lambda(x:\tau). e) \cup (\langle v, v \rangle)$

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and $\text{stat } b_0 (\text{mon } b_1 (\text{trace}_v^? \bar{b}_2 (\text{mon } b_3 v_0))) \blacktriangleright_A \text{trace } b_0 b_1 \bar{b}_2 (\text{mon } b_3 v_0)$

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and $b_3 = (\ell_4 \blacktriangleleft \tau_3 \blacktriangleleft \ell_5)$

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3.1. QED

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by inversion \vdash_A

$\cdot \vdash_A v_0 : \tau_3$

$\cdot \vdash_A \text{mon } b_3 v_0 : \mathcal{U}$

$\cdot \vdash_A \text{trace } b_0 b_1 \bar{b}_2 (\text{mon } b_3 v_0) : \mathcal{U}$

□

3693 4. CASE $\text{stat } b_0 \ i_0 \triangleright_{\mathcal{A}} \ i_0$

3694 4.1. QED

3695 $\cdot \vdash_{\mathcal{A}} \ i_0 : \mathcal{U}$

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LEMMA 7.24. *If $\cdot \vdash_{\mathcal{A}} \text{trace } \bar{b}_0 \ v_0 : \mathcal{U}$ then $\cdot \vdash_{\mathcal{A}} \text{add-trace}(\bar{b}_0, v_0) : \mathcal{U}$.*

PROOF. By case analysis of *add-trace*.

1. CASE $\text{add-trace}(\cdot, v_0) = v_0$

1.1. QED

2. CASE $\text{add-trace}(\bar{b}_0, ((\text{trace}_{\sqrt{\cdot}} \bar{b}_1 \ v_1))^{\bar{e}_2}) = \text{trace}_{\sqrt{\cdot}} \bar{b}_0 \bar{b}_1 ((v_1))^{\bar{e}_2}$

2.1. QED

by $\cdot \vdash_{\mathcal{A}} \ v_1 : \mathcal{U}$

3. CASE $\text{add-trace}(\bar{b}_0, v_1) = \text{trace}_{\sqrt{\cdot}} \bar{b}_0 \ v_1$

3.1. QED

by $\cdot \vdash_{\mathcal{A}} \ v_1 : \mathcal{U}$

LEMMA 7.25.

- *If $(x_0 : \tau_0), \Gamma_0 \vdash_{\mathcal{A}} \ e_1 : \tau?$ and $\cdot \vdash_{\mathcal{A}} \ v_0 : \tau_0$ then $\Gamma_0 \vdash_{\mathcal{A}} \ e_1[x_0 \leftarrow v_0] : \tau?$*
- *If $(x_0 : \mathcal{U}), \Gamma_0 \vdash_{\mathcal{A}} \ e_1 : \tau?$ and $\cdot \vdash_{\mathcal{A}} \ v_0 : \mathcal{U}$ then $\Gamma_0 \vdash_{\mathcal{A}} \ e_1[x_0 \leftarrow v_0] : \tau?$*

PROOF. By induction on e_1 .

1. CASE $e_1 = x_2$

1.1. SCASE $x_0 = x_2$

1.1.1. QED

$e_1[x_0 \leftarrow v_0] = v_0$

1.2. SCASE $x_0 \neq x_2$

1.2.1. QED

$e_1[x_0 \leftarrow v_0] = e_1$

2. CASE $e_1 = i_0$

2.1. QED

$e_1[x_0 \leftarrow v_0] = e_1$

3. CASE $e_1 = \lambda x_2. e_2$

3.1. SCASE $x_0 = x_2$

3.1.1. QED

by the induction hypothesis

3.2. SCASE $x_0 \neq x_2$

3.2.1. QED

$e_1[x_0 \leftarrow v_0] = e_1$

4. CASE $e_1 = \lambda(x_2 : \tau_2). e_2$

4.1. SCASE $x_0 = x_2$

- 3745 4.1.1. QED
 3746 by the induction hypothesis
 3747
 3748 4.2. SCASE $x_0 \neq x_2$
 3749 4.2.1. QED
 3750 $e_1[x_0 \leftarrow v_0] = e_1$
 3751 5. CASE $e_1 = \langle e_2, e_3 \rangle$
 3752 5.1. QED
 3753 by the induction hypothesis
 3754
 3755 6. CASE $e_1 = \text{app}\{\tau?\} e_2 e_3$
 3756 6.1. QED
 3757 by the induction hypothesis
 3758
 3759 7. CASE $e_1 = \text{unop}\{\tau?\} e_2$
 3760 7.1. QED
 3761 by the induction hypothesis
 3762
 3763 8. CASE $e_1 = \text{binop}\{\tau?\} e_2 e_3$
 3764 8.1. QED
 3765 by the induction hypothesis
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 3767 9. CASE $e_1 = \text{dyn } b_2 e_2$
 3768 9.1. QED
 3769 by the induction hypothesis
 3770
 3771 10. CASE $e_1 = \text{stat } b_2 e_2$
 3772 10.1. QED
 3773 by the induction hypothesis
 3774
 3775 11. CASE $e_1 = \text{trace}_v \bar{b}_2 v_2$
 3776 11.1. QED
 3777 by the induction hypothesis
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 3779 12. CASE $e_1 = \text{trace } \bar{b}_2 e_2$
 3780 12.1. QED
 3781 by the induction hypothesis

□

3783 LEMMA 7.26 ($\overline{\text{IF}}_{\Lambda}$ -PROGRESS). *If $\cdot \vdash_{\Lambda} e_0 : \tau?$ and $\cdot; \ell \bullet \overline{\text{IF}}_{\Lambda} e_0$ then one of the following holds:*

- 3785 • $e_0 \in ((v))^{\bar{\ell}}$
- 3786 • $e_0 \in E[\text{Err}]^{\bar{\ell}}$
- 3787 • $\exists e_1$ such that $e_0 \rightarrow_{\Lambda} e_1$

3789 PROOF. By case analysis of e_0 .

3791 By lemma 7.30, it suffices to consider the following cases.

- 3792 1. CASE $e_0 \in ((v))^{\bar{\ell}}$
 3793 1.1. QED
 3794
 3795 2. CASE $e_0 \in E[\text{Err}]^{\bar{\ell}}$

3797 2.1. QED
3798 3. CASE $e_0 = E[\text{app}\{\tau_0\}((v_0))^{\bar{\tau}_0} v_1]^{\ell_1}$
3799 3.1. $v_0 \in (\lambda(x:\tau). e) \cup (\text{mon } b \ v)$
3800 by \vdash_A inversion and canonical forms
3801 3.2. SCASE $v_0 = \lambda(x_2:\tau_2). e_2$
3802 3.2.1. QED
3803 $e_0 \triangleright_A E[(e_2[x_2 \leftarrow (v_1)^{\ell_1 \text{rev}(\bar{\tau}_0)}])]^{\bar{\tau}_0}_{\ell_1}$
3804 3.3. SCASE $v_0 = \text{mon}(\ell_2 \blacktriangleleft (\tau_2 \Rightarrow \tau_3) \blacktriangleleft \ell_3)(v_2)^{\ell_4}$
3805 3.3.1. LET $b_3 = (\ell_2 \blacktriangleleft \tau_3 \blacktriangleleft \ell_3)$
3806 and $b_4 = (\ell_3 \blacktriangleleft \tau_2 \blacktriangleleft \ell_2)$
3807 3.3.2. QED
3808 $e_0 \triangleright_A E[(\text{dyn } b_3(\text{app}\{\mathcal{U}\} v_2(\text{stat } b_4((v_1))^{\ell_1 \ell_1 \text{rev}(\bar{\tau}_0)})^{\ell_4}))^{\bar{\tau}_0}]^{\ell_1}$
3809 4. CASE $e_0 = E[\text{app}\{\mathcal{U}\}((v_0))^{\bar{\tau}_0} v_1]^{\ell_1}$
3810 4.1. SCASE $v_0 = \text{trace}_v^? \bar{b}_2((\lambda x_2. e_2))^{\bar{\tau}_3}$
3811 4.1.1. LET $v_2 = \text{add-trace}(\text{rev}(\bar{b}_2), (v_1))^{\ell_1 \text{rev}(\bar{\tau}_0) \text{rev}(\bar{\tau}_3)}$
3812 4.1.2. QED
3813 $e_0 \blacktriangleright_A E[(\text{trace } \bar{b}_2((e_2[x_2 \leftarrow v_1]))^{\bar{\tau}_3})^{\bar{\tau}_0}]^{\ell_1}$
3814 4.2. SCASE $v_0 = \text{trace}_v^? \bar{b}_2((\text{mon}(\ell_3 \blacktriangleleft (\tau_2 \Rightarrow \tau_3)^{\ell_4} \blacktriangleleft \ell_4)(v_2)^{\ell_5}))^{\bar{\tau}_6}$
3815 4.2.1. LET $b_7 = (\ell_3 \blacktriangleleft \tau_3 \blacktriangleleft \ell_4)$
3816 and $b_8 = (\ell_4 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3)$
3817 and $\tau_4 = \text{forget}(\tau_3)$
3818 4.2.2. QED
3819 $e_0 \blacktriangleright_A E[(\text{trace } \bar{b}_2((\text{stat } b_7(\text{app}\{\tau_4\} v_2(\text{dyn } b_8((v_2))^{\text{last}(\bar{\tau}_6)})^{\ell_3}))^{\bar{\tau}_6})^{\bar{\tau}_0})]^{\ell_1}$
3820 4.3. SCASE $v_0 \notin (\lambda x. e) \cup (\text{mon } b \ v)$
3821 4.3.1. QED
3822 $e_0 \blacktriangleright_A E[\text{TagErr } \bullet]^{\ell_0}$
3823 5. CASE $e_0 = E[\text{unop}\{\tau?\}((v_0))^{\bar{\tau}_0}]^{\ell_1}$
3824 5.1. QED
3825 by lemma 7.16 and lemma 7.33
3826 6. CASE $e_0 = E[\text{binop}\{\tau?\}((v_0))^{\bar{\tau}_0} ((v_1))^{\bar{\tau}_1}]^{\ell_2}$
3827 6.1. QED
3828 by lemma 7.16 and lemma 7.33
3829 7. CASE $e_0 = E[\text{dyn } b_0((v_1))^{\bar{\tau}_1}]^{\ell_2}$
3830 7.1. QED
3831 by lemma 7.16 and lemma 7.35
3832 8. CASE $e_0 = E[\text{stat } b_0((v_1))^{\bar{\tau}_1}]^{\ell_2}$
3833 8.1. QED
3834 by lemma 7.16 and lemma 7.36
3835 9. CASE $e_0 = E[\text{trace } \bar{b}_0 v_0]^{\ell_2}$
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3849 9.1. QED

3850 $e_0 \triangleright_A E[\text{add-trace}(\bar{b}_0, v_0)]^{\ell_2}$

3855 LEMMA 7.27 ($\overline{\Gamma}_A$ -PRESERVATION). *If $\cdot \vdash_A e_0 : \tau?$ and $\cdot; \ell_\bullet \overline{\Gamma}_A e_0$ and $e_0 \rightarrow_A e_1$ then $\cdot; \ell_\bullet \overline{\Gamma}_A e_1$*

3858 PROOF. By lemma 7.28 and lemma 7.29.

3863 LEMMA 7.28. *If $\cdot \vdash_A e_0 : \tau_0$ and $\cdot; \ell_0 \overline{\Gamma}_A e_0$ and $e_0 \triangleright_A e_1$ then $\cdot; \ell_0 \overline{\Gamma}_A e_1$*

3866 PROOF. By case analysis of \triangleright_A .

3867 1. CASE $\delta_A(\text{unop}, v_0)$ is defined

3868 and $(\text{unop}\{\tau_1\} v_0)^{\ell_0} \triangleright_A (\delta_A(\text{unop}, v_0))^{\ell_0}$

3869 1.1. QED

3871 by lemma 7.34

3872 2. CASE $\delta_A(\text{binop}, v_0, v_1)$ is defined

3873 and $(\text{binop}\{\tau_1\} v_0 v_1)^{\ell_0} \triangleright_A (\delta_A(\text{binop}, v_0, v_1))^{\ell_0}$

3875 2.1. QED

3876 by lemma 7.34

3877 3. CASE $(\text{app}\{\tau_0\} ((\lambda(x_0 : \tau_1). e_0))^{\bar{\ell}_0} v_1)^{\ell_1} \triangleright_A (e_0[x_0 \leftarrow ((v_1))^{\ell_1 \text{rev}}(\bar{\ell}_0)])^{\bar{\ell}_0 \ell_1}$

3879 3.1. $\bar{\ell}_0 = \ell_1 \cdots \ell_1$

3880 by inversion $\overline{\Gamma}_A$

3881 3.2. $\cdot; \ell_0 \overline{\Gamma}_A ((v_1))^{\ell_1 \text{rev}}(\bar{\ell}_0)$

3882 by 3.1 and inversion $\overline{\Gamma}_A$

3883 3.3. QED

3885 by lemma 7.40

3886 by lemma 7.40

3887 $\frac{\cdot; \ell_1 \overline{\Gamma}_A e_0[x_0 \leftarrow ((v_1))^{\ell_1 \text{rev}}(\bar{\ell}_0)]}{\cdot; \ell_1 \overline{\Gamma}_A (e_0[x_0 \leftarrow ((v_1))^{\ell_1 \text{rev}}(\bar{\ell}_0)])^{\bar{\ell}_0 \ell_1}}$

3888 $\cdot; \ell_1 \overline{\Gamma}_A (e_0[x_0 \leftarrow ((v_1))^{\ell_1 \text{rev}}(\bar{\ell}_0)])^{\bar{\ell}_0 \ell_1}$

3891 4. CASE $(\text{app}\{\tau_0\} ((\text{mon}(\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2))^{\ell_1} \blacktriangleleft \ell_1) (v_0)^{\ell_2})^{\bar{\ell}_3} v_1)^{\ell_4}$

3892 $\triangleright_A (\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{app}\{\mathcal{U}\} v_0 (\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) ((v_1))^{\ell_4 \text{rev}}(\bar{\ell}_3)))^{\ell_2} \bar{\ell}_3 \ell_4)$

3893 4.1. $\ell_1 = \ell_2$

3894 and $\bar{\ell}_3 = \ell_4 \cdots \ell_4$

3895 by inversion $\overline{\Gamma}_A$

3896 4.2. QED

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$$\begin{array}{c}
\text{by inversion } \overline{\Gamma}_A \\
\frac{}{\vdash \ell_0 \overline{\Gamma}_A v_1} \\
\frac{}{\vdash \ell_0 \overline{\Gamma}_A ((v_1))^{\ell_4 \text{rev}(\overline{\ell}_3)}} \\
\text{by inversion } \overline{\Gamma}_A \quad \frac{}{\vdash \ell_1 \overline{\Gamma}_A v_0} \quad \frac{}{\vdash \ell_1 \overline{\Gamma}_A \text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) ((v_1))^{\ell_4 \text{rev}(\overline{\ell}_3)}} \\
\frac{}{\vdash \ell_1 \overline{\Gamma}_A \text{app}\{\mathcal{U}\} v_0 (\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) ((v_1))^{\ell_4 \text{rev}(\overline{\ell}_3)})} \\
\frac{}{\vdash \ell_1 \overline{\Gamma}_A (\text{app}\{\mathcal{U}\} v_0 (\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) ((v_1))^{\ell_4 \text{rev}(\overline{\ell}_3)}))^{\ell_2}} \\
\frac{}{\vdash \ell_0 \overline{\Gamma}_A \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{app}\{\mathcal{U}\} v_0 (\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) ((v_1))^{\ell_4 \text{rev}(\overline{\ell}_3)}))^{\ell_2}} \\
\frac{}{\vdash \ell_0 \overline{\Gamma}_A (\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{app}\{\mathcal{U}\} v_0 (\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) ((v_1))^{\ell_4 \text{rev}(\overline{\ell}_3)}))^{\ell_2})^{\overline{\ell}_3 \ell_4}}
\end{array}$$

5. CASE $(\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) ((v_0))^{\overline{\ell}_2})^{\overline{\ell}_3} \triangleright_A e_2$

by lemma 7.37

□

LEMMA 7.29. *If $\vdash_A e_0 : \mathcal{U}$ and $\vdash \ell_0 \overline{\Gamma}_A e_0$ and $e_0 \blacktriangleright_A e_1$ then $\vdash \ell_0 \overline{\Gamma}_A e_1$*

PROOF. By case analysis of \blacktriangleright_A .

1. CASE $v_0 \notin \text{mon}(\ell \blacktriangleleft \tau \times \tau \blacktriangleleft \ell) v$

and $\delta_A(\text{unop}, v_0)$ is not defined

and $(\text{unop}\{\mathcal{U}\} v_0)^{\ell_0} \blacktriangleright_A (\text{TagErr} \bullet)^{\ell_0}$

1.1. QED

2. CASE $\delta_A(\text{unop}, v_0)$ is defined

and $(\text{unop}\{\mathcal{U}\} v_0)^{\ell_0} \blacktriangleright_A (\delta_A(\text{unop}, v_0))^{\ell_0}$

2.1. QED

by lemma 7.34

3. CASE $(\text{fst}\{\mathcal{U}\} ((\text{trace}_v^2 \overline{b}_0 ((\text{mon}(\ell_1 \blacktriangleleft (\tau_0 \times \tau_1)^{\ell_2} \blacktriangleleft \ell_2) (v_1)^{\ell_3}))^{\overline{\ell}_4})^{\overline{\ell}_5 \ell_6})) \blacktriangleright_A$

$(\text{trace} \overline{b}_0 ((\text{stat}(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) (\text{fst}\{\tau_0\} v_1)^{\ell_3}))^{\overline{\ell}_4})^{\overline{\ell}_5 \ell_6}$

3.1. $\overline{\ell}_5 = \ell_6 \cdots \ell_6$

and $\overline{b}_0 \simeq \overline{\ell}_4$

and $\text{last}(\overline{\ell}_4) = \ell_1$

and $\ell_2 = \ell_3$

by inversion $\overline{\Gamma}_A$

3.2. QED

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- $$\frac{\frac{\frac{\text{by inversion } \overline{\Gamma}_A}{\vdash; \ell_2 \overline{\Gamma}_A v_1}}{\vdash; \ell_2 \overline{\Gamma}_A \text{fst}\{\tau_0\} v_1}}{\vdash; \ell_2 \overline{\Gamma}_A (\text{fst}\{\tau_0\} v_1)^{\ell_3}}}{\vdash; \text{last}(\overline{\ell}_4) \overline{\Gamma}_A \text{stat}(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) (\text{fst}\{\tau_0\} v_1)^{\ell_3}}}$$
- $$\frac{\vdash; \ell_6 \overline{\Gamma}_A \text{trace } \overline{b}_0 ((\text{stat}(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) (\text{fst}\{\tau_0\} v_1)^{\ell_3}))^{\overline{\ell}_4}}{\vdash; \ell_6 \overline{\Gamma}_A (\text{trace } \overline{b}_0 ((\text{stat}(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) (\text{fst}\{\tau_0\} v_1)^{\ell_3}))^{\overline{\ell}_4})^{\overline{\ell}_5 \ell_6}}$$
4. CASE $(\text{snd}\{\mathcal{U}\} ((\text{trace}_v^? \overline{b}_0 ((\text{mon}(\ell_1 \blacktriangleleft (\tau_0 \times \tau_1)^{\ell_2} \blacktriangleleft \ell_2) (v_1)^{\ell_3}))^{\overline{\ell}_4}))^{\overline{\ell}_5 \ell_6}) \blacktriangleright_A$
 $(\text{trace } \overline{b}_0 ((\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2) (\text{snd}\{\tau_1\} v_1)^{\ell_3}))^{\overline{\ell}_4 \overline{\ell}_5 \ell_6})$
- 4.1. QED
similar to fst
5. CASE $\delta_A(\text{binop}, v_0, v_1)$ is not defined
and $(\text{binop}\{\mathcal{U}\} v_0 v_1)^{\ell_0} \blacktriangleright_A (\text{TagErr } \bullet)^{\ell_0}$
- 5.1. QED
6. CASE $\delta_A(\text{binop}, v_0, v_1)$ is defined
and $(\text{binop}\{\mathcal{U}\} v_0 v_1)^{\ell_0} \blacktriangleright_A (\delta_A(\text{binop}, v_0, v_1))^{\ell_0}$
- 6.1. QED
by lemma 7.34
7. CASE $(\text{app}\{\mathcal{U}\} ((\text{trace}_v^? \overline{b}_0 ((\lambda x_0. e_0))^{\overline{\ell}_1})^{\overline{\ell}_2} v_1))^{\ell_3} \blacktriangleright_A$
 $(\text{trace } \overline{b}_0 ((e_0[x_0 \leftarrow \text{add-trace}(\text{rev}(\overline{b}_0), (v_1))^{\ell_3 \text{rev}(\overline{\ell}_2) \text{rev}(\overline{\ell}_1)})])^{\overline{\ell}_1 \overline{\ell}_2 \ell_3})$
- 7.1. $\overline{\ell}_2 = \ell_3 \cdots \ell_3$
and $\overline{b}_0 \simeq \overline{\ell}_1$
and $\vdash; \text{last}(\overline{\ell}_1) \overline{\Gamma}_A \lambda x_0. e_0$
- 7.2. $\vdash; \text{last}(\overline{\ell}_1) \overline{\Gamma}_A \text{add-trace}(\text{rev}(\overline{b}_0), (v_1))^{\ell_3 \text{rev}(\overline{\ell}_2) \text{rev}(\overline{\ell}_1)}$
- 7.3. QED
- $$\frac{\frac{\text{by lemma 7.40}}{\vdash; \text{last}(\overline{\ell}_1) \overline{\Gamma}_A e_0[x_0 \leftarrow \text{add-trace}(\text{rev}(\overline{b}_0), (v_1))^{\ell_3 \text{rev}(\overline{\ell}_2) \text{rev}(\overline{\ell}_1)}]}}{\vdash; \ell_3 \overline{\Gamma}_A \text{trace } \overline{b}_0 ((e_0[x_0 \leftarrow \text{add-trace}(\text{rev}(\overline{b}_0), (v_1))^{\ell_3 \text{rev}(\overline{\ell}_2) \text{rev}(\overline{\ell}_1)})])^{\overline{\ell}_1}}}$$
- $$\vdash; \ell_3 \overline{\Gamma}_A (\text{trace } \overline{b}_0 ((e_0[x_0 \leftarrow \text{add-trace}(\text{rev}(\overline{b}_0), (v_1))^{\ell_3 \text{rev}(\overline{\ell}_2) \text{rev}(\overline{\ell}_1)})])^{\overline{\ell}_1})^{\overline{\ell}_2 \ell_3}$$
8. CASE $(\text{app}\{\mathcal{U}\} ((\text{trace}_v^? \overline{b}_0 ((\text{mon}(\ell_1 \blacktriangleleft (\tau_0 \Rightarrow \tau_1)^{\ell_2} \blacktriangleleft \ell_2) (v_0)^{\ell_3}))^{\overline{\ell}_4}))^{\overline{\ell}_5 \ell_6} v_1))^{\ell_3} \blacktriangleright_A$
 $(\text{trace } \overline{b}_0 ((\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2) (\text{app}\{\tau_1\} v_0 (\text{dyn}(\ell_2 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (v_2)^{\text{last}(\overline{\ell}_4)})^{\ell_3}))^{\overline{\ell}_4 \overline{\ell}_5 \ell_6}))$

- 4057 2. CASE $e_0 \in ((v))^{\bar{\ell}}$
 4058 2.1. QED
 4059 3. CASE $e_0 = \langle e_1, e_2 \rangle$
 4060 3.1. $\cdot \vdash_A e_1 : \tau?$ and $\cdot \vdash_A e_2 : \tau?$
 4061 by inversion \vdash_A
 4062 3.2. $;\ell_0 \Vdash_A e_1$ and $;\ell_0 \Vdash_A e_2$
 4063 by inversion \Vdash_A
 4064 3.3. SCASE $e_1 \notin ((v))^{\bar{\ell}}$
 4065 3.3.1. QED
 4066 by the induction hypothesis
 4067 3.4. SCASE $e_1 \in ((v))^{\bar{\ell}}$ and $e_2 \notin ((v))^{\bar{\ell}}$
 4068 3.4.1. QED
 4069 by the induction hypothesis
 4070 3.5. SCASE $e_1 \in ((v))^{\bar{\ell}}$ and $e_2 \in ((v))^{\bar{\ell}}$
 4071 3.5.1. QED
 4072 $e_0 \in v$
 4073 4. CASE $e_0 = \text{app}\{\tau?\} e_1 e_2$
 4074 4.1. QED
 4075 by the induction hypothesis
 4076 5. CASE $e_0 = \text{unop}\{\tau?\} e_1$
 4077 5.1. QED
 4078 by the induction hypothesis
 4079 6. CASE $e_0 = \text{binop}\{\tau?\} e_1 e_2$
 4080 6.1. QED
 4081 by the induction hypothesis
 4082 7. CASE $e_0 = \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)(e_1)^{\ell_1}$
 4083 7.1. $;\ell_1 \Vdash_A e_1$
 4084 by inversion \Vdash_A
 4085 7.2. QED
 4086 by the induction hypothesis
 4087 8. CASE $e_0 = \text{stat } b_1 e_1$
 4088 8.1. QED
 4089 by the induction hypothesis
 4090 9. CASE $e_0 = \text{trace } \bar{b}_1((e_1))^{\bar{\ell}_2}$
 4091 9.1. QED
 4092 by the induction hypothesis
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LEMMA 7.31. *If $;\ell_0 \Vdash_A E_0[e_0]$ then $\exists \ell_1$ such that $;\ell_1 \Vdash_A e_0$*

PROOF. By induction on the structure of E_0 .

1. $E_0 = []$

- 4109 1.1. QED
 4110 2. $E_0 = \langle E_1, e_2 \rangle$
 4111 2.1. QED
 4112 by the induction hypothesis
 4113 3. $E_0 = \langle e_1, E_2 \rangle$
 4114 3.1. QED
 4115 by the induction hypothesis
 4116 4. $E_0 = \text{unop}\{\tau?\} E_1$
 4117 4.1. QED
 4118 by the induction hypothesis
 4119 5. $E_0 = \text{binop}\{\tau?\} E_1 e_2$
 4120 5.1. QED
 4121 by the induction hypothesis
 4122 6. $E_0 = \text{binop}\{\tau?\} e_1 E_2$
 4123 6.1. QED
 4124 by the induction hypothesis
 4125 7. $E_0 = \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) (E_1)^{\ell_3}$
 4126 7.1. $\cdot; \ell_2 \Vdash_{\mathbb{A}} E_1[e_0]$
 4127 7.2. QED
 4128 by the induction hypothesis
 4129 8. $E_0 = \text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) (E_1)^{\ell_3}$
 4130 8.1. QED
 4131 by the induction hypothesis
 4132 9. $E_0 = (E_1)^{\ell_0}$
 4133 9.1. QED
 4134 by the induction hypothesis
 4135 10. $E_0 = \text{trace } \bar{b}_1 ((E_1))^{\bar{\ell}_1}$
 4136 10.1. QED
 4137 by the induction hypothesis

□

4147 LEMMA 7.32 ($\Vdash_{\mathbb{A}}$ REPLACEMENT). *If $\cdot; \ell_0 \Vdash_{\mathbb{A}} E_0[e_0]$ and the derivation contains a proof of $\cdot; \ell_1 \Vdash_{\mathbb{A}} e_0$ and $\cdot; \ell_1 \Vdash_{\mathbb{A}} e_1$*
 4148 *then $L_0; \ell_0 \Vdash_{\mathbb{A}} E_0[e_1]$*
 4149

4150 PROOF. By induction on the structure of E_0 .

- 4151 1. $E_0 = []$
 4152 1.1. QED
 4153 2. $E_0 = \langle E_1, e_2 \rangle$
 4154 2.1. QED
 4155 by the induction hypothesis
 4156 3. $E_0 = \langle e_1, E_2 \rangle$
 4157 3.1. QED

4160

4161 by the induction hypothesis

4162 4. $E_0 = \text{unop}\{\tau?\} E_1$

4163 4.1. QED

4164 by the induction hypothesis

4165 5. $E_0 = \text{binop}\{\tau?\} E_1 e_2$

4166 5.1. QED

4167 by the induction hypothesis

4168 6. $E_0 = \text{binop}\{\tau?\} e_1 E_2$

4169 6.1. QED

4170 by the induction hypothesis

4171 7. $E_0 = \text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)(E_1)^{\ell_3}$

4172 7.1. $\cdot; \ell_2 \Vdash_{\mathbb{A}} E_1[e_0]$

4173 7.2. QED

4174 by the induction hypothesis

4175 8. $E_0 = \text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)(E_1)^{\ell_3}$

4176 8.1. QED

4177 by the induction hypothesis

4178 9. $E_0 = (E_1)^{\ell_0}$

4179 9.1. QED

4180 by the induction hypothesis

4181 10. $E_0 = \text{trace } \bar{b}_1((E_1))^{\bar{\ell}_1}$

4182 10.1. QED

4183 by the induction hypothesis

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LEMMA 7.33 ($\delta_{\mathbb{A}}$ LABEL PROGRESS).

- If $\cdot \vdash_{\mathbb{A}} \text{unop}\{\tau_1\} v_0 : \tau_0$ and $\cdot; \ell_0 \Vdash_{\mathbb{A}} \text{unop}\{\tau_1\} v_0$ and $(\text{unop}\{\tau_1\} v_0)^{\ell_0} \triangleright_{\mathbb{A}} (e_1)^{\ell_0}$.
- if $\cdot \vdash_{\mathbb{A}} \text{binop}\{\tau_1\} v_0 v_1 : \tau_0$ and $\cdot; \ell_0 \Vdash_{\mathbb{A}} \text{binop}\{\tau_1\} v_0 v_1$ and $(\text{binop}\{\tau_1\} v_0 v_1)^{\ell_0} \triangleright_{\mathbb{A}} (e_2)^{\ell_0}$.
- If $\cdot \vdash_{\mathbb{A}} \text{unop}\{\mathcal{U}\} v_0 : \mathcal{U}$ and $\cdot; \ell_0 \Vdash_{\mathbb{A}} \text{unop}\{\mathcal{U}\} v_0$ then $(\text{unop}\{\mathcal{U}\} v_0)^{\ell_0} \blacktriangleright_{\mathbb{A}} (e_1)^{\ell_0}$.
- if $\cdot \vdash_{\mathbb{A}} \text{binop}\{\mathcal{U}\} v_0 v_1 : \mathcal{U}$ and $\cdot; \ell_0 \Vdash_{\mathbb{A}} \text{binop}\{\mathcal{U}\} v_0 v_1$ then $(\text{binop}\{\mathcal{U}\} v_0 v_1)^{\ell_0} \blacktriangleright_{\mathbb{A}} (e_2)^{\ell_0}$.

PROOF. By case analysis of $\delta_{\mathbb{A}}$, $\vdash_{\mathbb{A}}$, $\Vdash_{\mathbb{A}}$, and $\blacktriangleright_{\mathbb{A}}$.

1. CASE $\cdot \vdash_{\mathbb{A}} \text{unop}\{\tau_1\} v_0 : \tau_0$

1.1. $v_0 \in (((v, v))^{\bar{\ell}}) \cup (((\text{mon}(\ell \blacktriangleleft (\tau \times \tau)^{\ell} \blacktriangleleft \ell) v))^{\bar{\ell}})$

by $\vdash_{\mathbb{A}}$ inversion and canonical forms

1.2. SCASE $v_0 = (((v_1, v_2))^{\bar{\ell}_0})$

1.2.1. QED

$(\text{unop}\{\tau_1\} v_0)^{\ell_0} \triangleright_{\mathbb{A}} (\delta_{\mathbb{A}}(\text{unop}, v_0))^{\ell_0}$

1.3. SCASE $v_0 = ((\text{mon}(\ell_1 \blacktriangleleft (\tau_1 \times \tau_2)^{\ell_2} \blacktriangleleft \ell_2) v_1))^{\bar{\ell}_3}$

1.3.1. QED

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□

4213 $(\text{fst}\{\tau_0\} v_0)^{\ell_0} \triangleright_A ((\text{dyn}(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) (\text{fst}\{\mathcal{U}\} v_1)))^{\bar{\ell}_3 \ell_1}$
 4214 (and similarly for snd)

4215 2. CASE $\cdot \vdash_A \text{binop}\{\tau_1\} v_0 v_1 : \tau_0$

4216 2.1. $v_0 \in ((i))^{\bar{\ell}}$ and $v_1 \in ((i))^{\bar{\ell}}$

4218 by \vdash_A inversion and canonical forms

4219 2.2. QED

4220 $(\text{binop}\{\tau_1\} v_0 v_1)^{\ell_0} \triangleright_A (\delta_A(\text{binop}, v_0, v_1))^{\ell_0}$

4222 3. CASE $\cdot \vdash_A \text{unop}\{\mathcal{U}\} v_0 : \mathcal{U}$

4223 3.1. SCASE $v_0 \in \text{trace}_v^? \bar{b}((\text{mon } b(v)^\ell))^{\bar{\ell}}$

4224 3.1.1. QED

4225 by definition \blacktriangleright_A

4227 3.2. SCASE $v_0 \in \text{trace}_v^? \bar{b}(\langle v, v \rangle)^{\bar{\ell}}$

4228 3.2.1. QED

4229 $(\text{unop}\{\mathcal{U}\} v_0)^{\ell_0} \blacktriangleright_A (\delta_A(\text{unop}, v_0))^{\ell_0}$

4230 3.3. SCASE $\text{rem-trace}(v_0) \notin \langle v, v \rangle \cup (\text{mon } b(v)^\ell)$

4232 3.3.1. QED

4233 $(\text{unop}\{\mathcal{U}\} v_0)^{\ell_0} \blacktriangleright_A (\text{TagErr} \bullet)^{\ell_0}$

4234 4. CASE $\cdot \vdash_A \text{binop}\{\mathcal{U}\} v_0 v_1 : \mathcal{U}$

4235 4.1. QED

4236 by definition \blacktriangleright_A

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4242 LEMMA 7.34 (δ_A LABEL PRESERVATION).

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4245 • If $\cdot; \ell_0 \Vdash_A \text{unop}\{\tau?\} v_0$ and $(\text{unop}\{\tau?\} v_0)^{\ell_0} (\triangleright_A \cup \blacktriangleright_A) (e_1)^{\ell_0}$ then $\cdot; \ell_0 \Vdash_A e_1$.

4246 • If $\cdot; \ell_0 \Vdash_A \text{binop}\{\tau?\} v_0 v_1$ and $(\text{binop}\{\tau?\} v_0 v_1)^{\ell_0} (\triangleright_A \cup \blacktriangleright_A) (e_1)^{\ell_0}$ then $\cdot; \ell_0 \Vdash_A e_1$.

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4248

4249 PROOF. By case analysis of $(\triangleright_A \cup \blacktriangleright_A)$.

4250 1. $(\text{fst}\{\tau_0\} (\langle v_1, v_2 \rangle))^{\bar{\ell}_1 \ell_0} \triangleright_A ((v_1))^{\bar{\ell}_1 \ell_0}$

4251 1.1. $\cdot; \ell_0 \Vdash_A v_0$

4253 and $\bar{\ell}_1 = \ell_0 \cdots \ell_0$

4254 by inversion \Vdash_A

4255 1.2. QED

4257 2. $(\text{fst}\{\tau_0\} (\text{mon}(\ell_1 \blacktriangleleft (\tau_1 \times \tau_2)^{\ell_2} \blacktriangleleft \ell_2) (v_1)^{\ell_2}))^{\bar{\ell}_3 \ell_0} \triangleright_A (\text{dyn}(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) (\text{fst}\{\mathcal{U}\} v_0)^{\ell_2})^{\bar{\ell}_3 \ell_0}$

4258 2.1. $\bar{\ell}_3 = \ell_0 \cdots \ell_0$

4259 and $\ell_0 = \ell_1$

4261 by inversion \Vdash_A

4262 2.2. QED

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□

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- $$\frac{\frac{\text{by inversion } \overline{\Gamma}_A}{\cdot; \ell_0 \overline{\Gamma}_A v_0}}{\cdot; \ell_0 \overline{\Gamma}_A \text{fst}\{\mathcal{U}\} v_0}}{\cdot; \ell_0 \overline{\Gamma}_A \text{dyn}(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) (\text{fst}\{\mathcal{U}\} v_0)^{\ell_2}}}{\cdot; \ell_0 \overline{\Gamma}_A (\text{dyn}(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) (\text{fst}\{\mathcal{U}\} v_0)^{\ell_2})^{\overline{\ell}_3 \ell_0}}$$
3. $(\text{fst}\{\mathcal{U}\} (\text{trace}_v^? \overline{b}_0 ((v_1, v_2))^{\overline{\ell}_1})^{\overline{\ell}_2})^{\ell_0} \blacktriangleright_A (\text{add-trace}(\overline{b}_0, (v_1))^{\overline{\ell}_1})^{\overline{\ell}_2 \ell_0})$
- 3.1. $\overline{\ell}_2 = \ell_0 \cdots \ell_0$
and $\overline{b}_0 \simeq \overline{\ell}_1$
and $\cdot; \text{last}(\overline{\ell}_1) \overline{\Gamma}_A v_1$
by inversion $\overline{\Gamma}_A$
- 3.2. QED
by lemma 7.39
- $$4. (\text{fst}\{\mathcal{U}\} (\text{trace}_v^? \overline{b}_0 ((\text{mon}(\ell_1 \blacktriangleleft (\tau_1 \times \tau_2)^{\ell_2} \blacktriangleleft \ell_2) (v_1)^{\ell_3}))^{\overline{\ell}_4})^{\overline{\ell}_5})^{\ell_0} \blacktriangleright_A (\text{trace} \overline{b}_0 ((\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2) (\text{fst}\{\text{forget}(\tau_1)\} v_1)^{\ell_3}))^{\overline{\ell}_4})^{\overline{\ell}_5 \ell_0})$$
- 4.1. $\overline{\ell}_5 = \ell_0 \cdots \ell_0$
and $\overline{b}_0 \simeq \overline{\ell}_4$
and $\text{last}(\overline{\ell}_4) = \ell_1$
and $\cdot; \ell_2 \overline{\Gamma}_A v_1$
by inversion $\overline{\Gamma}_A$
- 4.2. QED
- $$\frac{\frac{\text{by inversion } \overline{\Gamma}_A}{\cdot; \ell_0 \overline{\Gamma}_A v_1}}{\cdot; \ell_0 \overline{\Gamma}_A \text{fst}\{\text{forget}(\tau_1)\} v_1}}{\cdot; \ell_0 \overline{\Gamma}_A \text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2) (\text{fst}\{\text{forget}(\tau_1)\} v_1)^{\ell_3}}}{\cdot; \ell_0 \overline{\Gamma}_A \text{trace} \overline{b}_0 ((\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2) (\text{fst}\{\text{forget}(\tau_1)\} v_1)^{\ell_3}))^{\overline{\ell}_4}}}{\cdot; \ell_0 \overline{\Gamma}_A (\text{trace} \overline{b}_0 ((\text{stat}(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2) (\text{fst}\{\text{forget}(\tau_1)\} v_1)^{\ell_3}))^{\overline{\ell}_4})^{\overline{\ell}_5 \ell_0}}$$
5. $(\text{snd}\{\tau?\} v_0)^{\ell_0} (\triangleright_A \cup \blacktriangleright_A) ((e_2))^{\ell_0}$
- 5.1. QED
similar to fst cases
6. $(\text{sum}\{\tau?\} v_0 v_1)^{\ell_0} (\triangleright_A \cup \blacktriangleright_A) (i_2)^{\ell_0}$
- 6.1. QED
7. $(\text{quotient}\{\tau?\} v_0 v_1)^{\ell_0} (\triangleright_A \cup \blacktriangleright_A) (\text{DivErr})^{\ell_0}$
- 7.1. QED
8. $(\text{quotient}\{\tau?\} ((i_1))^{\ell_1} ((i_2))^{\ell_2})^{\ell_0} (\triangleright_A \cup \blacktriangleright_A) ([i_1/i_2])^{\ell_0}$

4317 8.1. QED

4318

4319 □

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LEMMA 7.35. *If $\cdot \vdash_A \text{dyn } b_0 v_0 : \tau_0$ and $\cdot; \ell_0 \Vdash_A \text{dyn } b_0 v_0$ then $(\text{dyn } b_0 v_0)^{\ell_0} \triangleright_A (e_1)^{\ell_0}$.*

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4323

PROOF. By inversion of \vdash_A and case analysis of $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$.

4324

1. $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$

4325

and $\ell_0; \ell_1 \Vdash \tau_0$

4326

and $\cdot; \ell_1 \Vdash_A v_0$

4327

and $v_0 = ((v_1))^{\ell_1}$

4328

by inversion \Vdash_A

4329

2. CASE $\text{tag-match}(\lfloor \tau_0 \rfloor, v_1)$

4330

and $\text{rem-trace}(v_1) \in ((\lambda x. e))^{\bar{\ell}} \cup ((\langle v, v \rangle))^{\bar{\ell}} \cup ((\text{mon } b v))^{\bar{\ell}}$

4331

4332

2.1. QED

4333

$(\text{dyn } b_0 v_0)^{\ell_0} \triangleright_A (\text{mon } b_0 v_0)^{\ell_0}$

4334

3. CASE $v_1 \in i$ and $\text{tag-match}(\lfloor \text{Int} \rfloor, v_1)$

4335

3.1. QED

4336

$(\text{dyn } b_0 v_0)^{\ell_0} \triangleright_A (v_1)^{\ell_0}$

4337

4338

4. CASE $v_1 \in n$ and $\text{tag-match}(\lfloor \text{Nat} \rfloor, v_1)$

4339

4.1. QED

4340

$(\text{dyn } b_0 v_0)^{\ell_0} \triangleright_A (v_1)^{\ell_0}$

4341

4342

5. CASE $\neg \text{tag-match}(\lfloor \tau_0 \rfloor, v_1)$

4343

5.1. QED

4344

$(\text{dyn } b_0 v_0)^{\ell_0} \triangleright_A (\text{BndryErr}(b_0, v_0))^{\ell_0}$

4345

4346

4347 □

4348

4349

LEMMA 7.36. *If $\cdot \vdash_A \text{stat } b_0 v_0 : \mathcal{U}$ and $\cdot; \ell_0 \Vdash_A \text{stat } b_0 v_0$ then $(\text{stat } b_0 v_0)^{\ell_0} \blacktriangleright_A (e_1)^{\ell_0}$.*

4350

4351

PROOF. By case analysis on v_0 .

4352

1. $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$

4353

and $\ell_0; \ell_1 \Vdash \tau_0$

4354

and $\cdot; \ell_1 \Vdash_A v_0$

4355

and $v_0 = ((v_1))^{\bar{\ell}_2}$

4356

by inversion \Vdash_A

4357

2. CASE $v_1 \in \lambda x. e$

4358

2.1. CONTRADICTION:

4359

$\cdot \vdash_A \text{stat } b_0 v_0 : \mathcal{U}$

4360

4361

3. CASE $v_1 \in \lambda(x:\tau). e$

4362

3.1. QED

4363

$(\text{stat } b_0 v_0)^{\ell_0} \blacktriangleright_A (\text{mon } b_0 v_0)^{\ell_0}$

4364

4365

4. CASE $v_1 \in \langle v, v \rangle$

4366

4.1. QED

4367

4368

4369 $(\text{stat } b_0 \ v_0)^{\ell_0} \triangleright_A (\text{mon } b_0 \ v_0)^{\ell_0}$
 4370
 4371 5. CASE $v_1 = \text{mon } b_1 ((\text{trace}_v^? \bar{b}_2 ((v_2))^{\bar{\ell}_3})^{\bar{\ell}_4})$
 4372 5.1. SCASE $v_2 \in (\lambda x. e) \cup \langle v, v \rangle$
 4373 5.1.1. QED
 4374 $(\text{stat } b_0 \ v_0)^{\ell_0} \triangleright_A (\text{trace } b_0 b_1 \bar{b}_2 ((v_2))^{\bar{\ell}_3 \bar{\ell}_4 \bar{\ell}_2})^{\ell_0}$
 4375 5.2. SCASE $v_2 \in (\lambda(x:\tau). e)$
 4376 5.2.1. CONTRADICTION:
 4377 $\cdot \vdash_A v_0 : \tau_0$
 4378
 4379 5.3. SCASE $v_2 = (\text{mon } b_5 ((v_3))^{\bar{\ell}_6})$
 4380 5.3.1. SSCASE $v_3 \in (\lambda(x:e).) \cup \langle v, v \rangle$
 4381 $(\text{stat } b_0 \ v_0)^{\ell_0} \triangleright_A (\text{trace } b_0 b_1 \bar{b}_2 ((v_2))^{\bar{\ell}_3 \bar{\ell}_4 \bar{\ell}_2})^{\ell_0}$
 4382 5.3.2. SSCASE $v_3 \notin (\lambda(x:e).) \cup \langle v, v \rangle$
 4383 5.3.2.1. CONTRADICTION:
 4384 $\cdot \vdash_A v_1 : \tau_0$
 4385
 4386 5.4. SCASE otherwise
 4387 5.4.1. CONTRADICTION:
 4388 $\cdot \vdash_A v_1 : \tau_0$
 4389
 4390 6. CASE $v_1 = \text{trace}_v^? \bar{b}_0 ((i_1))^{\bar{\ell}_1}$
 4391 6.1. QED
 4392 $(\text{stat } b_0 \ v_0)^{\ell_0} \triangleright_A (i_1)^{\ell_0}$
 4393
 4394

□

LEMMA 7.37. *If $\cdot \vdash_A \text{dyn } b_0 \ v_0 : \tau_0$ and $;\ell_0 \overline{\vdash}_A \text{dyn } b_0 \ v_0$ and $(\text{dyn } b_0 \ v_0)^{\ell_0} \triangleright_A (e_1)^{\ell_0}$ then $;\ell_0 \overline{\vdash}_A e_1$.*

PROOF. By case analysis of \triangleright_A .

4400 1. $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
 4401 and $\ell_0; \ell_1 \Vdash \tau_0$
 4402 and $;\ell_1 \overline{\vdash}_A v_0$
 4403 by inversion $\overline{\vdash}_A$
 4404 2. CASE $(\text{dyn } b_0 \ v_0)^{\ell_0} \triangleright_A (\text{mon } b_0 \ v_0)^{\ell_0}$
 4405 2.1. QED

$$\frac{\text{by inversion } \overline{\vdash}_A}{\frac{;\ell_1 \overline{\vdash}_A v_0}{;\ell_0 \overline{\vdash}_A \text{mon } b_0 \ v_0}}$$

4413 3. CASE $(\text{dyn } b_0 \ v_0)^{\ell_0} \triangleright_A (i_1)^{\ell_0}$
 4414 3.1. QED
 4415 4. CASE $(\text{dyn } b_0 \ v_0)^{\ell_0} \triangleright_A (\text{BndryErr}(b_0, v_0))^{\ell_0}$
 4416 4.1. QED
 4417
 4418

□

4421 LEMMA 7.38 (A-stat PRESERVATION). *If $\cdot \vdash_A \text{stat } b_0 v_0 : \mathcal{U}$ and $\cdot; \ell_0 \overline{\Gamma}_A \text{stat } b_0 v_0$ and $(\text{stat } b_0 v_0)^{\ell_0} \blacktriangleright_A (e_1)^{\ell_0}$ then*
 4422 *$\cdot; \ell_0 \overline{\Gamma}_A e_1$.*
 4423

4424 PROOF. By case analysis of \blacktriangleright_A .

- 4425 1. $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
 4426 and $\cdot; \ell_1 \overline{\Gamma}_A v_0$
 4427 by inversion $\overline{\Gamma}_A$
 4428 2. CASE $(\text{stat } b_0 v_0)^{\ell_0} \blacktriangleright_A (\text{mon } b_0 v_0)^{\ell_0}$
 4429 2.1. QED
 4430
 4431

by inversion $\overline{\Gamma}_A$

$\cdot; \ell_1 \overline{\Gamma}_A v_0$

$\cdot; \ell_0 \overline{\Gamma}_A \text{mon } b_0 v_0$

- 4432
 4433
 4434
 4435
 4436
 4437 3. CASE $(\text{stat } b_0 ((\text{mon } b_1 ((\text{trace}_v^? \bar{b}_2 v_2))^{\bar{\ell}_4})^{\bar{\ell}_5})^{\ell_0}) \blacktriangleright_A (\text{trace } b_0 b_1 \bar{b}_2 ((v_2))^{\bar{\ell}_4 \bar{\ell}_5 \ell_0})^{\ell_0}$
 4438 3.1. $\bar{b}_2 \simeq \bar{\ell}_4$
 4439 by inversion $\overline{\Gamma}_A$
 4440 3.2. QED
 4441
 4442
 4443

by inversion $\overline{\Gamma}_A$

$\cdot; \text{last}(\bar{\ell}_4) \overline{\Gamma}_A v_2$

$\cdot; \ell_0 \overline{\Gamma}_A \text{trace } b_0 b_1 \bar{b}_2 ((v_2))^{\bar{\ell}_4 \bar{\ell}_5 \ell_0}$

$\cdot; \ell_0 \overline{\Gamma}_A (\text{trace } b_0 b_1 \bar{b}_2 ((v_2))^{\bar{\ell}_4 \bar{\ell}_5 \ell_0})^{\ell_0}$

- 4444
 4445
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 4448
 4449 4. CASE $(\text{stat } b_0 v_0)^{\ell_0} \blacktriangleright_A (i_1)^{\ell_0}$
 4450 qedstep
 4451
 4452

□

4453
 4454
 4455 LEMMA 7.39. *If $\cdot \vdash_A \text{trace } \bar{b}_0 v_0 : \mathcal{U}$ and $\cdot; \ell_0 \overline{\Gamma}_A \text{trace } \bar{b}_0 v_0$ then $\cdot; \ell_0 \overline{\Gamma}_A \text{add-trace}(\bar{b}_0, v_0)$.*
 4456

4457 PROOF. By case analysis of *add-trace*.

- 4458 1. CASE $\text{add-trace}(\cdot, v_0) = v_0$
 4459 1.1. QED
 4460 2. CASE $\text{add-trace}(\bar{b}_0, ((\text{trace}_v \bar{b}_1 v_1))^{\bar{\ell}_2}) = \text{trace}_v \bar{b}_0 \bar{b}_1 ((v_1))^{\bar{\ell}_2}$
 4461 2.1. QED
 4462 2.1.1. $\bar{b}_0 \simeq \bar{\ell}_2$
 4463 by inversion $\overline{\Gamma}_A$
 4464 2.1.2. QED
 4465 3. CASE $\text{add-trace}(\bar{b}_0, v_1) = \text{trace}_v \bar{b}_0 v_1$
 4466 and $v_0 \notin \text{trace}_v \bar{b} v$
 4467 3.1. $v_1 = ((v_2))^{\bar{\ell}_2}$
 4468 and $\bar{b}_0 \simeq \bar{\ell}_2$
 4469
 4470
 4471
 4472

4473 by inversion $\cdot; \ell_0 \Vdash_{\mathbb{A}} \text{trace } \bar{b}_0 v_1$

4474 3.2. QED

4475
4476 □

4477 LEMMA 7.40. *If $(x_0 : \tau_0), \Gamma_0 \vdash_{\mathbb{A}} e_1 : \tau?$ and $(x_0 : \ell_0), L_0; \ell_1 \Vdash_{\mathbb{A}} e_1$ and $\cdot \vdash_{\mathbb{A}} v_0 : \tau?'$ and $\cdot; \ell_0 \Vdash_{\mathbb{A}} v_0$ then $\Gamma_0 \vdash_{\mathbb{A}} e_1[x_0 \leftarrow$*
4478 *$v_0] : \tau?$ and $L_0 \Vdash_{\mathbb{A}} e_1[x_0 \leftarrow v_0]$.*

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4480 PROOF. By induction on the structure of e_0 .

4481 1. $e_0 = x_2$

4482 1.1. SCASE $x_0 = x_2$

4483 1.1.1. QED

4484 1.2. SCASE $x_0 \neq x_2$

4485 1.2.1. QED

4486
$$e_1[x_0 \leftarrow v_0] = e_1$$

4487
4488 2. CASE $e_0 \in i$

4489 2.1. QED

4490
$$e_1[x_0 \leftarrow v_0] = e_1$$

4491 3. CASE $e_0 = \lambda x_2. e_2$

4492
$$\text{or } e_0 = \lambda(x_2 : \tau_2). e_2$$

4493 3.1. SCASE $x_0 = x_2$

4494 3.1.1. QED

4495
$$e_1[x_0 \leftarrow v_0] = e_1$$

4496 3.2. SCASE $x_0 \neq x_2$

4497 3.2.1. QED

4498 by the induction hypothesis

4499 4. CASE $e_0 = \langle e_1, e_2 \rangle$

4500 4.1. QED

4501 by the induction hypothesis

4502 5. CASE $e_0 = \text{app}\{\tau?\} e_1 e_2$

4503 5.1. QED

4504 by the induction hypothesis

4505 6. CASE $e_0 = \text{unop}\{\tau?\} e_1$

4506 6.1. QED

4507 by the induction hypothesis

4508 7. CASE $e_0 = \text{binop}\{\tau?\} e_1 e_2$

4509 7.1. QED

4510 by the induction hypothesis

4511 8. CASE $e_0 = \text{dyn } b_1 e_1$

4512 8.1. QED

4513 by the induction hypothesis

4514 9. CASE $e_0 = \text{stat } b_1 e_1$

4515 9.1. QED

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4524 2019-10-03 17:26. Page 87 of 1-148.

4525 by the induction hypothesis

4526 10. CASE $e_0 = (e_1)^{\ell_1}$

4527 10.1. $\ell_0 = \ell_1$

4528 by inversion $\overline{\Gamma}_A$

4530 10.2. QED

4531 by the induction hypothesis

4532 11. CASE $e_0 = \text{trace } \bar{b}_1 ((e_1))^{\bar{\ell}_1}$

4533 11.1. $\cdot; \text{last}(\bar{\ell}_1) \overline{\Gamma}_A e_1$

4535 by inversion $\overline{\Gamma}_A$

4536 11.2. QED

4537 by the induction hypothesis

4538 12. CASE $e_0 = \text{trace}_v \bar{b}_1 ((e_1))^{\bar{\ell}_1}$

4539 12.1. QED

4541 by the induction hypothesis

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LEMMA 7.41 (BOUNDARY PRESERVATION). *If $e_0 : \tau? \overline{\mathbf{wf}}$ and $e_0 \rightarrow_A^* E_0[\text{dyn } b_1 v_1]$ then either $\text{has-boundary}(e_0, b_1)$ or $\text{has-boundary}(e_0, \text{flip}(b_1))$.*

PROOF. By case analysis of \triangleright_A and \blacktriangleright_A , evaluation does not create new labels and only creates a new boundary by flipping an existing boundary.

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LEMMA 7.42. *If $e_0 : \tau? \overline{\mathbf{wf}}$ and $e_0 \rightarrow_A^* E[\text{mon } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2) v_0]$ and $\text{tag-match}([\tau_1 \times \tau_2], v_0)$ then $\tau_0 \in (\tau \times \tau)^\ell$*

PROOF. Surface expressions do not contain monitors, and \blacktriangleright_A and \triangleright_A only create monitors with compatible types and values.

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8 N/A SIMULATION

$$v \lesssim v$$

$$\frac{}{i_0 \lesssim i_0} \quad \frac{}{i_0 \lesssim \text{trace}_v \bar{b}_0 i_0} \quad \frac{v_0 \lesssim v_2 \quad v_1 \lesssim v_3}{\langle v_0, v_1 \rangle \lesssim \langle v_2, v_3 \rangle} \quad \frac{v_0 \lesssim \text{mon}^? \text{fst}(b_0) v_2 \quad v_1 \lesssim \text{mon}^? \text{snd}(b_0) v_3}{\langle v_0, v_1 \rangle \lesssim \text{mon } b_0 \langle v_2, v_3 \rangle}$$

$$\frac{v_0 \lesssim \text{add-trace}(\text{fst}(\bar{b}_0), v_2) \quad v_1 \lesssim \text{add-trace}(\text{snd}(\bar{b}_0), v_3)}{\langle v_0, v_1 \rangle \lesssim \text{trace}_v \bar{b}_0 \langle v_2, v_3 \rangle} \quad \frac{v_0 \lesssim \text{add-trace}(\text{fst}(\bar{b}_0), (\text{mon}^? \text{fst}(b_1) v_2)) \quad v_1 \lesssim \text{add-trace}(\text{snd}(\bar{b}_0), (\text{mon}^? \text{snd}(b_1) v_3))}{\langle v_0, v_1 \rangle \lesssim \text{trace}_v \bar{b}_0 (\text{mon } b_1 \langle v_2, v_3 \rangle)}$$

$$\frac{v_0 \lesssim \text{mon}^? \text{fst}(b_0) (\text{add-trace}(\text{fst}(\bar{b}_1), v_2)) \quad v_1 \lesssim \text{mon}^? \text{snd}(b_1) (\text{add-trace}(\text{snd}(\bar{b}_1), v_3))}{\langle v_0, v_1 \rangle \lesssim \text{mon } b_0 (\text{trace}_v \bar{b}_1 \langle v_2, v_3 \rangle)}$$

$$\frac{v_0 \lesssim \text{mon}^? \text{fst}(b_0) (\text{mon}^? \text{fst}(b_1) v_2) \quad v_1 \lesssim \text{mon}^? \text{snd}(b_0) (\text{mon}^? \text{snd}(b_1) v_3)}{\langle v_0, v_1 \rangle \lesssim \text{mon } b_0 (\text{mon } b_1 \langle v_2, v_3 \rangle)}$$

$$\frac{v_0 \lesssim \text{mon}^? \text{fst}(b_0) (\text{add-trace}(\text{fst}(\bar{b}_1), (\text{mon } \text{fst}(b_2) v_2))) \quad v_1 \lesssim \text{mon}^? \text{snd}(b_0) (\text{add-trace}(\text{snd}(\bar{b}_1), (\text{mon } \text{snd}(b_2) v_3)))}{\langle v_0, v_1 \rangle \lesssim \text{mon } b_0 (\text{trace}_v \bar{b}_1 (\text{mon } b_2 \langle v_2, v_3 \rangle))}$$

$$\frac{v_0 \lesssim \text{mon}^{+?} \text{fst}(\bar{b}_0) (\text{mon}^? \text{fst}(b_1) (\text{mon}^? \text{fst}(b_2) v_2)) \quad v_1 \lesssim \text{mon}^{+?} \text{snd}(\bar{b}_0) (\text{mon}^? \text{snd}(b_1) (\text{mon}^? \text{snd}(b_2) v_3))}{\langle v_0, v_1 \rangle \lesssim \text{mon}^+ \bar{b}_0 (\text{mon } b_1 (\text{mon } b_2 \langle v_2, v_3 \rangle))}$$

$$\frac{v_0 \lesssim \text{mon}^{+?} \text{fst}(\bar{b}_0) (\text{mon}^? \text{fst}(b_1) (\text{add-trace}(\text{fst}(\bar{b}_2), (\text{mon}^? \text{fst}(b_3) v_2)))) \quad v_1 \lesssim \text{mon}^{+?} \text{snd}(\bar{b}_0) (\text{mon}^? \text{snd}(b_1) (\text{add-trace}(\text{snd}(\bar{b}_2), (\text{mon}^? \text{snd}(b_3) v_3))))}{\langle v_0, v_1 \rangle \lesssim \text{mon}^+ \bar{b}_0 (\text{mon } b_1 (\text{trace}_v \bar{b}_2 (\text{mon } b_3 \langle v_2, v_3 \rangle))} \quad \frac{e_0 \lesssim e_1}{\lambda x_0. e_0 \lesssim \lambda x_0. e_1}$$

$$\frac{e_0 \lesssim e_1}{\lambda(x_0 : \tau_0). e_0 \lesssim \lambda(x_0 : \tau_0). e_1} \quad \frac{v_1 \in \lambda x. e \cup \text{mon } b (\lambda(x : \tau). e) \quad b_0 \leq b_2 \quad b_1 \leq b_3 \quad v_0 \lesssim v_1}{\text{mon } b_0 (\text{mon } b_1 v_0) \lesssim \text{trace}_v b_2 b_3 \cdot v_1}$$

$$\frac{v_1 \in \lambda x. e \cup \text{mon } b (\lambda(x : \tau). e) \quad b_0 \leq b_2 \quad b_1 \leq b_3 \quad v_0 \lesssim \text{trace}_v \bar{b}_4 v_1}{\text{mon } b_0 (\text{mon } b_1 v_0) \lesssim \text{trace}_v b_2 b_3 \bar{b}_4 v_1} \quad \frac{b_0 \leq b_1 \quad v_0 \lesssim v_1}{\text{mon } b_0 v_0 \lesssim \text{mon } b_1 v_1}$$

4629	$e \lesssim e$			
4630				
4631		$e_0 \lesssim e_2 \quad e_1 \lesssim e_3$	$e_0 \lesssim e_2 \quad e_1 \lesssim e_3$	$e_0 \lesssim e_1$
4632	$x_0 \lesssim x_0$	$\langle e_0, e_1 \rangle \lesssim \langle e_2, e_3 \rangle$	$\text{app}\{\tau?_0\} e_0 e_1 \lesssim \text{app}\{\tau?_0\} e_2 e_3$	$\text{unop}\{\tau?\} e_0 \lesssim \text{unop}\{\tau?\} e_1$
4633				
4634		$e_0 \lesssim e_2 \quad e_1 \lesssim e_3$	$b_0 \leqslant b_1 \quad e_0 \lesssim e_1$	$b_0 \leqslant b_1 \quad e_0 \lesssim e_1$
4635		$\text{binop}\{\tau?\} e_0 e_1 \lesssim \text{binop}\{\tau?\} e_2 e_3$	$\text{dyn } b_0 e_0 \lesssim \text{dyn } b_1 e_1$	$\text{stat } b_0 e_0 \lesssim \text{stat } b_1 e_1$
4636				
4637				
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4639		$b_0 \leqslant b_2 \quad b_1 \leqslant b_3$		
4640		$e_0 \lesssim \text{trace } \bar{b}_4 e_1$		
4641	$\text{stat } b_0 (\text{dyn } b_1 e_0) \lesssim \text{trace } b_2 b_3 \bar{b}_4 e_1$	$\text{TagErr } \circ \lesssim \text{TagErr } \circ$	$\text{TagErr } \bullet \lesssim \text{TagErr } \bullet$	$\text{DivErr} \lesssim \text{DivErr}$
4642				
4643				
4644		$\text{BndryErr } (b_0, v_0) \lesssim e_1$		
4645				
4646	$E \lesssim E$			
4647				
4648		$E_0 \lesssim E_2 \quad e_1 \lesssim e_3$	$v_0 \lesssim v_2 \quad E_1 \lesssim E_3$	$E_0 \lesssim E_2 \quad e_1 \lesssim e_3$
4649	$[] \lesssim []$	$\langle E_0, e_1 \rangle \lesssim \langle E_2, e_3 \rangle$	$\langle v_0, E_1 \rangle \lesssim \langle v_2, E_3 \rangle$	$\text{app}\{\tau?_0\} E_0 e_1 \lesssim \text{app}\{\tau?_0\} E_2 e_3$
4650				
4651				
4652		$v_0 \lesssim v_2 \quad E_1 \lesssim E_3$	$E_0 \lesssim E_1$	$E_0 \lesssim E_2 \quad e_1 \lesssim e_3$
4653	$\text{app}\{\tau?_0\} v_0 E_1 \lesssim \text{app}\{\tau?_0\} v_2 E_3$		$\text{unop}\{\tau?\} E_0 \lesssim \text{unop}\{\tau?\} E_1$	$\text{binop}\{\tau?\} E_0 e_1 \lesssim \text{binop}\{\tau?\} E_2 e_3$
4654				
4655				
4656		$v_0 \lesssim v_2 \quad E_1 \lesssim E_3$	$b_0 \leqslant b_1 \quad E_0 \lesssim E_1$	$b_0 \leqslant b_1 \quad E_0 \lesssim E_1$
4657	$\text{binop}\{\tau?\} v_0 E_1 \lesssim \text{binop}\{\tau?\} v_2 E_3$		$\text{dyn } b_0 E_0 \lesssim \text{dyn } b_1 E_1$	$\text{stat } b_0 E_0 \lesssim \text{stat } b_1 E_1$
4658				
4659				
4660		$b_0 \leqslant b_2 \quad b_1 \leqslant b_3 \quad E_0 \lesssim \text{trace } \bar{b}_4 E_1$		
4661	$\text{stat } b_0 (\text{dyn } b_1 E_0) \lesssim \text{trace } b_2 b_3 \bar{b}_4 E_1$			
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4681 COROLLARY 8.1. *If $e_0 : \tau?$ wf and $e_0 \rightarrow_{\mathbb{A}}^* \text{BndryErr}(\bar{b}_2, v_2)$ then $e_0 \rightarrow_{\mathbb{N}}^* \text{BndryErr}(b_1, v_1)$*

4682
4683 PROOF. By lemma 8.6 and the fact that $e \lesssim \text{BndryErr}(\bar{b}_2, v_2)$ implies $e \in \text{BndryErr}(b, v)$

□

4684
4685
4686 *Example 8.2.* There exists $e_0 : \tau?$ wf such that $e_0 \rightarrow_{\mathbb{A}}^* \text{BndryErr}(\bar{b}_2, v_2)$ and $e_0 \rightarrow_{\mathbb{N}}^* \text{BndryErr}(b_1, v_1)$ and $b_1 \notin \bar{b}_2$

4687
4688 PROOF. Choose any e_0 where Natural detects an error that Amnesic misses, and then Amnesic detects an error at a
4689 boundary between two different components, e.g.:

$$4690 e_0 = \text{sum}(\text{fst}\{\text{Nat}\}(\text{dyn}(\ell_0 \blacktriangleleft (\text{Nat} \times \text{Nat})^{\ell_1} \blacktriangleleft \ell_1) \langle 0, -1 \rangle))$$

$$4691 (\text{dyn}(\ell_0 \blacktriangleleft (\text{Nat})^{\ell_1} \blacktriangleleft \ell_1) (\text{stat}(\ell_1 \blacktriangleleft (\text{Nat})^{\ell_2} \blacktriangleleft \ell_2) (\text{dyn}(\ell_2 \blacktriangleleft (\text{Nat})^{\ell_3} \blacktriangleleft \ell_3) -1)))$$

□

4692
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4695 THEOREM 8.3. *If $e_0 : \tau?$ wf and $e_0 \rightarrow_{\mathbb{N}}^* e_1 \rightarrow_{\mathbb{N}} \text{BndryErr}(b_1, v_1)$ then $\exists e_2$ such that $e_0 \rightarrow_{\mathbb{A}}^* e_2$ and one of the following*

4696 holds:

- 4697 • $e_1 \lesssim e_2$, or
- 4698 • $e_1 = E_1[\text{app}\{\tau_2\} v_0 (\text{dyn } b_1 v_1)]$ and $e_2 = E_2[\text{app}\{\tau_2\} (\text{trace}_v \bar{b}_2 v_2) v_3]$ and $E_1 \lesssim E_2$ and $\text{flip}(b_1) \in \bar{b}_2$

4699
4700 PROOF. by lemma 8.6 and lemma 8.26.

□

4701
4702
4703
4704 *Example 8.4.* There exists $e_0 : \tau?$ wf such that $e_0 \rightarrow_{\mathbb{N}}^* \text{BndryErr}(b_1, v_1)$ and $e_0 \rightarrow_{\mathbb{A}}^* v_1$

4705
4706 PROOF. $e_0 = \text{dyn}(\ell_0 \blacktriangleleft \text{Nat} \times \text{Nat} \blacktriangleleft \ell_1) \langle -1, -2 \rangle$

□

4707
4708
4709
4710 LEMMA 8.5 (\lesssim SURFACE REFLEXIVITY). *If $e_0 : \tau?$ wf then $e_0 \lesssim e_0$*

4711 PROOF. By induction on e_0 .

4712
4713 1. CASE $e_0 = x_0$

4714 1.1. QED

$$4715 x_0 \lesssim x_0$$

4716
4717 2. CASE $e_0 = i_0$

4718 2.1. QED

$$4719 i_0 \lesssim i_0$$

4720
4721 3. CASE $e_0 = \lambda x_1. e_1$

4722 3.1. $e_1 \lesssim e_1$

4723 by the induction hypothesis

4724 3.2. QED

$$4725 \lambda x_1. e_1 \lesssim \lambda x_1. e_1$$

4726
4727 4. CASE $e_0 = \langle e_1, e_2 \rangle$

4728 4.1. QED

4729 by the induction hypothesis

4730
4731 5. CASE $e_0 = \text{app}\{\tau?\} e_1 e_2$

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4733 5.1. QED
 4734 by the induction hypothesis

4735 6. CASE $e_0 = unop\{\tau?\} e_1$
 4736

4737 6.1. QED
 4738 by the induction hypothesis

4739 7. CASE $e_0 = binop\{\tau?\} e_1 e_2$
 4740

4741 7.1. QED
 4742 by the induction hypothesis

4743 8. CASE $e_0 = dyn b_0 e_1$

4744 8.1. $b_0 \leqslant b_0$
 4745 by reflexivity \leqslant :
 4746

4747 8.2. QED
 4748 by the induction hypothesis

4749 9. CASE $e_0 = stat b_0 e_1$
 4750

4751 9.1. QED
 4752 by the induction hypothesis

4753

□

4754

4755 LEMMA 8.6 (SIMULATION).

- 4756 • if $e_0 \lesssim e_2$ and $e_0 \rightarrow_N e_1$ then $\exists e_3, e_4$ such that $e_1 \rightarrow_N^* e_3$ and $e_2 \rightarrow_A^* e_4$ and $e_3 \lesssim e_4$
- 4757 • if $e_0 \lesssim e_2$ and $e_2 \rightarrow_A e_3$ then $\exists e_1, e_4$ such that $e_3 \rightarrow_A^* e_4$ and $e_0 \rightarrow_N^* e_1$ and $e_1 \lesssim e_4$

4758

4759 PROOF. By lemma 8.8 and lemma 8.9.
 4760

4761

□

4762

4763 *Definition 8.7 (reduced WF expressions).* Expressions e_1 and e_2 are *reduced WF expressions* if $e_0 : \tau? \text{ wf}$ and $e_0 \rightarrow_N^* e_1$
 4764 and $e_0 \rightarrow_A^* e_2$
 4765

4766 LEMMA 8.8 (H-STEP SIMULATION). If e_0 and e_2 are reduced WF expressions and $e_0 \lesssim e_2$ and $e_0 \rightarrow_N e_1$ then $\exists e_3, e_4$
 4767 such that $e_1 \rightarrow_N^* e_3$ and $e_2 \rightarrow_A^* e_4$ and $e_3 \lesssim e_4$
 4768

4769 PROOF. By case analysis of $e_0 \rightarrow_N e_1$
 4770

4771 1. CASE $e_0 = E_0[\text{BndryErr}(b_0, v_0)]$ and $e_0 \rightarrow_N \text{BndryErr}(b_0, v_0)$

4772 1.1. QED
 4773 trivial, $\text{BndryErr}(b_0, v_0) \lesssim e_2$

4774 2. CASE $e_0 = E_0[\text{Err}_0]$ and $\text{Err}_0 \notin \text{BndryErr}(\bar{b}, v)$ and $e_0 \rightarrow_N \text{Err}_0$

4775 2.1. $e_2 = E_2[\text{Err}_1]$ and $E_0 \lesssim E_2$ and $\text{Err}_0 \lesssim \text{Err}_1$
 4776

4777 by lemma 8.10

4778 2.2. $e_2 \rightarrow_A \text{Err}_1$
 4779 by definition \rightarrow_A
 4780

4781 2.3. QED
 4782 $\text{Err}_0 \lesssim \text{Err}_1$

4783 3. CASE $e_0 = E_0[\text{dyn } b_0 v_0]$ and $\text{dyn } b_0 v_0 \triangleright_N e_3$
 4784

- 4785 3.1. $e_2 = E_2[\text{dyn } b_2 v_2]$ and $E_0 \lesssim E_2$ and $\text{dyn } b_0 v_0 \lesssim \text{dyn } b_2 v_2$
4786 by lemma 8.10
- 4787 3.2. $\text{dyn } b_0 v_0 \rightarrow_N^* e_3$ and $\text{dyn } b_2 v_2 \rightarrow_A^* e_4$ and $e_3 \lesssim e_4$
4788 by lemma 8.12
- 4789 3.3. QED
4790 by lemma 8.31
- 4791 4. CASE $e_0 = E_0[\text{stat } b_0 v_0]$ and $\text{stat } b_0 v_0 \blacktriangleright_N e_3$
4792 4.1. $e_2 = E_2[\text{stat } b_2 v_2]$ and $E_0 \lesssim E_2$ and $\text{stat } b_0 v_0 \lesssim \text{stat } b_2 v_2$
4793 by lemma 8.10
- 4794 4.2. $\text{stat } b_0 v_0 \rightarrow_N^* e_3$ and $\text{stat } b_2 v_2 \rightarrow_A^* e_4$ and $e_3 \lesssim e_4$
4795 by lemma 8.14
- 4796 4.3. QED
4797 by lemma 8.31
- 4800 5. CASE $e_0 = E_0[\text{unop}\{\tau?\} v_0]$ and $\text{unop}\{\tau?\} v_0 \blacktriangleright_N \delta_N(\text{unop}, v_0)$
4801 5.1. $e_2 = E_2[\text{unop}\{\tau?\} v_2]$ and $E_0 \lesssim E_2$ and $\text{unop}\{\tau?\} v_0 \lesssim \text{unop}\{\tau?\} v_2$
4802 by lemma 8.10
- 4803 5.2. $\cdot \vdash_N \text{unop}\{\tau?\} v_0 : \tau_0$ and $\cdot \vdash_A \text{unop}\{\tau?\} v_2 : \tau_0$
4804 by type soundness
- 4805 5.3. $\text{unop}\{\tau?\} v_2 \rightarrow_A^* v_3$ and $\delta_N(\text{unop}, v_0) \lesssim v_3$
4806 by lemma 8.22
- 4807 5.4. QED
4808 by lemma 8.31
- 4809 6. CASE $e_0 = E_0[\text{unop}\{\tau?\} v_0]$ and $\text{unop}\{\tau?\} v_0 \blacktriangleright_N e_1$
4810 6.1. QED
4811 by lemma 8.23
- 4812 7. CASE $e_0 = E_0[\text{app}\{\tau_0\} (\text{mon } (\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2) \blacktriangleleft \ell_1) v_0) v_1]$
4813 and $\text{app}\{\tau_0\} (\text{mon } (\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2) \blacktriangleleft \ell_1) v_0) v_1 \blacktriangleright_N \text{dyn } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{app}\{\mathcal{U}\} v_0 (\text{stat } (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1))$
4814 7.1. $e_2 = E_2[\text{app}\{\tau_0\} v_3 v_4]$
4815 by lemma 8.10
- 4816 7.2. SCASE $v_3 = \text{mon } (\ell_0 \blacktriangleleft (\tau_3 \Rightarrow \tau_4) \blacktriangleleft \ell_1) v_5$
4817 7.2.1. $\text{app}\{\tau_0\} (\text{mon } (\ell_0 \blacktriangleleft (\tau_3 \Rightarrow \tau_4) \blacktriangleleft \ell_1) v_5) v_4 \blacktriangleright_A \text{dyn } (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{app}\{\mathcal{U}\} v_5 (\text{stat } (\ell_1 \blacktriangleleft \tau_3 \blacktriangleleft \ell_0) v_4))$
4818 by definition \blacktriangleright_A
- 4819 7.2.2. QED
4820 by lemma 8.31
- 4821 7.3. SCASE $v_3 = \text{trace}_v \bar{b}_0 v_5$
4822 7.3.1. CONTRADICTION:
4823 $\cdot \vdash_A \text{app}\{\tau_0\} v_3 v_4 : \tau_0$
- 4824 8. CASE $e_0 = E_0[\text{app}\{\mathcal{U}\} (\text{mon } (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1) v_0) v_1]$
4825 and $\text{app}\{\mathcal{U}\} (\text{mon } (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1) v_0) v_1 \blacktriangleright_N \text{stat } (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{app}\{\tau_1\} v_0 (\text{stat } (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0) v_1))$
4826 8.1. $e_2 = E_2[\text{app}\{\mathcal{U}\} v_3 v_4]$
4827 by lemma 8.10
- 4828 2019-10-03 17:26. Page 93 of 1-148.

4837 8.2. SCASE $v_3 = \text{mon}(\ell_0 \blacktriangleleft (\tau_3 \Rightarrow \tau_4) \blacktriangleleft \ell_1) v_3$ and $v_0 \lesssim v_3$

4838 8.2.1. QED

4839 by definition $\blacktriangleright_{\mathbb{A}}$ and \lesssim

4840 8.3. SCASE $v_3 = \text{trace}_v \bar{b}_2 v_5$

4842 8.3.1. QED

4843 by lemma 8.26

4844 9. CASE $e_0 = E_0[\text{app}\{\tau_0\}(\lambda(x_0 : \tau_1). e_0) v_1]$ and $e_0 \triangleright_{\mathbb{N}} E_0[e_0[x_0 \leftarrow v_1]]$

4845 9.1. $e_2 = E_2[\text{app}\{\tau_0\} v_2 v_3]$

4846 by lemma 8.10

4848 9.2. $v_2 = \lambda(x_0 : \tau_1). e_4$ and $e_0 \lesssim e_4$

4849 by inversion \lesssim

4850 9.3. QED

4852 by lemma 8.30

4853 10. CASE $e_0 = E_0[\text{app}\{\mathcal{U}\}(\lambda x_0. e_0) v_1]$ and $e_0 \blacktriangleright_{\mathbb{N}} E_0[e_0[x_0 \leftarrow v_1]]$

4854 10.1. $e_2 = E_2[\text{app}\{\tau_0\} v_2 v_3]$

4855 by lemma 8.10

4857 10.2. $v_2 = \lambda x_0. e_4$ and $e_0 \lesssim e_4$

4858 by inversion \lesssim

4859 10.3. QED

4860 by lemma 8.30

4862 11. CASE $e_0 = E_0[\text{binop}\{\tau?\} v_0 v_1]$ and $e_0 \rightarrow_{\mathbb{N}} E_0[e_1]$

4863 11.1. $e_2 = E_2[\text{binop}\{\tau?\} v_2 v_3]$

4864 by lemma 8.10

4866 11.2. QED

4867 by lemma 8.29

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LEMMA 8.9 (A-STEP SIMULATION). *If e_0 and e_2 are reduced WF expressions and $e_0 \lesssim e_2$ and $e_2 \rightarrow_{\mathbb{A}} e_3$ then $\exists e_1, e_4$ such that $e_3 \rightarrow_{\mathbb{A}}^* e_4$ and $e_0 \rightarrow_{\mathbb{N}}^* e_1$ and $e_1 \lesssim e_4$*

PROOF. By case analysis of $e_2 \rightarrow_{\mathbb{A}} e_3$

1. CASE e_0 contains a boundary error; $e_0 = E_0[\text{BdryErr}(b_0, v_0)]$

1.1. $e_0 \rightarrow_{\mathbb{N}} \text{BdryErr}(b_0, v_0)$

by definition $\rightarrow_{\mathbb{N}}$

1.2. QED

$\text{BdryErr}(b_0, v_0) \lesssim e_3$

2. CASE $e_2 = E_2[\text{Err}_2]$ and $e_2 \rightarrow_{\mathbb{A}} \text{Err}_2$

2.1. $e_0 = E_0[\text{Err}_0]$ and $E_0 \lesssim E_2$ and $\text{Err}_0 \lesssim \text{Err}_2$

by lemma 8.11

2.2. QED

$e_0 \rightarrow_{\mathbb{N}} \text{Err}_0 \lesssim \text{Err}_2$

3. CASE $e_2 = E_2[\text{dyn } b_2 v_2]$ and $\text{dyn } b_2 v_2 \triangleright_{\mathbb{A}} e_3$

- 4889 3.1. $e_0 = E_0[\text{dyn } b_0 v_0]$ and $\text{dyn } b_0 v_0 \lesssim \text{dyn } b_2 v_2$
4890 by lemma 8.11
4891 3.2. $\text{dyn } b_0 v_0 \rightarrow_N^* e_1$ and $\text{dyn } b_2 v_2 \rightarrow_A^* e_4$ and $e_1 \lesssim e_4$
4892 by lemma 8.12
4893
4894 3.3. QED
4895 by lemma 8.31
4896
4897 4. CASE $e_2 = E_2[\text{stat } b_2 v_2]$ and $\text{stat } b_2 v_2 \triangleright_A e_3$
4898 4.1. $e_0 = E_0[\text{stat } b_0 v_0]$ and $\text{stat } b_0 v_0 \lesssim \text{stat } b_2 v_2$
4899 by lemma 8.11
4900 4.2. $\text{stat } b_0 v_0 \rightarrow_N^* e_1$ and $\text{stat } b_2 v_2 \rightarrow_A^* e_4$ and $e_1 \lesssim e_4$
4901 by lemma 8.14
4902
4903 4.3. QED
4904 by lemma 8.31
4905
4906 5. CASE $e_2 = E_2[\text{unop}\{\tau?\} v_2]$ and $\text{unop}\{\tau?\} v_2 \triangleright_A e_3$
4907 5.1. $e_0 = E_0[\text{unop}\{\tau?\} v_0]$ and $E_0 \lesssim E_2$ and $v_0 \lesssim v_2$
4908 by lemma 8.11
4909 5.2. $\cdot \vdash_N \text{unop}\{\tau?\} v_0 : \tau_0$ and $\cdot \vdash_A \text{unop}\{\tau?\} v_2 : \tau_0$
4910 by type soundness
4911 5.3. $\text{unop}\{\tau?\} v_2 \rightarrow_A^* v_4$ and $\delta_N(\text{unop}, v_0) \lesssim v_4$
4912 by lemma 8.22
4913
4914 5.4. QED
4915 by lemma 8.31
4916
4917 6. CASE $e_2 = E_2[\text{unop}\{\tau?\} v_2]$ and $\text{unop}\{\tau?\} v_2 \triangleright_A e_3$
4918 6.1. QED
4919 by lemma 8.23
4920
4921 7. CASE $e_2 = E_2[\text{app}\{\tau_2\} (\text{mon } (\ell_0 \blacktriangleleft (\tau_3 \Rightarrow \tau_4) \blacktriangleleft \ell_1) v_2) v_3]$
4922 and $e_2 \rightarrow_A E_2[\text{dyn } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{app}\{\mathcal{U}\} v_2 (\text{stat } (\ell_1 \blacktriangleleft \tau_3 \blacktriangleleft \ell_0) v_3))]$
4923 7.1. $\tau_4 \leq \tau_2$
4924 by inversion \vdash_A
4925 7.2. $e_0 = E_0[\text{app}\{\tau_2\} v_0 v_1]$
4926 by lemma 8.11
4927 7.3. $v_0 = \text{mon } (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1) v_4$
4928 by inversion \lesssim
4929
4930 7.4. QED
4931 $\text{app}\{\tau_2\} v_0 v_1 \triangleright_N \text{dyn } (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{app}\{\mathcal{U}\} v_4 (\text{stat } (\ell_1 \blacktriangleleft \tau_3 \blacktriangleleft \ell_0) v_1))$
4932
4933 8. CASE $e_2 = E_2[\text{app}\{\mathcal{U}\} (\text{trace}_v b_0 \bar{b}_1 v_2) v_3]$
4934 8.1. $e_0 = E_0[\text{app}\{\mathcal{U}\} v_0 v_1]$
4935 by lemma 8.11
4936
4937 8.2. QED
4938 by lemma 8.26
4939
4940 9. CASE $e_2 = E_2[\text{app}\{\tau_0\} (\lambda(x_1 : \tau_1). e_2) v_3]$

- 4941 9.1. $e_0 = E_0[\text{app}\{\tau_0\} v_0 v_1]$
 4942 by lemma 8.11
 4943 9.2. $v_0 = \lambda x_0. e_1$ and $e_1 \lesssim e_2$
 4944 by inversion \lesssim
 4945 9.3. QED
 4946 by lemma 8.30
 4947 10. CASE $e_2 = E_2[\text{binop}\{\tau?\} v_2 v_3]$ and $e_2 \rightarrow_{\Lambda} E_2[e_4]$
 4948 10.1. $e_0 = E_0[\text{binop}\{\tau?\} v_0 v_1]$
 4949 by lemma 8.11
 4950 10.2. QED
 4951 by lemma 8.29
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 4953
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□

4956 LEMMA 8.10 (H CONTEXT-MATCHING). *If $E_0[e_0] \lesssim e_1$ and $e_0 \notin v$ then $\exists E_2, e_2$ such that $e_1 = E_2[e_2]$ and $E_0 \lesssim E_2$ and*
 4957 $e_0 \lesssim e_2$
 4958
 4959

4960 PROOF. By induction on the structure of E_0

- 4961 1. CASE $E_0 = []$
 4962 trivial, $E_2 = []$ and $e_2 = e_1$
 4963 2. CASE $E_0 = \langle E_3, e_4 \rangle$
 4964 2.1. $E[e_0] = \langle E_3[e_0], e_4 \rangle \lesssim e_1$
 4965 by assumption
 4966 2.2. $e_1 = \langle e_5, e_6 \rangle$ and $E_3[e_0] \lesssim e_5$ and $e_4 \lesssim e_6$
 4967 by inversion \lesssim
 4968 2.3. $e_4 = E_7[e_8]$ and $E_3 \lesssim E_7$ and $e_0 \lesssim e_8$
 4969 by the induction hypothesis
 4970 2.4. QED
 4971 $E_1 = \langle E_7, e_6 \rangle$ and $e_2 = e_8$
 4972 3. CASE $E_0 = \langle v_3, E_4 \rangle$
 4973 3.1. $e_1 = \langle e_5, e_6 \rangle$ and $v_3 \lesssim e_5$ and $E_4[e_0] \lesssim e_6$
 4974 by inversion \lesssim
 4975 3.2. QED
 4976 by the induction hypothesis
 4977 4. CASE $E_0 = \text{app}\{\tau?\} E_3 e_4$
 4978 4.1. $e_1 = \text{app}\{\tau?\} e_5 e_6$ and $E_3[e_0] \lesssim e_5$ and $e_4 \lesssim e_6$
 4979 by inversion \lesssim
 4980 4.2. QED
 4981 by the induction hypothesis
 4982 5. CASE $E_0 = \text{app}\{\tau?\} v_3 E_4$
 4983 5.1. $e_1 = \text{app}\{\tau?\} e_5 e_6$ and $v_3 \lesssim e_5$ and $E_4[e_0] \lesssim e_6$
 4984 by inversion \lesssim
 4985 5.2. QED
 4986
 4987
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 4992

- 4993 by the induction hypothesis
 4994 6. CASE $E_0 = \text{unop}\{\tau?\} E_3$
 4995 6.1. $e_1 = \text{unop}\{\tau?\} e_4$ and $E_3[e_0] \lesssim e_4$
 4996 by inversion \lesssim
 4998 6.2. QED
 4999 by the induction hypothesis
 5000 7. CASE $E_0 = \text{binop}\{\tau?\} E_3 e_4$
 5001 7.1. $e_1 = \text{binop}\{\tau?\} e_5 e_6$ and $E_3[e_0] \lesssim e_5$ and $e_4 \lesssim e_6$
 5002 by inversion \lesssim
 5004 7.2. QED
 5005 by the induction hypothesis
 5006 8. CASE $E_0 = \text{binop}\{\tau?\} v_3 E_4$
 5007 8.1. $e_1 = \text{binop}\{\tau?\} v_5 e_6$ and $v_3 \lesssim v_5$ and $E_4[e_0] \lesssim e_6$
 5008 by inversion \lesssim
 5009 8.2. QED
 5010 by the induction hypothesis
 5011 9. CASE $E_0 = \text{dyn } b_0 E_3$
 5012 9.1. $e_1 = \text{dyn } b_1 e_4$ and $E_3[e_0] \lesssim e_4$
 5013 by inversion \lesssim
 5014 9.2. QED
 5015 by the induction hypothesis
 5016 10. CASE $E_0 = \text{stat } b_0 E_3$
 5017 10.1. $e_1 = \text{stat } b_1 e_4$ and $E_3[e_0] \lesssim e_4$
 5018 by inversion \lesssim
 5019 10.2. QED
 5020 by the induction hypothesis

□

5027 LEMMA 8.11 (A CONTEXT-MATCHING). *If $e_0 \lesssim E_2[e_2]$ and $e_2 \notin v$ and e_0 does not contain a subterm $\text{BndryErr}(b, v)$*
 5028 *then $\exists E_1, e_1$ such that $e_0 = E_1[e_1]$ and $E_1 \lesssim E_2$ and $e_1 \lesssim e_2$*
 5029

5030 PROOF. By induction on the structure of E_2

- 5031 1. CASE $E_2 = []$
 5032 trivial, $E_1 = []$ and $e_1 = e_0$
 5033 2. CASE $E_2 = \langle E_3, e_3 \rangle$
 5034 2.1. $e_0 = \langle e_4, e_5 \rangle \lesssim \langle E_3[e_2], e_3 \rangle$
 5035 by inversion \lesssim
 5036 2.2. QED
 5037 by the induction hypothesis
 5038 3. CASE $E_2 = \langle v_3, E_3 \rangle$
 5039 3.1. $e_0 = \langle e_4, e_5 \rangle \lesssim \langle v_3, E_3[e_2] \rangle$
 5040 by inversion \lesssim
 5041

5045 3.2. QED
5046 by the induction hypothesis
5047 4. CASE $E_2 = \text{app}\{\tau?_0\} E_3 e_3$
5048 4.1. $e_0 = \text{app}\{\tau?_0\} e_4 e_5 \lesssim \text{app}\{\tau?_0\} E_3[e_2] e_3$
5049 by inversion \lesssim
5050 4.2. QED
5051 by the induction hypothesis
5052 5. CASE $E_2 = \text{app}\{\tau?_0\} v_3 E_3$
5053 5.1. $e_0 = \text{app}\{\tau?_0\} e_4 e_5 \lesssim \text{app}\{\tau?_0\} v_3 E_3[e_2]$
5054 by inversion \lesssim
5055 5.2. QED
5056 by the induction hypothesis
5057 6. CASE $E_2 = \text{unop}\{\tau?\} E_3$
5058 6.1. $e_0 = \text{unop}\{\tau?\} e_4 \lesssim \text{unop}\{\tau?\} E_3[e_2]$
5059 by inversion \lesssim
5060 6.2. QED
5061 by the induction hypothesis
5062 7. CASE $E_2 = \text{binop}\{\tau?\} E_3 e_3$
5063 7.1. $e_0 = \text{binop}\{\tau?\} e_4 e_5 \lesssim \text{binop}\{\tau?\} E_3[e_2] e_3$
5064 by inversion \lesssim
5065 7.2. QED
5066 by the induction hypothesis
5067 8. CASE $E_2 = \text{binop}\{\tau?\} v_3 E_3$
5068 8.1. $e_0 = \text{binop}\{\tau?\} e_4 e_5 \lesssim \text{binop}\{\tau?\} v_3 E_3[e_2]$
5069 by inversion \lesssim
5070 8.2. QED
5071 by the induction hypothesis
5072 9. CASE $E_2 = \text{dyn } b_0 E_3$
5073 9.1. $e_0 = \text{dyn } b_0 e_4 \lesssim \text{dyn } b_2 E_3[e_2]$
5074 by inversion \lesssim
5075 9.2. QED
5076 by the induction hypothesis
5077 10. CASE $E_2 = \text{stat } b_0 E_3$
5078 10.1. $e_0 = \text{stat } b_0 e_4 \lesssim \text{stat } b_1 E_3[e_2]$
5079 by inversion \lesssim
5080 10.2. QED
5081 by the induction hypothesis
5082 11. CASE $E_2 = \text{trace } b_0 b_1 \bar{b}_2 E_3$
5083 11.1. $e_0 = \text{stat } b_3 \text{ dyn } b_4 e_4$ and $e_4 \lesssim E_3[e_0]$
5084 by inversion \lesssim
5085 11.2. QED
5086
5087
5088
5089
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5092
5093
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by the induction hypothesis

□

LEMMA 8.12. *If $(\text{dyn } b_0 \ v_0)$ and $(\text{dyn } b_1 \ v_1)$ are reduced WF expressions and $\text{dyn } b_0 \ v_0 \lesssim \text{dyn } b_1 \ v_1$ then $\text{dyn } b_0 \ v_0 \rightarrow_{\mathbb{N}}^* e_2$ and $\text{dyn } b_1 \ v_1 \rightarrow_{\mathbb{A}}^* e_3$ then $e_2 \lesssim e_3$*

PROOF. By case analysis of $v_0 \lesssim v_1$.

1. CASE $v_0 = i_0 \lesssim i_0 = v_1$

1.1. SCASE $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$

1.1.1. $\text{dyn } b_0 \ v_0 \triangleright_{\mathbb{N}} v_0$

by definition $\triangleright_{\mathbb{N}}$

1.1.2. $\text{tag-match}(\lfloor \tau_1 \rfloor, v_1)$

by lemma 8.17

1.1.3. QED

$\text{dyn } b_1 \ v_1 \triangleright_{\mathbb{A}} v_1$

1.2. SCASE $\neg \text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$

1.2.1. QED

$\text{dyn } b_0 \ v_0 \triangleright_{\mathbb{N}} \text{BndryErr}(b_0, v_0) \lesssim \text{dyn } b_1 \ v_1$

2. CASE $v_0 = i_0 \lesssim \text{trace}_v \bar{b}_2 \ i_0 = v_1$

2.1. SCASE $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$

2.1.1. $\text{dyn } b_0 \ v_0 \triangleright_{\mathbb{N}} v_0$

by definition $\triangleright_{\mathbb{N}}$

2.1.2. $\text{tag-match}(\lfloor \tau_1 \rfloor, v_1)$

by lemma 8.17

2.1.3. QED

$\text{dyn } b_1 \ v_1 \triangleright_{\mathbb{A}} v_1$

2.2. SCASE $\neg \text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$

2.2.1. QED

$\text{dyn } b_0 \ v_0 \triangleright_{\mathbb{N}} \text{BndryErr}(b_0, v_0) \lesssim \text{dyn } b_1 \ v_1$

3. CASE $v_0 = \langle v_2, v_3 \rangle \lesssim \langle v_4, v_5 \rangle = v_1$

3.1. SCASE $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$

3.1.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)$

3.1.2. $v_2 \lesssim v_4$ and $v_3 \lesssim v_5$

by inversion \lesssim

3.1.3. $\tau_0 = \tau_2 \times \tau_3 \leq \tau_4 \times \tau_5 = \tau_1$

by 3.1 and inversion \leq :

3.1.4. $\text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \ v_0 \triangleright_{\mathbb{N}} \langle \text{dyn } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) \ v_2, \text{dyn } (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) \ v_3 \rangle$

by definition $\triangleright_{\mathbb{N}}$

3.1.5. $\text{tag-match}(\lfloor \tau_1 \rfloor, v_1)$

by lemma 8.17

3.1.6. $\text{dyn } b_1 \ v_1 \triangleright_{\mathbb{A}} \text{mon } b_1 \ v_1$

by definition $\triangleright_{\mathbb{A}}$

5149 3.1.7. $\text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_N^* e_6 \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) v_4$
5150 and $\text{dyn}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_N^* e_7 \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) v_5$
5151 by lemma 8.13
5152
5153 3.1.8. QED
5154 either e_6 or $e_7 \in \text{BndryErr}(b, v)$ or:
5155
$$\frac{e_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) v_4 \quad e_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) v_5}{\langle e_6, e_7 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) \langle v_4, v_5 \rangle}$$

5156
5157
5158 3.2. SCASE $\neg \text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$
5159 3.2.1. QED
5160 $\text{dyn } b_0 v_0 \triangleright_N \text{BndryErr}(b_0, v_0) \lesssim \text{dyn } b_1 v_1$
5161 4. CASE $v_0 = \langle v_2, v_3 \rangle \lesssim \text{mon}(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) \langle v_4, v_5 \rangle = v_1$
5162 4.1. SCASE $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$
5163 4.1.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
5164 and $b_1 = (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)$
5165
5166 4.1.2. $v_2 \lesssim \text{mon}^?(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) v_4$ and $v_3 \lesssim \text{mon}^?(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) v_5$
5167 by inversion \lesssim
5168
5169 4.1.3. $\tau_0 = \tau_2 \times \tau_3 \leq \tau_4 \times \tau_5 = \tau_1$
5170 by 4.1 and inversion \leq :
5171
5172 4.1.4. $\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \triangleright_N \langle \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{dyn}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$
5173 by definition \triangleright_N
5174
5175 4.1.5. $\text{tag-match}(\lfloor \tau_1 \rfloor, v_1)$
5176 by lemma 8.17
5177
5178 4.1.6. $\text{dyn } b_1 v_1 \triangleright_A \text{mon } b_1 v_1$
5179 by definition \triangleright_A
5180 4.1.7. $\text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_N^* e_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) v_4)$
5181 and $\text{dyn}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_N^* e_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) v_5)$
5182 by lemma 8.13
5183
5184 4.1.8. QED
5185 either e_6 or $e_7 \in \text{BndryErr}(b, v)$ or:
5186
$$\frac{e_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) v_4) \quad e_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) v_5)}{\langle e_6, e_7 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{mon}(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) \langle v_4, v_5 \rangle)}$$

5187
5188 4.2. SCASE $\neg \text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$
5189 4.2.1. QED
5190 $\text{dyn } b_0 v_0 \triangleright_N \text{BndryErr}(b_0, v_0) \lesssim \text{dyn } b_1 v_1$
5191 5. CASE $v_0 = \langle v_2, v_3 \rangle \lesssim \text{trace}_\sqrt{\bar{b}_2} \langle v_4, v_5 \rangle = v_1$
5192 5.1. SCASE $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$
5193 5.1.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
5194 and $b_1 = (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)$
5195
5196 5.1.2. $v_2 \lesssim \text{add-trace}(\bar{b}_2, v_4)$ and $v_3 \lesssim \text{add-trace}(\bar{b}_2, v_5)$
5197 by inversion \lesssim
5198
5199
5200

5201 5.1.3. $\tau_0 = \tau_2 \times \tau_3 \leq: \tau_4 \times \tau_5 = \tau_1$
5202 by 5.1 and inversion \leq :
5203 5.1.4. $\text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \triangleright_N \langle \text{dyn } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{dyn } (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$
5204 by definition \triangleright_N
5205 5.1.5. $\text{tag-match}(\lfloor \tau_1 \rfloor, v_1)$
5206 by lemma 8.17
5207 5.1.6. $\text{dyn } b_1 v_1 \triangleright_A \text{mon } b_1 v_1$
5208 by definition \triangleright_A
5209 5.1.7. $\text{dyn } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_N^* e_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, v_4))$
5210 and $\text{dyn } (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_N^* e_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, v_5))$
5211 by lemma 8.13
5212 5.1.8. QED
5213 either e_6 or $e_7 \in \text{BdryErr}(b, v)$ or:
5214
$$\frac{e_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, v_4)) \quad e_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, v_5))}{\langle e_6, e_7 \rangle \lesssim \text{mon} (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{trace}_v \bar{b}_2 \langle v_4, v_5 \rangle)}$$

5215 5.2. SCASE $\neg \text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$
5216 5.2.1. QED
5217 $\text{dyn } b_0 v_0 \triangleright_N \text{BdryErr}(b_0, v_0) \lesssim \text{dyn } b_1 v_1$
5218 6. CASE $v_0 = \langle v_2, v_3 \rangle \lesssim \text{trace}_v \bar{b}_2 (\text{mon} (\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) \langle v_4, v_5 \rangle) = v_1$
5219 6.1. SCASE $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$
5220 6.1.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
5221 and $b_1 = (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)$
5222 6.1.2. $v_2 \lesssim \text{add-trace}(\bar{b}_2, (\text{mon}^? (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) v_4))$ and $v_3 \lesssim \text{add-trace}(\bar{b}_2, (\text{mon}^? (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) v_5))$
5223 by inversion \lesssim
5224 6.1.3. $\tau_0 = \tau_2 \times \tau_3 \leq: \tau_4 \times \tau_5 = \tau_1$
5225 by 6.1 and inversion \leq :
5226 6.1.4. $\text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \triangleright_N \langle \text{dyn } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{dyn } (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$
5227 by definition \triangleright_N
5228 6.1.5. $\text{tag-match}(\lfloor \tau_1 \rfloor, v_1)$
5229 by lemma 8.17
5230 6.1.6. $\text{dyn } b_1 v_1 \triangleright_A \text{mon } b_1 v_1$
5231 by definition \triangleright_A
5232 6.1.7. $\text{dyn } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_N^* e_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, (\text{mon}^? (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) v_4)))$
5233 and $\text{dyn } (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_N^* e_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, (\text{mon}^? (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) v_5)))$
5234 by lemma 8.13
5235 6.1.8. QED
5236 either e_6 or $e_7 \in \text{BdryErr}(b, v)$ or:
5237
$$\frac{e_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, (\text{mon}^? (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) v_4))) \quad e_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, (\text{mon}^? (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) v_5)))}{\langle e_6, e_7 \rangle \lesssim \text{mon} (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{trace}_v \bar{b}_2 (\text{mon} (\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) \langle v_4, v_5 \rangle))}$$

5238 5239 5240 5241 5242 5243 5244 5245 5246 5247 5248 5249 5250 5251 5252

5253 6.2. SCASE $\neg\text{tag-match}(\lfloor\tau_0\rfloor, v_0)$
5254 6.2.1. QED
5255 $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \text{BdryErr}(b_0, v_0) \lesssim \text{dyn } b_1 v_1$
5256
5257 7. CASE $v_0 = \langle v_2, v_3 \rangle \lesssim \text{mon } b_2 (\text{trace}_{\sqrt{}} \bar{b}_3 \langle v_4, v_5 \rangle) = v_1$
5258 7.1. CONTRADICTION:
5259 $\cdot \vdash_{\mathbb{A}} v_1 : \mathcal{U}$
5260
5261 8. CASE $v_0 = \langle v_2, v_3 \rangle \lesssim \text{mon } b_2 (\text{mon } b_3 \langle v_4, v_5 \rangle) = v_1$
5262 8.1. CONTRADICTION:
5263 by lemma 7.10
5264
5265 9. CASE $v_0 = \langle v_2, v_3 \rangle \lesssim \text{mon } b_2 (\text{trace}_{\sqrt{}} \bar{b}_3 (\text{mon } b_4 \langle v_4, v_5 \rangle)) = v_1$
5266 9.1. CONTRADICTION:
5267 by lemma 7.10
5268
5269 10. CASE $v_0 = \langle v_2, v_3 \rangle \lesssim \text{mon}^+ \bar{b}_2 (\text{mon } b_3 (\text{mon } b_4 \langle v_4, v_5 \rangle)) = v_1$
5270 10.1. CONTRADICTION:
5271 by lemma 7.10
5272
5273 11. CASE $v_0 = \langle v_2, v_3 \rangle \lesssim \text{mon}^+ \bar{b}_2 (\text{mon } b_3 (\text{trace}_{\sqrt{}} \bar{b}_4 (\text{mon } b_5 \langle v_4, v_5 \rangle))) = v_1$
5274 11.1. CONTRADICTION:
5275 by lemma 7.10
5276
5277 12. CASE $v_0 = \lambda x_2. e_2 \lesssim \lambda x_2. e_3 = v_1$
5278 12.1. SCASE $\text{tag-match}(\lfloor\tau_0\rfloor, v_0)$
5279 12.1.1. QED
5280 $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \text{mon } b_0 v_0$ and $\text{dyn } b_1 v_1 \triangleright_{\mathbb{A}} \text{mon } b_1 v_1$
5281 12.2. SCASE $\neg\text{tag-match}(\lfloor\tau_0\rfloor, v_0)$
5282 12.2.1. QED
5283 $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \text{BdryErr}(b_0, v_0) \lesssim \text{dyn } b_1 v_1$
5284
5285 13. CASE $v_0 = \lambda(x_2 : \tau_2). e_2 \lesssim \lambda(x_2 : \tau_2). e_3 = v_1$
5286 13.1. CONTRADICTION:
5287 $\cdot \vdash_{\mathbb{N}} v_0 : \mathcal{U}$
5288
5289 14. CASE $v_0 = \text{mon } b_2 (\text{mon } b_3 v_2) \lesssim \text{trace}_{\sqrt{}} b_4 b_5 (\lambda x_0. e_3) = v_1$
5290 14.1. SCASE $\text{tag-match}(\lfloor\tau_0\rfloor, v_0)$
5291 14.1.1. $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \text{mon } b_0 v_0$
5292 by definition $\triangleright_{\mathbb{N}}$
5293 14.1.2. $\text{tag-match}(\lfloor\tau_1\rfloor, v_1)$
5294 by lemma 8.17
5295 14.1.3. $\text{dyn } b_1 v_1 \triangleright_{\mathbb{A}} \text{mon } b_1 v_1$
5296 by definition $\triangleright_{\mathbb{A}}$
5297 14.1.4. QED
5298
5299
5300
$$\frac{\text{mon } b_2 (\text{mon } b_3 v_2) \lesssim \text{trace}_{\sqrt{}} b_4 b_5 (\lambda x_0. e_3)}{\text{mon } b_0 (\text{mon } b_2 (\text{mon } b_3 v_2)) \lesssim \text{mon } b_1 (\text{trace}_{\sqrt{}} b_4 b_5 (\lambda x_0. e_3))}$$

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5305 14.2. SCASE $\neg\text{tag-match}(\lfloor\tau_0\rfloor, v_0)$

5306 14.2.1. QED

5307 $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \text{BdryErr}(b_0, v_0) \lesssim \text{dyn } b_1 v_1$

5308 15. CASE $v_0 = \text{mon } b_2 (\text{mon } b_3 v_2) \lesssim (\text{trace}_v b_4 b_5 \bar{b}_6 (\lambda x_3. e_3)) = v_1$

5309 15.1. $\text{tag-match}(\lfloor\tau_0\rfloor, v_0)$

5310 15.1.1. $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \text{mon } b_0 v_0$

5311 by definition $\triangleright_{\mathbb{N}}$

5312 15.1.2. $\text{tag-match}(\lfloor\tau_1\rfloor, v_1)$

5313 by lemma 8.17

5314 15.1.3. $\text{dyn } b_1 v_1 \triangleright_{\Lambda} \text{mon } b_1 v_1$

5315 by definition \triangleright_{Λ}

5316 15.1.4. QED

$$\frac{\text{mon } b_2 (\text{mon } b_3 v_2) \lesssim (\text{trace}_v b_4 b_5 \bar{b}_6 (\lambda x_3. e_3))}{\text{mon } b_0 (\text{mon } b_2 (\text{mon } b_3 v_2)) \lesssim \text{mon } b_1 (\text{trace}_v b_4 b_5 \bar{b}_6 (\lambda x_3. e_3))}$$

5317 15.2. $\neg\text{tag-match}(\lfloor\tau_0\rfloor, v_0)$

5318 15.2.1. QED

5319 $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \text{BdryErr}(b_0, v_0) \lesssim \text{dyn } b_1 v_1$

5320 16. CASE $v_0 = \text{mon } b_2 v_2 \lesssim \text{mon } b_3 v_3 = v_1$

5321 16.1. SCASE $\text{tag-match}(\lfloor\tau_0\rfloor, v_0)$

5322 16.1.1. $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \text{mon } b_0 v_0$

5323 by definition $\triangleright_{\mathbb{N}}$

5324 16.1.2. $\text{tag-match}(\lfloor\tau_1\rfloor, v_1)$

5325 by lemma 8.17

5326 16.1.3. $\text{dyn } b_1 v_1 \triangleright_{\Lambda} \text{mon } b_1 v_1$

5327 by definition \triangleright_{Λ}

5328 16.1.4. QED

$$\frac{\text{mon } b_2 v_2 \lesssim \text{mon } b_3 v_3}{\text{mon } b_0 (\text{mon } b_2 v_2) \lesssim \text{mon } b_1 (\text{mon } b_3 v_3)}$$

5329 16.2. SCASE $\neg\text{tag-match}(\lfloor\tau_0\rfloor, v_0)$

5330 16.2.1. QED

5331 $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \text{BdryErr}(b_0, v_0) \lesssim \text{dyn } b_1 v_1$

5332 \square

5333 LEMMA 8.13. *If $\cdot \vdash_{\mathbb{N}} v_0 : \mathcal{U}$ and $\cdot \vdash_{\Lambda} v_1 : \mathcal{U}$ and $v_0 \lesssim v_1$ and $b_0 \leq b_1$ then $\text{dyn } b_0 v_0 \rightarrow_{\mathbb{N}}^* e_2$ and $e_2 \lesssim \text{mon}^? b_1 v_1$.*

5334 PROOF. By induction on v_0 via case analysis of $v_0 \lesssim v_1$.

5335 1. CASE $i_0 \lesssim i_0$

5336 1.1. SCASE $\text{tag-match}(\lfloor\tau_0\rfloor, v_0)$

5337 1.1.1. $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} v_0$

5338 by definition $\triangleright_{\mathbb{N}}$

5357 1.1.2. QED
5358 $\text{mon}^? b_1 v_1 = v_1$
5359 1.2. SCASE $\neg \text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$
5360 1.2.1. QED
5361 $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \text{BndryErr}(b_0, v_0) \lesssim \text{mon}^? b_1 v_1$
5362 2. CASE $i_0 \lesssim \text{trace}_v \bar{b}_2 i_0$
5363 2.1. SCASE $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$
5364 2.1.1. $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} v_0$
5365 by definition $\triangleright_{\mathbb{N}}$
5366 2.1.2. QED
5367 $\text{mon}^? b_1 v_1 = v_1$
5368 2.2. SCASE $\neg \text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$
5369 2.2.1. QED
5370 $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \text{BndryErr}(b_0, v_0) \lesssim \text{mon}^? b_1 v_1$
5371 3. CASE $\langle v_2, v_3 \rangle \lesssim \langle v_4, v_5 \rangle$
5372 3.1. SCASE $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$
5373 3.1.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_4 \times \tau_5 \blacktriangleleft \ell_1)$
5374 3.1.2. $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \langle \text{dyn } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{dyn } (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$
5375 by definition $\triangleright_{\mathbb{N}}$
5376 3.1.3. $v_2 \lesssim v_4$ and $v_3 \lesssim v_5$
5377 by inversion \lesssim
5378 3.1.4. $\text{dyn } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \xrightarrow{*}_{\mathbb{N}} e_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) v_4$
5379 and $\text{dyn } (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \xrightarrow{*}_{\mathbb{N}} e_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) v_5$
5380 by the induction hypothesis
5381 3.1.5. QED
5382 either e_6 or $e_7 \in \text{BndryErr}(b, v)$, or:
5383
$$\frac{e_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) v_4 \quad e_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) v_5}{\langle e_6, e_7 \rangle \lesssim \text{mon } b_1 \langle v_4, v_5 \rangle}$$

5384 3.2. SCASE $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$
5385 3.2.1. QED
5386 $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \text{BndryErr}(b_0, v_0) \lesssim \text{mon}^? b_1 v_1$
5387 4. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) \langle v_4, v_5 \rangle$
5388 4.1. SCASE $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$
5389 4.1.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_4 \times \tau_5 \blacktriangleleft \ell_1)$
5390 4.1.2. $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \langle \text{dyn } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{dyn } (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$
5391 by definition $\triangleright_{\mathbb{N}}$
5392 4.1.3. $v_2 \lesssim \text{mon}^? (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) v_4$ and $v_3 \lesssim \text{mon}^? (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) v_5$
5393 by inversion \lesssim
5394 4.1.4. $\text{dyn } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \xrightarrow{*}_{\mathbb{N}} e_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^? (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) v_4)$
5395 and $\text{dyn } (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \xrightarrow{*}_{\mathbb{N}} e_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^? (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) v_5)$
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by the induction hypothesis

4.1.5. QED

either e_6 or $e_7 \in \text{BndryErr}(b, v)$, or:

$$e_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) v_4) \quad e_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) v_5)$$

$$\langle e_6, e_7 \rangle \lesssim \text{mon } b_1 (\text{mon}(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) \langle v_4, v_5 \rangle)$$

4.2. SCASE $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$

4.2.1. QED

$$\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \text{BndryErr}(b_0, v_0) \lesssim \text{mon}^? b_1 v_1$$

5. CASE $\langle v_2, v_3 \rangle \lesssim \text{trace}_v \bar{b}_2 \langle v_4, v_5 \rangle$

5.1. SCASE $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$

5.1.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_4 \times \tau_5 \blacktriangleleft \ell_1)$

5.1.2. $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \langle \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{dyn}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$

by definition $\triangleright_{\mathbb{N}}$

5.1.3. $v_2 \lesssim \text{add-trace}(\bar{b}_2, v_4)$ and $v_3 \lesssim \text{add-trace}(\bar{b}_2, v_5)$

by inversion \lesssim

5.1.4. $\text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_{\mathbb{N}}^* e_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, v_4))$

and $\text{dyn}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_{\mathbb{N}}^* e_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, v_5))$

by the induction hypothesis

5.1.5. QED

either e_6 or $e_7 \in \text{BndryErr}(b, v)$, or:

$$e_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, v_4)) \quad e_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, v_5))$$

$$\langle e_6, e_7 \rangle \lesssim \text{mon } b_1 (\text{trace}_v \bar{b}_2 \langle v_4, v_5 \rangle)$$

5.2. SCASE $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$

5.2.1. QED

$$\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \text{BndryErr}(b_0, v_0) \lesssim \text{mon}^? b_1 v_1$$

6. CASE $\langle v_2, v_3 \rangle \lesssim \text{trace}_v \bar{b}_2 \langle v_4, v_5 \rangle$

6.1. SCASE $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$

6.1.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_4 \times \tau_5 \blacktriangleleft \ell_1)$

6.1.2. $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \langle \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{dyn}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$

by definition $\triangleright_{\mathbb{N}}$

6.1.3. $v_2 \lesssim \text{add-trace}(\bar{b}_2, v_4)$ and $v_3 \lesssim \text{add-trace}(\bar{b}_2, v_5)$

by inversion \lesssim

6.1.4. $\text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_{\mathbb{N}}^* e_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, v_4))$

and $\text{dyn}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_{\mathbb{N}}^* e_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, v_5))$

by the induction hypothesis

6.1.5. QED

either e_6 or $e_7 \in \text{BndryErr}(b, v)$, or:

$$e_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, v_4)) \quad e_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, v_5))$$

$$\langle e_6, e_7 \rangle \lesssim \text{mon } b_1 (\text{trace}_v \bar{b}_2 \langle v_4, v_5 \rangle)$$

6.2. SCASE $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$

5461 6.2.1. QED
5462 $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \text{BndryErr}(b_0, v_0) \lesssim \text{mon}^? b_1 v_1$
5463 7. CASE $\langle v_2, v_3 \rangle \lesssim \text{trace}_v \bar{b}_2 (\text{mon} (\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_1) \langle v_4, v_5 \rangle)$
5464 7.1. SCASE $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$
5465 7.1.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_4 \times \tau_5 \blacktriangleleft \ell_1)$
5466 7.1.2. $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \langle \text{dyn} (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{dyn} (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$
5467 by definition $\triangleright_{\mathbb{N}}$
5468 7.1.3. $v_2 \lesssim \text{add-trace}(\bar{b}_2, (\text{mon}^? (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_7) v_4))$ and $v_3 \lesssim \text{add-trace}(\bar{b}_2, (\text{mon}^? (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_7) v_5))$
5469 by inversion \lesssim
5470 7.1.4. $\text{dyn} (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_{\mathbb{N}}^* e_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, (\text{mon}^? (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_7) v_4)))$
5471 and $\text{dyn} (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_{\mathbb{N}}^* e_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, (\text{mon}^? (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_7) v_5)))$
5472 by the induction hypothesis
5473 7.1.5. QED
5474 either e_6 or $e_7 \in \text{BndryErr}(b, v)$, or:
5475
$$e_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, (\text{mon}^? (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_7) v_4)))$$

5476
$$e_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, (\text{mon}^? (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_7) v_5)))$$

5477
$$\frac{\langle e_6, e_7 \rangle \lesssim \text{mon } b_1 (\text{trace}_v \bar{b}_2 (\text{mon} (\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) \langle v_4, v_5 \rangle))}{\langle v_6, v_7 \rangle \lesssim \text{mon } b_1 (\text{trace}_v \bar{b}_2 (\text{mon} (\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) \langle v_4, v_5 \rangle))}$$

5478 7.2. SCASE $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$
5479 7.2.1. QED
5480 $\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \text{BndryErr}(b_0, v_0) \lesssim \text{mon}^? b_1 v_1$
5481 8. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon } b_2 (\text{trace}_v \bar{b}_3 \langle v_4, v_5 \rangle)$
5482 8.1. CONTRADICTION:
5483 $\cdot \vdash_{\mathcal{A}} v_1 : \mathcal{U}$
5484 9. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon } b_2 (\text{mon } b_3 \langle v_4, v_5 \rangle)$
5485 9.1. $v_2 \lesssim \text{mon}^? b_2 (\text{mon}^? b_3 v_4)$
5486 and $v_3 \lesssim \text{mon}^? b_2 (\text{mon}^? b_3 v_5)$
5487 by inversion \lesssim
5488 9.2. $\text{dyn} (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \blacktriangleright_{\mathbb{N}} \langle \text{dyn} (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{dyn} (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$
5489 by definition $\blacktriangleright_{\mathbb{N}}$
5490 9.3. $\text{dyn} (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_{\mathbb{N}}^* v_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^? b_2 (\text{mon}^? b_3 v_4))$
5491 and $\text{dyn} (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_{\mathbb{N}}^* v_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^? b_2 (\text{mon}^? b_3 v_5))$
5492 by the induction hypothesis
5493 9.4. QED
5494
$$\frac{v_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^? b_2 (\text{mon}^? b_3 v_4)) \quad v_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^? b_2 (\text{mon}^? b_3 v_5))}{\langle v_6, v_7 \rangle \lesssim \text{mon} (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{mon } b_2 (\text{mon } b_3 \langle v_4, v_5 \rangle))}$$

5495 10. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon } b_2 (\text{trace}_v \bar{b}_3 (\text{mon } b_4 \langle v_4, v_5 \rangle))$
5496 10.1. $v_2 \lesssim \text{mon}^? b_2 (\text{trace}_v \bar{b}_3 (\text{mon}^? b_4 v_4))$
5497 and $v_3 \lesssim \text{mon}^? b_2 (\text{trace}_v \bar{b}_3 (\text{mon}^? b_4 v_5))$
5498 by inversion \lesssim
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- 5513 10.2. $\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \blacktriangleright_{\mathbb{N}} \langle \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{dyn}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$
 5514 by definition $\blacktriangleright_{\mathbb{N}}$
 5515 10.3. $\text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_{\mathbb{N}}^* v_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^? b_2 (\text{trace}_v \bar{b}_3 (\text{mon}^? b_4 v_4)))$
 5516 and $\text{dyn}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_{\mathbb{N}}^* v_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^? b_2 (\text{trace}_v \bar{b}_3 (\text{mon}^? b_4 v_5)))$
 5517 by the induction hypothesis
 5518 10.4. QED
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$$\frac{v_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^? b_2 (\text{trace}_v \bar{b}_3 (\text{mon}^? b_4 v_4)))}{v_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^? b_2 (\text{trace}_v \bar{b}_3 (\text{mon}^? b_4 v_5)))}$$

$$\langle v_6, v_7 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{mon} b_2 (\text{trace}_v \bar{b}_3 (\text{mon} b_4 \langle v_4, v_5 \rangle)))$$

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 5525 11. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}^+ \bar{b}_2 (\text{mon} b_3 (\text{mon} b_4 \langle v_4, v_5 \rangle))$
 5526 11.1. $v_2 \lesssim \text{mon}^{+?} \bar{b}_2 (\text{mon}^? b_3 (\text{mon}^? b_4 v_4))$
 5527 and $v_3 \lesssim \text{mon}^{+?} \bar{b}_2 (\text{mon}^? b_3 (\text{mon}^? b_4 v_5))$
 5528 by inversion \lesssim
 5529 11.2. $\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \blacktriangleright_{\mathbb{N}} \langle \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{dyn}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$
 5530 by definition $\blacktriangleright_{\mathbb{N}}$
 5531 11.3. $\text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_{\mathbb{N}}^* v_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^{+?} \bar{b}_2 (\text{mon}^? b_3 (\text{mon}^? b_4 v_4)))$
 5532 and $\text{dyn}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_{\mathbb{N}}^* v_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^{+?} \bar{b}_2 (\text{mon}^? b_3 (\text{mon}^? b_4 v_5)))$
 5533 by the induction hypothesis
 5534 11.4. QED
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$$\frac{v_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^{+?} \bar{b}_2 (\text{mon}^? b_3 (\text{mon}^? b_4 v_4)))}{v_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^{+?} \bar{b}_2 (\text{mon}^? b_3 (\text{mon}^? b_4 v_5)))}$$

$$\langle v_6, v_7 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{mon}^+ \bar{b}_2 (\text{mon} b_3 (\text{mon} b_4 \langle v_4, v_5 \rangle)))$$

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 5541 12. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}^+ \bar{b}_2 (\text{mon} b_3 (\text{trace}_v \bar{b}_4 (\text{mon} b_5 \langle v_4, v_5 \rangle)))$
 5542 12.1. $v_2 \lesssim \text{mon}^+ \bar{b}_2 (\text{mon} b_3 (\text{trace}_v \bar{b}_4 (\text{mon} b_5 v_4)))$
 5543 and $v_3 \lesssim \text{mon}^+ \bar{b}_2 (\text{mon} b_3 (\text{trace}_v \bar{b}_4 (\text{mon} b_5 v_5)))$
 5544 by inversion \lesssim
 5545 12.2. $\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \blacktriangleright_{\mathbb{N}} \langle \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{dyn}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$
 5546 by definition $\blacktriangleright_{\mathbb{N}}$
 5547 12.3. $\text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_{\mathbb{N}}^* v_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^+ \bar{b}_2 (\text{mon} b_3 (\text{trace}_v \bar{b}_4 (\text{mon} b_5 v_4))))$
 5548 and $\text{dyn}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_{\mathbb{N}}^* v_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^+ \bar{b}_2 (\text{mon} b_3 (\text{trace}_v \bar{b}_4 (\text{mon} b_5 v_5))))$
 5549 by the induction hypothesis
 5550 12.4. QED
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$$\frac{v_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^+ \bar{b}_2 (\text{mon} b_3 (\text{trace}_v \bar{b}_4 (\text{mon} b_5 v_4))))}{v_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^+ \bar{b}_2 (\text{mon} b_3 (\text{trace}_v \bar{b}_4 (\text{mon} b_5 v_5))))}$$

$$\langle v_6, v_7 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{mon}^+ \bar{b}_2 (\text{mon} b_3 (\text{trace}_v \bar{b}_4 (\text{mon} b_5 \langle v_4, v_5 \rangle))))$$

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 5557 13. CASE $\lambda x_2. e_2 \lesssim \lambda x_2. e_3$
 5558 13.1. SCASE $\text{tag-match}([\tau_0], v_0)$
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5565 13.1.1. QED
5566 $\text{dyn } b_0 v_0 \triangleright_N \text{mon } b_0 v_0 \lesssim \text{mon } b_1 v_1$
5567 13.2. SCASE $\neg \text{tag-match}([\tau_0], v_0)$
5568 13.2.1. QED
5569 $\text{dyn } b_0 v_0 \triangleright_N \text{BdryErr}(b_0, v_0) \lesssim \text{dyn } b_1 v_1$
5570 14. CASE $\lambda(x_2 : \tau_2). e_2 \lesssim \lambda(x_2 : \tau_2). e_3$
5571 14.1. CONTRADICTION:
5572 $\cdot \vdash_N v_0 : \mathcal{U}$
5573 15. CASE $\text{mon } b_2 (\text{mon } b_3 v_2) \lesssim \text{trace}_v b_4 b_5 (\lambda x_0. e_3)$
5574 15.1. SCASE $\text{tag-match}([\tau_0], v_0)$
5575 15.1.1. $\text{dyn } b_0 v_0 \triangleright_N \text{mon } b_0 v_0$
5576 by definition \triangleright_N
5577 15.1.2. QED
5578
$$\frac{\text{mon } b_2 (\text{mon } b_3 v_2) \lesssim \text{trace}_v b_4 b_5 (\lambda x_0. e_3)}{\text{mon } b_0 (\text{mon } b_2 (\text{mon } b_3 v_2)) \lesssim \text{mon } b_1 (\text{trace}_v b_4 b_5 (\lambda x_0. e_3))}$$

5579 15.2. SCASE $\neg \text{tag-match}([\tau_0], v_0)$
5580 15.2.1. QED
5581 $\text{dyn } b_0 v_0 \triangleright_N \text{BdryErr}(b_0, v_0) \lesssim \text{dyn } b_1 v_1$
5582 16. CASE $\text{mon } b_2 (\text{mon } b_3 v_2) \lesssim (\text{trace}_v b_4 b_5 \bar{b}_6 (\lambda x_3. e_3))$
5583 16.1. $\text{tag-match}([\tau_0], v_0)$
5584 16.1.1. $\text{dyn } b_0 v_0 \triangleright_N \text{mon } b_0 v_0$
5585 by definition \triangleright_N
5586 16.1.2. QED
5587
$$\frac{\text{mon } b_2 (\text{mon } b_3 v_2) \lesssim (\text{trace}_v b_4 b_5 \bar{b}_6 (\lambda x_3. e_3))}{\text{mon } b_0 (\text{mon } b_2 (\text{mon } b_3 v_2)) \lesssim \text{mon } b_1 (\text{trace}_v b_4 b_5 \bar{b}_6 (\lambda x_3. e_3))}$$

5588 16.2. $\neg \text{tag-match}([\tau_0], v_0)$
5589 16.2.1. QED
5590 $\text{dyn } b_0 v_0 \triangleright_N \text{BdryErr}(b_0, v_0) \lesssim \text{dyn } b_1 v_1$
5591 17. CASE $v_0 = \text{mon } b_2 v_2 \lesssim \text{mon } b_3 v_3 = v_1$
5592 17.1. SCASE $\text{tag-match}([\tau_0], v_0)$
5593 17.1.1. $\text{dyn } b_0 v_0 \triangleright_N \text{mon } b_0 v_0$
5594 by definition \triangleright_N
5595 17.1.2. QED
5596
$$\frac{\text{mon } b_2 v_2 \lesssim \text{mon } b_3 v_3}{\text{mon } b_0 (\text{mon } b_2 v_2) \lesssim \text{mon } b_1 (\text{mon } b_3 v_3)}$$

5597 17.2. SCASE $\neg \text{tag-match}([\tau_0], v_0)$
5598 17.2.1. QED

$$\text{dyn } b_0 v_0 \triangleright_{\mathbb{N}} \text{BndryErr}(b_0, v_0) \lesssim \text{dyn } b_1 v_1$$

□

LEMMA 8.14. *If $\text{stat } b_0 v_0$ and $\text{stat } b_1 v_1$ are reduced WF expressions and $\text{stat } b_0 v_0 \lesssim \text{stat } b_1 v_1$ and $\text{stat } b_0 v_0 \triangleright_{\mathbb{N}} e_2$ and $\text{stat } b_1 v_1 \triangleright_{\mathbb{A}} e_3$ then $e_2 \lesssim e_3$*

PROOF. By case analysis of $v_0 \lesssim v_1$.

1. CASE $i_0 \lesssim i_0$

1.1. $\text{stat } b_0 v_0 \triangleright_{\mathbb{N}} v_0$ and $\text{stat } b_1 v_1 \triangleright_{\mathbb{A}} v_1$

by definition $\triangleright_{\mathbb{N}}, \triangleright_{\mathbb{A}}$

1.2. QED

2. CASE $i_0 \lesssim \text{trace}_v \bar{b}_2 i_0$

2.1. CONTRADICTION:

$\cdot \vdash_{\mathbb{A}} v_1 : \tau_1$

3. CASE $\langle v_2, v_3 \rangle \lesssim \langle v_4, v_5 \rangle$

3.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$

3.2. $v_2 \lesssim v_4$ and $v_3 \lesssim v_5$

by inversion \lesssim

3.3. $\text{stat } b_0 v_0 \triangleright_{\mathbb{N}} \langle \text{stat } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{stat } (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$

and $\text{stat } b_1 v_1 \triangleright_{\mathbb{A}} \text{mon } b_1 v_1$

by definition $\triangleright_{\mathbb{N}}$ and $\triangleright_{\mathbb{A}}$

3.4. $\text{stat } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_{\mathbb{N}}^* v_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) v_4$

and $\text{stat } (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_{\mathbb{N}}^* v_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) v_5$

by lemma 8.15

3.5. QED

$$\frac{v_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) v_4 \quad v_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) v_5}{\langle v_6, v_7 \rangle \lesssim \text{mon } b_1 \langle v_4, v_5 \rangle}$$

4. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon} (\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) \langle v_4, v_5 \rangle$

4.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$

4.2. $v_2 \lesssim \text{mon}^? (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) v_4$

and $v_3 \lesssim \text{mon}^? (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) v_5$

by inversion \lesssim

4.3. $\text{stat } b_0 v_0 \triangleright_{\mathbb{N}} \langle \text{stat } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{stat } (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$

and $\text{stat } b_1 v_1 \rightarrow_{\mathbb{A}}^* \text{trace}_v b_1 (\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) \langle v_4, v_5 \rangle$

by definition $\triangleright_{\mathbb{N}}$ and $\triangleright_{\mathbb{A}}$

4.4. $\text{stat } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_{\mathbb{N}}^* v_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^? (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) v_4)$

and $\text{stat } (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_{\mathbb{N}}^* v_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^? (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) v_5)$

by lemma 8.15

4.5. $v_6 \lesssim \text{add-trace}((\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3), v_4)$

and $v_7 \lesssim \text{add-trace}((\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3), v_5)$

by lemma 8.16

5669 4.6. QED

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$$v_6 \lesssim \text{add-trace}((\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3), v_4) \quad v_7 \lesssim \text{add-trace}((\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3), v_5)$$

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$$\langle v_6, v_7 \rangle \lesssim \text{trace}_v b_1(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) \langle v_4, v_5 \rangle$$

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5674 5. CASE $\langle v_2, v_3 \rangle \lesssim \text{trace}_v \bar{b}_2 \langle v_4, v_5 \rangle$

5675

5676 5.1. CONTRADICTION:

5677

$$\cdot \vdash_A v_1 : \tau_1$$

5678

5679 6. CASE $\langle v_2, v_3 \rangle \lesssim \text{trace}_v \bar{b}_2 (\text{mon } b_3 \langle v_4, v_5 \rangle)$

5680

5681 6.1. CONTRADICTION:

5682

$$\cdot \vdash_A v_1 : \tau_1$$

5683

5684 7. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) (\text{trace}_v \bar{b}_4 \langle v_4, v_5 \rangle)$

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5686 7.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$

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5688 7.2. $v_2 \lesssim \text{mon}^?(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) (\text{add-trace}(\bar{b}_4, v_4))$

5689

5690 and $v_3 \lesssim \text{mon}^?(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) (\text{add-trace}(\bar{b}_4, v_5))$

5691

5692 by inversion \lesssim

5693

5694 7.3. $\text{stat } b_0 v_0 \blacktriangleright_N \langle \text{stat}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{stat}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$

5695

5696 and $\text{stat } b_1 v_1 \xrightarrow{*}_A \text{trace}_v b_1(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) \bar{b}_4 \langle v_4, v_5 \rangle$

5697

5698 by definition \blacktriangleright_N and \blacktriangleright_A

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5700 7.4. $\text{stat}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \xrightarrow{*}_N v_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) (\text{add-trace}(\bar{b}_4, v_4)))$

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5702 and $\text{stat}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \xrightarrow{*}_N v_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) (\text{add-trace}(\bar{b}_4, v_5)))$

5703

5704 by lemma 8.15

5705

5706 7.5. $v_6 \lesssim \text{add-trace}((\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3), (\text{add-trace}(\bar{b}_4, v_4)))$

5707

5708 and $v_7 \lesssim \text{add-trace}((\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3), (\text{add-trace}(\bar{b}_4, v_5)))$

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5710 by lemma 8.16

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5712 7.6. QED

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$$v_6 \lesssim \text{add-trace}((\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) \bar{b}_4, v_4) \quad v_7 \lesssim \text{add-trace}((\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) \bar{b}_4, v_5)$$

5715

$$\langle v_6, v_7 \rangle \lesssim \text{trace}_v b_1(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) \bar{b}_4 \langle v_4, v_5 \rangle$$

5716

5717 8. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) (\text{mon}(\ell_4 \blacktriangleleft \tau_8 \times \tau_9 \blacktriangleleft \ell_5) \langle v_4, v_5 \rangle)$

5718

5719 8.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$

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5721 8.2. $v_2 \lesssim \text{mon}^?(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) (\text{mon}^?(\ell_4 \blacktriangleleft \tau_8 \blacktriangleleft \ell_5) v_4)$

5722

5723 and $v_3 \lesssim \text{mon}^?(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) (\text{mon}^?(\ell_4 \blacktriangleleft \tau_9 \blacktriangleleft \ell_5) v_5)$

5724

5725 by inversion \lesssim

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5727 8.3. $\text{stat } b_0 v_0 \blacktriangleright_N \langle \text{stat}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{stat}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$

5728

5729 and $\text{stat } b_1 v_1 \xrightarrow{*}_A \text{trace}_v b_1(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) (\text{mon}(\ell_4 \blacktriangleleft \tau_8 \times \tau_9 \blacktriangleleft \ell_5) \langle v_4, v_5 \rangle)$

5730

5731 by definition \blacktriangleright_N and \blacktriangleright_A

5732

5733 8.4. $\text{stat}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \xrightarrow{*}_N v_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) (\text{mon}^?(\ell_4 \blacktriangleleft \tau_8 \blacktriangleleft \ell_5) v_4))$

5734

5735 and $\text{stat}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \xrightarrow{*}_N v_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) (\text{mon}^?(\ell_4 \blacktriangleleft \tau_9 \blacktriangleleft \ell_5) v_5))$

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5737 by lemma 8.15

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5721 8.5. $v_6 \lesssim \text{add-trace}((\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3), (\text{mon}^?(\ell_4 \blacktriangleleft \tau_8 \blacktriangleleft \ell_5) v_4))$
 5722 and $v_7 \lesssim \text{add-trace}((\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3), (\text{mon}^?(\ell_4 \blacktriangleleft \tau_9 \blacktriangleleft \ell_5) v_5))$
 5723 by lemma 8.16
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5725 8.6. QED
 5726

$$\begin{aligned} &v_6 \lesssim \text{add-trace}((\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3), (\text{mon}^?(\ell_4 \blacktriangleleft \tau_8 \blacktriangleleft \ell_5) v_4)) \\ &v_7 \lesssim \text{add-trace}((\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3), (\text{mon}^?(\ell_4 \blacktriangleleft \tau_9 \blacktriangleleft \ell_5) v_5)) \\ \hline &\langle v_6, v_7 \rangle \lesssim \text{trace}_v b_1(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) (\text{mon}(\ell_4 \blacktriangleleft \tau_8 \times \tau_9 \blacktriangleleft \ell_5) \langle v_4, v_5 \rangle) \end{aligned}$$

5731 9. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) (\text{trace}_v \bar{b}_4 (\text{mon}(\ell_5 \blacktriangleleft \tau_8 \times \tau_9 \blacktriangleleft \ell_6) \langle v_4, v_5 \rangle))$

5732 9.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$

5733 9.2. $v_2 \lesssim \text{mon}^?(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) (\text{trace}_v^? \bar{b}_4 (\text{mon}^?(\ell_5 \blacktriangleleft \tau_8 \blacktriangleleft \ell_6) v_4))$

5734 and $v_3 \lesssim \text{mon}^?(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) (\text{trace}_v^? \bar{b}_4 (\text{mon}^?(\ell_5 \blacktriangleleft \tau_9 \blacktriangleleft \ell_6) v_5))$

5735 by inversion \lesssim
 5736

5737 9.3. $\text{stat } b_0 v_0 \blacktriangleright_N (\text{stat } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{stat } (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3)$

5738 and $\text{stat } b_1 v_1 \rightarrow_A^* \text{trace}_v b_1(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) \bar{b}_4 (\text{mon}(\ell_5 \blacktriangleleft \tau_8 \times \tau_9 \blacktriangleleft \ell_6) \langle v_4, v_5 \rangle)$

5739 by definition \blacktriangleright_N and \blacktriangleright_A
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5741 9.4. $\text{stat } (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_N^* v_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) (\text{trace}_v^? \bar{b}_4 (\text{mon}^?(\ell_5 \blacktriangleleft \tau_8 \blacktriangleleft \ell_6) v_4)))$

5742 and $\text{stat } (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_N^* v_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) (\text{trace}_v^? \bar{b}_4 (\text{mon}^?(\ell_5 \blacktriangleleft \tau_9 \blacktriangleleft \ell_6) v_5)))$

5743 by lemma 8.15
 5744

5745 9.5. $v_6 \lesssim \text{add-trace}((\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3), (\text{trace}_v^? \bar{b}_4 (\text{mon}^?(\ell_5 \blacktriangleleft \tau_8 \blacktriangleleft \ell_6) v_4)))$

5746 and $v_7 \lesssim \text{add-trace}((\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3), (\text{trace}_v^? \bar{b}_4 (\text{mon}^?(\ell_5 \blacktriangleleft \tau_9 \blacktriangleleft \ell_6) v_5)))$

5747 by lemma 8.16
 5748

5749 9.6. QED
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$$\begin{aligned} &v_6 \lesssim \text{add-trace}(b_1(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) \bar{b}_4, (\text{mon}^?(\ell_5 \blacktriangleleft \tau_8 \blacktriangleleft \ell_6) v_4)) \\ &v_7 \lesssim \text{add-trace}(b_1(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) \bar{b}_4, (\text{mon}^?(\ell_5 \blacktriangleleft \tau_9 \blacktriangleleft \ell_6) v_5)) \end{aligned}$$

$$\langle v_6, v_7 \rangle \lesssim \text{trace}_v b_1(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) \bar{b}_4 (\text{mon}(\ell_5 \blacktriangleleft \tau_8 \times \tau_9 \blacktriangleleft \ell_6) \langle v_4, v_5 \rangle)$$

5755 10. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}^+ \bar{b}_2 (\text{mon } b_3 (\text{mon } b_4 \langle v_4, v_5 \rangle))$

5756 10.1. CONTRADICTION:

5757 by lemma 7.9
 5758

5759 11. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}^+ \bar{b}_2 (\text{mon } b_3 (\text{trace}_v \bar{b}_4 (\text{mon } b_5 \langle v_4, v_5 \rangle)))$

5760 11.1. CONTRADICTION:

5761 by lemma 7.9
 5762

5763 12. CASE $\lambda x_2. e_2 \lesssim \lambda x_2. e_3$

5764 12.1. CONTRADICTION:

5765 $\cdot \vdash_N v_0 : \tau_0$
 5766

5767 13. CASE $\lambda(x_2 : \tau_2). e_2 \lesssim \lambda(x_2 : \tau_2). e_3$

5768 13.1. QED
 5769

5770 $\text{stat } b_0 v_0 \blacktriangleright_N \text{mon } b_0 v_0$ and $\text{stat } b_1 v_1 \blacktriangleright_N \text{mon } b_1 v_1$

5771 14. CASE $\text{mon } b_2 (\text{mon } b_3 v_2) \lesssim \text{trace}_v \bar{b}_4 v_3$
 5772

5773 14.1. CONTRADICTION:

5774 $\cdot \vdash_A v_1 : \tau_1$

5775 15. CASE $\text{mon } b_2 v_2 \lesssim \text{mon } b_3 v_3$

5776 15.1. $\text{stat } b_0 v_0 \blacktriangleright_N \text{mon } b_0 v_0$

5777 by definition \blacktriangleright_N

5779 15.2. $\text{stat } b_1 v_1 \blacktriangleright_A \text{add-trace}(b_1 b_3 \text{get-trace}(v_3), \text{rem-trace}(v_3))$

5780 by definition \blacktriangleright_A

5781 15.3. QED

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$$\frac{b_0 \leq b_1 \quad b_1 \leq b_3 \quad v_2 \lesssim \text{trace}_v \text{get-trace}(v_3) \text{rem-trace}(v_3)}{\text{mon } b_0 (\text{mon } b_1 v_2) \lesssim \text{trace}_v b_1 b_3 \text{get-trace}(v_3) \text{rem-trace}(v_3)}$$

□

LEMMA 8.15. If v_0 and v_1 are reduced WF expressions and $v_0 \lesssim v_1$ and $\cdot \vdash_N v_0 : \tau_0$ and $\cdot \vdash_A v_1 : \tau_1$ and $\tau_0 \leq \tau_1$ then $\text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \rightarrow_N^* v_2$ and $v_2 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_3$

PROOF. By induction on v_0 via case analysis of $v_0 \lesssim v_1$.

1. CASE $i_0 \lesssim i_0$

1.1. QED

$\text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \blacktriangleright_N v_0 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_1$

1.2. QED

2. CASE $i_0 \lesssim \text{trace}_v \bar{b}_2 i_0$

2.1. CONTRADICTION:

$\cdot \vdash_A v_1 : \tau_1$

3. CASE $\langle v_2, v_3 \rangle \lesssim \langle v_4, v_5 \rangle$

3.1. $v_2 \lesssim v_4$ and $v_3 \lesssim v_5$

by inversion \lesssim

3.2. $\text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \blacktriangleright_N \langle \text{stat}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{stat}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$

by definition \blacktriangleright_N

3.3. $\text{stat}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_N^* v_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) v_4$

and $\text{stat}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_N^* v_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) v_5$

by the induction hypothesis

3.4. QED

$$\frac{v_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) v_4 \quad v_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) v_5}{\langle v_6, v_7 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) \langle v_4, v_5 \rangle}$$

4. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) \langle v_4, v_5 \rangle$

4.1. $v_2 \lesssim \text{mon}^?(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) v_4$

and $v_3 \lesssim \text{mon}^?(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) v_5$

by inversion \lesssim

4.2. $\text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \blacktriangleright_N \langle \text{stat}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{stat}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$

by definition \blacktriangleright_N

5825 4.3. $\text{stat}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_N^* v_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) v_4)$
 5826 and $\text{stat}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_N^* v_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) v_5)$
 5827 by the induction hypothesis
 5828

5829 4.4. QED

$$\frac{v_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) v_4) \quad v_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) v_5)}{\langle v_6, v_7 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_4 \times \tau_5 \blacktriangleleft \ell_1) (\text{mon}(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) \langle v_4, v_5 \rangle)}$$

5834 5. CASE $\langle v_2, v_3 \rangle \lesssim \text{trace}_v \bar{b}_2 \langle v_4, v_5 \rangle$

5835 5.1. CONTRADICTION:

5836 $\cdot \vdash_A v_1 : \tau_1$

5837 6. CASE $\langle v_2, v_3 \rangle \lesssim \text{trace}_v \bar{b}_2 (\text{mon } b_3 \langle v_4, v_5 \rangle)$

5838 6.1. CONTRADICTION:

5839 $\cdot \vdash_A v_1 : \tau_1$

5840 7. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) (\text{trace}_v \bar{b}_4 \langle v_4, v_5 \rangle)$

5841 7.1. $v_2 \lesssim \text{mon}^?(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) (\text{add-trace}(\bar{b}_4, v_4))$

5842 and $v_3 \lesssim \text{mon}^?(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) (\text{add-trace}(\bar{b}_4, v_5))$

5843 by inversion \lesssim

5844 7.2. $\text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \blacktriangleright_N \langle \text{stat}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{stat}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$

5845 by definition \blacktriangleright_N

5846 7.3. $\text{stat}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_N^* v_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) (\text{add-trace}(\bar{b}_4, v_4)))$

5847 and $\text{stat}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_N^* v_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) (\text{add-trace}(\bar{b}_4, v_5)))$

5848 by the induction hypothesis

5849 7.4. QED

$$v_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) (\text{add-trace}(\bar{b}_4, v_4)))$$

$$v_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) (\text{add-trace}(\bar{b}_4, v_5)))$$

$$\langle v_6, v_7 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_4 \times \tau_5 \blacktriangleleft \ell_1) (\text{mon}(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) (\text{trace}_v \bar{b}_4 \langle v_4, v_5 \rangle))$$

5854 8. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) (\text{mon}(\ell_4 \blacktriangleleft \tau_8 \times \tau_9 \blacktriangleleft \ell_5) \langle v_4, v_5 \rangle)$

5855 8.1. $v_2 \lesssim \text{mon}^?(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) (\text{mon}^?(\ell_4 \blacktriangleleft \tau_8 \blacktriangleleft \ell_5) v_4)$

5856 and $v_3 \lesssim \text{mon}^?(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) (\text{mon}^?(\ell_4 \blacktriangleleft \tau_9 \blacktriangleleft \ell_5) v_5)$

5857 by inversion \lesssim

5858 8.2. $\text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \blacktriangleright_N \langle \text{stat}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{stat}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$

5859 by definition \blacktriangleright_N

5860 8.3. $\text{stat}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_N^* v_6 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) (\text{mon}^?(\ell_4 \blacktriangleleft \tau_8 \blacktriangleleft \ell_5) v_4))$

5861 and $\text{stat}(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_N^* v_7 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) (\text{mon}^?(\ell_4 \blacktriangleleft \tau_9 \blacktriangleleft \ell_5) v_5))$

5862 by the induction hypothesis

5863 8.4. QED

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$$v_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^? (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) (\text{mon}^? (\ell_4 \blacktriangleleft \tau_8 \blacktriangleleft \ell_5) v_4))$$

$$v_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^? (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) (\text{mon}^? (\ell_4 \blacktriangleleft \tau_9 \blacktriangleleft \ell_5) v_5))$$

$$\frac{}{\langle v_6, v_7 \rangle \lesssim \text{mon} (\ell_0 \blacktriangleleft \tau_4 \times \tau_5 \blacktriangleleft \ell_1) (\text{mon} (\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) (\text{mon} (\ell_4 \blacktriangleleft \tau_8 \times \tau_9 \blacktriangleleft \ell_5) \langle v_4, v_5 \rangle))}$$

9. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon} (\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) (\text{trace}_v \bar{b}_4 (\text{mon} (\ell_5 \blacktriangleleft \tau_8 \times \tau_9 \blacktriangleleft \ell_6) \langle v_4, v_5 \rangle))$

9.1. $v_2 \lesssim \text{mon}^? (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) (\text{trace}_v^? \bar{b}_4 (\text{mon}^? (\ell_5 \blacktriangleleft \tau_8 \blacktriangleleft \ell_6) v_4))$

and $v_3 \lesssim \text{mon}^? (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) (\text{trace}_v^? \bar{b}_4 (\text{mon}^? (\ell_5 \blacktriangleleft \tau_9 \blacktriangleleft \ell_6) v_5))$

by inversion \lesssim

9.2. $\text{stat} (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \blacktriangleright_{\mathbb{N}} \langle \text{stat} (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{stat} (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$

by definition $\blacktriangleright_{\mathbb{N}}$

9.3. $\text{stat} (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_{\mathbb{N}}^* v_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^? (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) (\text{trace}_v^? \bar{b}_4 (\text{mon}^? (\ell_5 \blacktriangleleft \tau_8 \blacktriangleleft \ell_6) v_4))$

and $\text{stat} (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_{\mathbb{N}}^* v_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^? (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) (\text{trace}_v^? \bar{b}_4 (\text{mon}^? (\ell_5 \blacktriangleleft \tau_9 \blacktriangleleft \ell_6) v_5))$

by the induction hypothesis

9.4. QED

$$v_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^? (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) (\text{trace}_v^? \bar{b}_4 (\text{mon}^? (\ell_5 \blacktriangleleft \tau_8 \blacktriangleleft \ell_6) v_4)))$$

$$v_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^? (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) (\text{trace}_v^? \bar{b}_4 (\text{mon}^? (\ell_5 \blacktriangleleft \tau_9 \blacktriangleleft \ell_6) v_5)))$$

$$\frac{}{\langle v_6, v_7 \rangle \lesssim \text{mon} (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{mon} (\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3) (\text{trace}_v \bar{b}_4 (\text{mon} (\ell_5 \blacktriangleleft \tau_8 \times \tau_9 \blacktriangleleft \ell_6) \langle v_4, v_5 \rangle))}$$

10. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}^+ \bar{b}_2 (\text{mon} b_3 (\text{mon} b_4 \langle v_4, v_5 \rangle))$

10.1. $v_2 \lesssim \text{mon}^{+?} \bar{b}_2 (\text{mon}^? b_3 (\text{mon}^? b_4 v_4))$

and $v_3 \lesssim \text{mon}^{+?} \bar{b}_2 (\text{mon}^? b_3 (\text{mon}^? b_4 v_5))$

by inversion \lesssim

10.2. $\text{stat} (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \blacktriangleright_{\mathbb{N}} \langle \text{stat} (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{stat} (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$

by definition $\blacktriangleright_{\mathbb{N}}$

10.3. $\text{stat} (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_{\mathbb{N}}^* v_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^{+?} \bar{b}_2 (\text{mon}^? b_3 (\text{mon}^? b_4 v_4))$

and $\text{stat} (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_{\mathbb{N}}^* v_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^{+?} \bar{b}_2 (\text{mon}^? b_3 (\text{mon}^? b_4 v_5))$

by the induction hypothesis

10.4. QED

$$v_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^{+?} \bar{b}_2 (\text{mon}^? b_3 (\text{mon}^? b_4 v_4)))$$

$$v_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^{+?} \bar{b}_2 (\text{mon}^? b_3 (\text{mon}^? b_4 v_5)))$$

$$\frac{}{\langle v_6, v_7 \rangle \lesssim \text{mon} (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{mon}^+ \bar{b}_2 (\text{mon} b_3 (\text{mon} b_4 \langle v_4, v_5 \rangle))}$$

11. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}^+ \bar{b}_2 (\text{mon} b_3 (\text{trace}_v \bar{b}_4 (\text{mon} b_5 \langle v_4, v_5 \rangle)))$

11.1. $v_2 \lesssim \text{mon}^+ \bar{b}_2 (\text{mon} b_3 (\text{trace}_v \bar{b}_4 (\text{mon} b_5 v_4)))$

and $v_3 \lesssim \text{mon}^+ \bar{b}_2 (\text{mon} b_3 (\text{trace}_v \bar{b}_4 (\text{mon} b_5 v_5)))$

by inversion \lesssim

11.2. $\text{stat} (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \blacktriangleright_{\mathbb{N}} \langle \text{stat} (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2, \text{stat} (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rangle$

by definition $\blacktriangleright_{\mathbb{N}}$

11.3. $\text{stat} (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_2 \rightarrow_{\mathbb{N}}^* v_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^+ \bar{b}_2 (\text{mon} b_3 (\text{trace}_v \bar{b}_4 (\text{mon} b_5 v_4))))$

and $\text{stat} (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) v_3 \rightarrow_{\mathbb{N}}^* v_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^+ \bar{b}_2 (\text{mon} b_3 (\text{trace}_v \bar{b}_4 (\text{mon} b_5 v_5))))$

by the induction hypothesis

11.4. QED

$$\frac{v_6 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) (\text{mon}^+ \bar{b}_2 (\text{mon } b_3 (\text{trace}_v \bar{b}_4 (\text{mon } b_5 v_4))))}{v_7 \lesssim \text{mon}^? (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) (\text{mon}^+ \bar{b}_2 (\text{mon } b_3 (\text{trace}_v \bar{b}_4 (\text{mon } b_5 v_5))))} \\ \langle v_6, v_7 \rangle \lesssim \text{mon} (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{mon}^+ \bar{b}_2 (\text{mon } b_3 (\text{trace}_v \bar{b}_4 (\text{mon } b_5 \langle v_4, v_5 \rangle))))$$

12. CASE $\lambda x_2. e_2 \lesssim \lambda x_2. e_3$

12.1. CONTRADICTION:

$\cdot \vdash_N v_0 : \tau_0$

13. CASE $\lambda(x_2 : \tau_2). e_2 \lesssim \lambda(x_2 : \tau_2). e_3$

13.1. QED

$\text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \blacktriangleright_N \text{mon } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \lesssim \text{mon } (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_1$

14. CASE $\text{mon } b_2 (\text{mon } b_3 v_2) \lesssim \text{trace}_v \bar{b}_4 v_3$

14.1. CONTRADICTION:

$\cdot \vdash_A v_1 : \tau_1$

15. CASE $\text{mon } b_2 v_2 \lesssim \text{mon } b_3 v_3$

15.1. QED

$\text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \blacktriangleright_N \text{mon } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \lesssim \text{mon } (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_1$

□

LEMMA 8.16. *If $\cdot \vdash_N v_0 : \mathcal{U}$ and $\cdot \vdash_A \text{mon}^? b_0 (\text{mon}^? b_1 v_1) : \mathcal{U}$ and $v_0 \lesssim \text{mon}^? b_0 (\text{mon}^? b_1 v_1)$ then $v_0 \lesssim \text{add-trace}(b_0 b_1, v_1)$.*

PROOF. By induction on v_0 via case analysis of $v_0 \lesssim \text{mon}^? b_0 (\text{mon}^? b_1 v_1)$.

1. CASE $i_0 \lesssim i_0$

1.1. QED

$i_0 \lesssim \text{trace}_v b_0 b_1 i_0$

2. CASE $i_0 \lesssim \text{trace}_v \bar{b}_2 i_0$

2.1. QED

$i_0 \lesssim \text{trace}_v b_0 b_1 \bar{b}_2 i_0$

3. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon } b_0 (\text{mon } b_1 \langle v_4, v_5 \rangle)$

3.1. $v_2 \lesssim \text{mon}^? b_0 (\text{mon}^? b_1 v_4)$

and $v_3 \lesssim \text{mon}^? b_0 (\text{mon}^? b_1 v_5)$

by inversion \lesssim

3.2. $v_2 \lesssim \text{add-trace}(b_0 b_1, v_4)$

and $v_3 \lesssim \text{add-trace}(b_0 b_1, v_5)$

by the induction hypothesis

3.3. QED

$$\frac{v_2 \lesssim \text{add-trace}(b_0 b_1, v_4) \quad v_3 \lesssim \text{add-trace}(b_0 b_1, v_5)}{\langle v_2, v_3 \rangle \lesssim \text{trace}_v b_0 b_1 \langle v_4, v_5 \rangle}$$

4. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon } b_0 (\text{mon } b_1 (\text{trace}_v \bar{b}_2 \langle v_4, v_5 \rangle))$

- 5981 4.1. $v_2 \lesssim \text{mon}^? b_0 (\text{mon}^? b_1 (\text{add-trace}(\bar{b}_2, v_4)))$
 5982 and $v_3 \lesssim \text{mon}^? b_0 (\text{mon}^? b_1 (\text{add-trace}(\bar{b}_2, v_5)))$
 5983 by inversion \lesssim
 5984 4.2. $v_2 \lesssim \text{add-trace}(b_0 b_1, (\text{add-trace}(\bar{b}_2, v_4)))$
 5985 and $v_3 \lesssim \text{add-trace}(b_0 b_1, (\text{add-trace}(\bar{b}_2, v_5)))$
 5986 by the induction hypothesis
 5987 4.3. QED
 5988
 5989
 5990
 5991
$$\frac{v_2 \lesssim \text{add-trace}(b_0 b_1, (\text{add-trace}(\bar{b}_2, v_4))) \quad v_3 \lesssim \text{add-trace}(b_0 b_1, (\text{add-trace}(\bar{b}_2, v_5)))}{\langle v_2, v_3 \rangle \lesssim \text{trace}_v b_0 b_1 \bar{b}_2 \langle v_4, v_5 \rangle}$$

 5992
 5993 5. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon} b_0 (\text{mon} b_1 (\text{mon} b_2 \langle v_4, v_5 \rangle))$
 5994 5.1. $v_2 \lesssim \text{mon}^? b_0 (\text{mon}^? b_1 (\text{mon}^? b_2 v_4))$
 5995 and $v_3 \lesssim \text{mon}^? b_0 (\text{mon}^? b_1 (\text{mon}^? b_2 v_5))$
 5996 by inversion \lesssim
 5997 5.2. $v_2 \lesssim \text{add-trace}(b_0 b_1, (\text{mon}^? b_2 v_4))$
 5998 and $v_3 \lesssim \text{add-trace}(b_0 b_1, (\text{mon}^? b_2 v_5))$
 5999 by the induction hypothesis
 6000 5.3. QED
 6001
 6002
 6003
 6004
$$\frac{v_2 \lesssim \text{add-trace}(b_0 b_1, (\text{mon}^? b_2 v_4)) \quad v_3 \lesssim \text{add-trace}(b_0 b_1, (\text{mon}^? b_2 v_5))}{\langle v_2, v_3 \rangle \lesssim \text{trace}_v b_0 b_1 (\text{mon} b_2 \langle v_4, v_5 \rangle)}$$

 6005
 6006
 6007 6. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon} b_0 (\text{mon} b_1 (\text{trace}_v \bar{b}_2 (\text{mon} b_3 \langle v_4, v_5 \rangle)))$
 6008 6.1. $v_2 \lesssim \text{mon}^? b_0 (\text{mon}^? b_1 (\text{add-trace}(\bar{b}_2, (\text{mon}^? b_3 v_4))))$
 6009 and $v_3 \lesssim \text{mon}^? b_0 (\text{mon}^? b_1 (\text{add-trace}(\bar{b}_2, (\text{mon}^? b_3 v_5))))$
 6010 by inversion \lesssim
 6011 6.2. $v_2 \lesssim \text{add-trace}(b_0 b_1, (\text{add-trace}(\bar{b}_2, (\text{mon}^? b_3 v_4))))$
 6012 and $v_3 \lesssim \text{add-trace}(b_0 b_1, (\text{add-trace}(\bar{b}_2, (\text{mon}^? b_3 v_5))))$
 6013 by the induction hypothesis
 6014 6.3. QED
 6015
 6016
 6017
 6018
$$\frac{v_2 \lesssim \text{add-trace}(b_0 b_1 \bar{b}_2, (\text{mon}^? b_3 v_4)) \quad v_3 \lesssim \text{add-trace}(b_0 b_1 \bar{b}_2, (\text{mon}^? b_3 v_5))}{\langle v_2, v_3 \rangle \lesssim \text{trace}_v b_0 b_1 \bar{b}_2 (\text{mon} b_3 \langle v_4, v_5 \rangle)}$$

 6019
 6020
 6021 7. CASE $\text{mon} b_2 (\text{mon} b_3 v_2) \lesssim \text{mon} b_0 (\text{mon} b_1 \lambda x_3. e_3)$
 6022 7.1. $v_2 \lesssim \lambda x_3. e_3$
 6023 by inversion \lesssim
 6024 7.2. QED
 6025
 6026
 6027
$$\frac{v_2 \lesssim \lambda x_3. e_3}{\text{mon} b_2 (\text{mon} b_3 v_2) \lesssim (\text{add-trace}(b_0 b_1, \lambda x_3. e_3))}$$

 6028
 6029 8. CASE $\text{mon} b_2 (\text{mon} b_3 v_2) \lesssim \text{mon} b_0 (\text{mon} b_n (\text{trace}_v \bar{b}_6 \lambda x_3. e_3))$
 6030
 6031
 6032

6033 8.1. $v_2 \lesssim (\text{trace}_v \bar{b}_6 \lambda x_3. e_3)$
 6034 by inversion \lesssim
 6035

6036 8.2. QED

6037
 6038
$$\frac{v_2 \lesssim \lambda x_3. e_3}{\text{mon } b_2 (\text{mon } b_3 v_2) \lesssim (\text{add-trace}(b_0 b_1 \bar{b}_6, \lambda x_3. e_3))}$$

 6039

6040 9. CASE $\text{mon } b_2 (\text{mon } b_3 v_2) \lesssim \text{mon } b_0 (\text{mon } b_1 \lambda(x_3 : \tau_3). e_3)$
 6041

6042 9.1. CONTRADICTION:

6043 $\cdot \vdash_A v_1 : \mathcal{U}$
 6044
 6045

□

6046 LEMMA 8.17. *If $\cdot \vdash_N v_0 : \mathcal{U}$ and $\cdot \vdash_A v_1 : \mathcal{U}$ and $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$ and $v_0 \lesssim v_1$ then $\text{tag-match}(\lfloor \tau_0 \rfloor, v_1)$*
 6047

6048 PROOF. By induction on v_1 via case analysis of $v_0 \lesssim v_1$.

6049 1. CASE $i_0 \lesssim i_0$

6050 1.1. $\tau_0 \in \text{Nat} \cup \text{Int}$

6051 by inversion tag-match

6052 1.2. SCASE $\tau_0 = \text{Nat}$

6053 1.2.1. $i_0 \in \mathbb{N}$

6054 by inversion tag-match

6055 1.2.2. QED

6056 $\text{tag-match}(\lfloor \text{Nat} \rfloor, i_0)$

6057 1.3. SCASE $\tau_0 = \text{Int}$

6058 1.3.1. QED

6059 $\text{tag-match}(\lfloor \text{Int} \rfloor, i_0)$

6060 2. CASE $i_0 \lesssim \text{trace}_v \bar{b}_0 i_0$

6061 similar to previous case

6062 3. CASE $\langle v_2, v_3 \rangle \lesssim \langle v_4, v_5 \rangle$

6063 similar to previous case

6064 4. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_1) \langle v_4, v_5 \rangle$

6065 similar to previous case

6066 5. CASE $\langle v_2, v_3 \rangle \lesssim \text{trace}_v \bar{b}_0 \langle v_4, v_5 \rangle$

6067 similar to previous case

6068 6. CASE $\langle v_2, v_3 \rangle \lesssim \text{trace}_v \bar{b}_0 (\text{mon}(\ell_1 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_2) \langle v_4, v_5 \rangle)$

6069 similar to previous case

6070 7. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_1) (\text{trace}_v \bar{b}_2 \langle v_4, v_5 \rangle)$

6071 similar to previous case

6072 8. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_6 \times \tau_6 \blacktriangleleft \ell_1) (\text{mon } b_2 \langle v_4, v_5 \rangle)$

6073 similar to previous case

6074 9. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon } \ell_0 \tau_6 \times \tau_7 \ell_1 (\text{trace}_v \bar{b}_2 (\text{mon } b_3 \langle v_4, v_5 \rangle))$

6075 similar to previous case

6076 10. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}^+ \bar{b}_0 (\text{mon } b_1 (\text{mon } b_2 \langle v_4, v_5 \rangle))$

6077
 6078
 6079
 6080
 6081
 6082
 6083
 6084

- 6085 similar to previous case
 6086 11. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}^+ \bar{b}_0 (\text{mon } b_1 (\text{trace}_v \bar{b}_2 (\text{mon } b_3 \langle v_4, v_5 \rangle)))$
 6087 11.1. $\tau_0 \in \tau \times \tau$
 6088 by inversion *tag-match*
 6089 11.2. QED
 6090 12. CASE $\lambda x_2. e_2 \lesssim \lambda x_2. e_3$
 6091 12.1. $\tau_0 \in \tau \Rightarrow \tau$
 6092 by inversion *tag-match*
 6093 12.2. QED
 6094 13. CASE $\lambda(x_2 : \tau_2). e_2 \lesssim \lambda(x_2 : \tau_2). e_3$
 6095 13.1. CONTRADICTION:
 6096 $\cdot \vdash_{\mathbb{N}} v_0 : \mathcal{U}$
 6097 14. CASE $\text{mon}(\ell_0 \blacktriangleleft \tau_2 \Rightarrow \tau_3 \blacktriangleleft \ell_1) (\text{mon } b_2 v_2) \lesssim \text{trace}_v \bar{b}_3 \lambda x_3. e_3$
 6100 14.1. $\tau_0 \in \tau \Rightarrow \tau$
 6101 by inversion *tag-match*
 6102 14.2. QED
 6103 15. CASE $\text{mon}(\ell_0 \blacktriangleleft \tau_2 \Rightarrow \tau_3 \blacktriangleleft \ell_1) v_2 \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_4 \Rightarrow \tau_5 \blacktriangleleft \ell_1) v_3$
 6104 similar to previous case

□

6105 LEMMA 8.18. *If v_0 and $\text{mon}^2(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_1$ are reduced WF expressions and $\cdot \vdash_{\mathbb{N}} v_0 : \tau_0$ and $\cdot \vdash_{\mathbb{A}} \text{mon}^2(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_1 : \tau_1$ and $\tau_0 \leq \tau_1$ and $v_0 \lesssim \text{mon}^2(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_1$ then $\text{dyn}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_1 \rightarrow_{\mathbb{A}}^* v_2$ and $v_0 \lesssim v_2$.*

6112 PROOF. By case analysis of τ_0 and \lesssim .

- 6113 1. CASE $\tau_0 = \text{Nat}$
 6114 1.1. $v_0 \in \mathbb{N}$
 6115 by inversion $\vdash_{\mathbb{N}}$
 6116 1.2. SCASE $v_0 \lesssim v_0$
 6117 1.2.1. QED
 6118 $\text{dyn}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_1 \triangleright_{\mathbb{A}} v_0 = \text{mon}^2(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_1$
 6119 1.3. SCASE $v_0 \lesssim \text{trace}_v \bar{b}_0 v_0$
 6120 1.3.1. QED
 6121 $\text{dyn}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_1 \triangleright_{\mathbb{A}} \text{trace}_v \bar{b}_0 v_0 = \text{mon}^2(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_1$
 6122 2. CASE $\tau_0 = \text{Int}$
 6123 similar to previous case
 6124 3. CASE $\tau_0 = \tau_1 \times \tau_2$
 6125 3.1. $v_0 \in \langle v, v \rangle$
 6126 by inversion $\vdash_{\mathbb{N}}$
 6127 3.2. SCASE $v_0 \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) \langle v_2, v_3 \rangle$
 6128 3.2.1. QED
 6129 $\text{dyn}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) v_1 \triangleright_{\mathbb{A}} \text{mon}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) v_1$
 6130 3.3. SCASE $v_0 \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) (\text{trace}_v \bar{b}_2 \langle v_2, v_3 \rangle)$
 6131
 6132
 6133
 6134
 6135
 6136

3.3.1. QED

$$\text{dyn}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) v_1 \triangleright_{\Lambda} \text{mon}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) v_1$$

$$3.4. \text{ SCASE } v_0 \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) (\text{mon } b_2 \langle v_2, v_3 \rangle)$$

3.4.1. QED

$$\text{dyn}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) v_1 \triangleright_{\Lambda} \text{mon}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) v_1$$

$$3.5. \text{ SCASE } v_0 \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) (\text{trace}_v \bar{b}_2 (\text{mon } b_3 \langle v_2, v_3 \rangle))$$

3.5.1. CONTRADICTION:

$$\cdot \vdash_{\Lambda} v_1 : \mathcal{U}$$

$$3.6. \text{ SCASE } v_0 \lesssim \text{mon}^+ \bar{b}_0 (\text{mon } b_1 (\text{mon } b_2 \langle v_2, v_3 \rangle))$$

3.6.1. CONTRADICTION:

by lemma 7.10

$$3.7. \text{ SCASE } v_0 \lesssim \text{mon}^+ \bar{b}_0 (\text{mon } b_1 (\text{trace}_v \bar{b}_2 (\text{mon } b_3 \langle v_2, v_3 \rangle)))$$

3.7.1. CONTRADICTION:

by lemma 7.10

4. CASE $\tau_0 = \tau_1 \Rightarrow \tau_2$

$$4.1. v_0 \in \lambda x. e \cup \text{mon } b v$$

by inversion $\vdash_{\mathbb{N}}$

$$4.2. \text{ SCASE } v_0 \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_3 \Rightarrow \tau_4 \blacktriangleleft \ell_1) v_1$$

4.2.1. QED

$$\text{dyn}(\ell_0 \blacktriangleleft \tau_3 \Rightarrow \tau_4 \blacktriangleleft \ell_1) v_1 \triangleright_{\Lambda} \text{mon}(\ell_0 \blacktriangleleft \tau_3 \Rightarrow \tau_4 \blacktriangleleft \ell_1) v_1$$

□

LEMMA 8.19. *If v_0 and $\text{mon}^?(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3) v_1)$ are reduced WF expressions and $\cdot \vdash_{\mathbb{N}} v_0 : \tau_0$ and $\cdot \vdash_{\Lambda} \text{mon}^?(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3) v_1) : \tau_1$ and $v_0 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3) v_1)$ and $\tau_0 \leq \tau_1$ then $\text{dyn}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{stat}(\ell_2 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3) v_1) \rightarrow_{\Lambda}^* v_2$ and $v_0 \lesssim v_2$.*

PROOF. By case analysis of τ_0 and \lesssim .

1. CASE $\tau_0 = \text{Nat}$

$$1.1. v_0 \in \bar{\mathbb{N}}$$

by inversion $\vdash_{\mathbb{N}}$

$$1.2. \text{ SCASE } v_0 \lesssim v_0$$

1.2.1. QED

$$\text{dyn}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{stat}(\ell_2 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3) v_1) \rightarrow_{\Lambda}^* v_0 = \text{mon}^?(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3) v_1)$$

$$1.3. \text{ SCASE } v_0 \lesssim \text{trace}_v \bar{b}_0 v_0$$

1.3.1. QED

$$\text{dyn}(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{stat}(\ell_2 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3) v_1) \rightarrow_{\Lambda}^* v_0 = \text{mon}^?(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3) v_1)$$

2. CASE $\tau_0 = \text{Int}$

similar to previous case

3. CASE $\tau_0 = \tau_1 \times \tau_2$

$$3.1. v_0 \in \langle v, v \rangle$$

by inversion $\vdash_{\mathbb{N}}$

$$3.2. \text{ SCASE } v_0 \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) \langle v_2, v_3 \rangle$$

6189 3.2.1. QED
6190 $\text{dyn}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) v_1 \triangleright_{\Lambda} \text{mon}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) v_1$
6191 3.3. SCASE $v_0 \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) (\text{trace}_{\sqrt{}} \bar{b}_2 \langle v_2, v_3 \rangle)$
6192 3.3.1. QED
6193 $\text{dyn}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) v_1 \triangleright_{\Lambda} \text{mon}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) v_1$
6194 3.4. SCASE $v_0 \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) (\text{mon}(\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3) \langle v_2, v_3 \rangle)$
6195 3.4.1. $\tau_5 \in \tau \times \tau$
6196 by inversion \vdash_A
6197 3.4.2. QED
6200 $\text{dyn}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) (\text{stat}(\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3) \langle v_2, v_3 \rangle) \triangleright_{\Lambda} \text{mon}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) (\text{mon}(\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3) \langle v_2, v_3 \rangle)$
6201 3.5. SCASE $v_0 \lesssim \text{mon}^+ \bar{b}_0 (\text{mon } b_1 (\text{mon } b_2 \langle v_2, v_3 \rangle))$
6202 3.5.1. CONTRADICTION:
6203 by lemma 7.9
6204 3.6. SCASE $v_0 \lesssim \text{mon}^+ \bar{b}_0 (\text{mon } b_1 (\text{trace}_{\sqrt{}} \bar{b}_2 (\text{mon } b_3 \langle v_2, v_3 \rangle)))$
6205 3.6.1. CONTRADICTION:
6206 by lemma 7.9
6207 4. CASE $\tau_0 = \tau_1 \Rightarrow \tau_2$
6208 4.1. $v_0 \in \lambda x. e \cup \text{mon } b v$
6209 by inversion \vdash_N
6210 4.2. SCASE $v_0 \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_3 \Rightarrow \tau_4 \blacktriangleleft \ell_1) (\text{mon}(\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3) v_1)$
6211 4.2.1. $\tau_5 \in \tau \Rightarrow \tau$
6212 by inversion \vdash_A
6213 4.2.2. QED
6214 $\text{mon}(\ell_0 \blacktriangleleft \tau_3 \Rightarrow \tau_4 \blacktriangleleft \ell_1) (\text{mon}(\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3) v_1) \triangleright_{\Lambda} \text{mon}(\ell_0 \blacktriangleleft \tau_3 \Rightarrow \tau_4 \blacktriangleleft \ell_1) (\text{mon}(\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3) v_1)$
6215
6216
6217
6218
6219
6220 □

6221 LEMMA 8.20. *If $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$ and $\tau_0 \leq \tau_1$ then $\text{tag-match}(\lfloor \tau_1 \rfloor, v_0)$.*
6222
6223 PROOF. By induction on v_0 via case analysis of $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$.
6224 1. CASE $\text{tag-match}(\lfloor \text{Nat} \rfloor, i_0)$
6225 1.1. QED
6226 $\tau_1 \in \text{Nat} \cup \text{Int}$
6227 2. CASE $\text{tag-match}(\lfloor \text{Int} \rfloor, i_0)$
6228 2.1. QED
6229 $\tau_1 \in \text{Int}$
6230 3. CASE $\text{tag-match}(\lfloor \tau_2 \Rightarrow \tau_3 \rfloor, \lambda x_0. e_0)$
6231 3.1. QED
6232 $\tau_1 \in \tau \Rightarrow \tau$
6233 4. CASE $\text{tag-match}(\lfloor \tau_3 \Rightarrow \tau_4 \rfloor, \lambda(x_0 : \tau_2). e_0)$
6234 4.1. QED
6235 $\tau_1 \in \tau \Rightarrow \tau$
6236 5. CASE $\text{tag-match}(\lfloor \tau_2 \times \tau_3 \rfloor, \langle v_2, v_3 \rangle)$
6237
6238
6239
6240

- 6241 5.1. QED
6242 $\tau_1 \in \tau \times \tau$
6243 6. CASE $\text{tag-match}(\lfloor \tau_4 \Rightarrow \tau_5 \rfloor, \text{mon}(\ell_0 \blacktriangleleft \tau_2 \Rightarrow \tau_3 \blacktriangleleft \ell_1) v_1)$
6244 6.1. QED
6245 $\tau_1 \in \tau \Rightarrow \tau$
6246 7. CASE $\text{tag-match}(\lfloor \tau_4 \times \tau_5 \rfloor, \text{mon}(\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1) v_1)$
6247 7.1. QED
6248 $\tau_1 \in \tau \times \tau$
6249 8. CASE $\text{tag-match}(\lfloor \tau_0 \rfloor, \text{trace}_v \bar{b}_0 v_1)$
6250 8.1. $\text{tag-match}(\lfloor \tau_0 \rfloor, v_1)$
6251 by inversion tag-match
6252 8.2. $\text{tag-match}(\lfloor \tau_1 \rfloor, v_1)$
6253 by the induction hypothesis
6254 8.3. QED
6255 8.4. QED
6256 by definition tag-match

□

LEMMA 8.21. If $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$ and $\cdot \vdash_{\Lambda} v_0 : \mathcal{U}$ then $\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \triangleright_{\Lambda} \text{mon}^2(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0$.

PROOF. By case analysis of $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$.

- 6266 1. CASE $\text{tag-match}(\lfloor \text{Nat} \rfloor, n_0)$
6267 1.1. QED
6268 $\text{dyn}(\ell_0 \blacktriangleleft \text{Nat} \blacktriangleleft \ell_1) n_0 \triangleright_{\Lambda} n_0$
6269 2. CASE $\text{tag-match}(\lfloor \text{Int} \rfloor, i_0)$
6270 2.1. QED
6271 $\text{dyn}(\ell_0 \blacktriangleleft \text{Int} \blacktriangleleft \ell_1) i_0 \triangleright_{\Lambda} i_0$
6272 3. CASE $\text{tag-match}(\lfloor \tau_1 \Rightarrow \tau_2 \rfloor, \lambda x_0. e_0)$
6273 3.1. QED
6274 $\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \triangleright_{\Lambda} \text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0$
6275 4. CASE $\text{tag-match}(\lfloor \tau_2 \Rightarrow \tau_3 \rfloor, \lambda(x_0 : \tau_1). e_0)$
6276 4.1. CONTRADICTION:
6277 $\cdot \vdash_{\Lambda} v_0 : \mathcal{U}$
6278 5. CASE $\text{tag-match}(\lfloor \tau_1 \times \tau_2 \rfloor, \langle v_1, v_2 \rangle)$
6279 $\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \triangleright_{\Lambda} \text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0$
6280 6. CASE $\text{tag-match}(\lfloor \tau_3 \Rightarrow \tau_4 \rfloor, \text{mon}(\ell_0 \blacktriangleleft \tau_1 \Rightarrow \tau_2 \blacktriangleleft \ell_1) v_1)$
6281 $\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \triangleright_{\Lambda} \text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0$
6282 7. CASE $\text{tag-match}(\lfloor \tau_3 \times \tau_4 \rfloor, \text{mon}(\ell_0 \blacktriangleleft \tau_1 \times \tau_2 \blacktriangleleft \ell_1) v_1)$
6283 $\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \triangleright_{\Lambda} \text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0$
6284 8. CASE $\text{tag-match}(\lfloor \tau_1 \rfloor, \text{trace}_v \bar{b}_0 v_1)$
6285 8.1. $\text{tag-match}(\lfloor \tau_1 \rfloor, v_1)$
6286 by inversion tag-match

6293 8.2. $\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_1 \triangleright_{\mathbb{A}} \text{mon}^?(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_1$
 6294 by the induction hypothesis

6295 8.3. QED
 6296 $\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \text{trace}_v \bar{b}_0 v_1 \triangleright_{\mathbb{A}} \text{mon}^?(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{trace}_v \bar{b}_0 v_1)$

6298

□

6300 **LEMMA 8.22.** *If v_0 and v_1 are reduced WF expressions and $\text{unop}\{\tau?\} v_0 \lesssim \text{unop}\{\tau?\} v_1$ and $\cdot \vdash_N \text{unop}\{\tau?\} v_0 : \tau_0$ and*
 6301 *$\cdot \vdash_{\mathbb{A}} \text{unop}\{\tau?\} v_1 : \tau_1$ and $\tau_0 \leq \tau_1$ then $\text{unop}\{\tau?\} v_1 \rightarrow_{\mathbb{A}}^* v_2$ and $\delta_N(\text{unop}, v_0) \lesssim v_2$*

6302
 6303 **PROOF.** By induction on v_1 via case analysis of \lesssim . The cases for $\text{unop} = \text{fst}\{\tau_2\}$ and $\text{unop} = \text{snd}\{\tau_2\}$ are analogous;
 6304 we show only the fst cases.

6305 1. CASE $i_0 \lesssim i_0$

6306 1.1. CONTRADICTION:

6307 $\cdot \vdash_N \text{unop}\{\tau?\} v_0 : \tau_0$

6308 2. CASE $i_0 \lesssim \text{trace}_v \bar{b}_0 i_0$

6309 2.1. CONTRADICTION:

6310 $\cdot \vdash_N \text{unop}\{\tau?\} v_0 : \tau_0$

6311 3. CASE $\langle v_2, v_3 \rangle \lesssim \langle v_4, v_5 \rangle$

6312 3.1. $v_2 \lesssim v_4$

6313 by inversion \lesssim

6314 3.2. $\delta_N(\text{unop}, v_0) = v_2$

6315 by definition δ_N

6316 3.3. $\text{unop}\{\tau?\} v_1 \triangleright_{\mathbb{A}} v_4$

6317 by definition $\triangleright_{\mathbb{A}}$

6318 3.4. QED

6319 by 3.1

6320 4. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) \langle v_4, v_5 \rangle$

6321 4.1. $v_2 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_4$

6322 by inversion \lesssim and $\tau_3 \leq \tau_2$

6323 4.2. $\delta_N(\text{unop}, v_0) = v_2$

6324 by definition δ_N

6325 4.3. $\text{unop}\{\tau?\} v_1$

6326 $\rightarrow_{\mathbb{A}} \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{unop}\{\tau?\} \langle v_4, v_5 \rangle)$

6327 $\rightarrow_{\mathbb{A}} \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_4$

6328 by definition $\rightarrow_{\mathbb{A}}$

6329 4.4. QED

6330 by lemma 8.18

6331 5. CASE $\langle v_2, v_3 \rangle \lesssim \text{trace}_v \bar{b}_0 \langle v_4, v_5 \rangle$

6332 5.1. CONTRADICTION:

6333 $\cdot \vdash_{\mathbb{A}} v_1 : \tau_1$

6334 6. CASE $\langle v_2, v_3 \rangle \lesssim \text{trace}_v \bar{b}_0 (\text{mon}(\ell_1 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_2) \langle v_4, v_5 \rangle)$

6335 6.1. CONTRADICTION:

6336

6345 $\cdot \vdash_A v_1 : \tau_1$

6346 7. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) (\text{trace}_v \bar{b}_2 \langle v_4, v_5 \rangle)$

6347 7.1. $v_2 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, v_4))$

6348 by inversion \lesssim and $\tau_3 \leq \tau_2$

6349 7.2. $\delta_N(\text{unop}, v_0) = v_2$

6350 by definition δ_N

6351 7.3. $\text{unop}\{\tau?\} v_1$

6352 $\rightarrow_A \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{unop}\{\tau?\} (\text{trace}_v \bar{b}_2 \langle v_4, v_5 \rangle))$

6353 $\rightarrow_A \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, v_4))$

6354 by definition \rightarrow_A

6355 7.4. QED

6356 by lemma 8.18

6357 8. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1) (\text{mon}(\ell_2 \blacktriangleleft \tau_5 \times \tau_6 \blacktriangleleft \ell_3) \langle v_4, v_5 \rangle)$

6358 8.1. $v_2 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{mon}^?(\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3) v_4)$

6359 by inversion \lesssim and $\tau_3 \leq \tau_2$

6360 8.2. $\delta_N(\text{unop}, v_0) = v_2$

6361 by definition δ_N

6362 8.3. $\text{unop}\{\tau?\} v_1$

6363 $\rightarrow_A \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{unop}\{\tau?\} (\text{mon}(\ell_2 \blacktriangleleft \tau_5 \times \tau_6 \blacktriangleleft \ell_3) \langle v_4, v_5 \rangle))$

6364 $\rightarrow_A \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{stat}(\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3) (\text{unop}\{\tau?\} \langle v_4, v_5 \rangle))$

6365 $\rightarrow_A \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{stat}(\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3) v_4)$

6366 $\rightarrow_A \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{stat}(\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3) v_4)$

6367 by definition \rightarrow_A

6368 8.4. QED

6369 by lemma 8.19

6370 9. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1) (\text{trace}_v \bar{b}_2 (\text{mon}(\ell_3 \blacktriangleleft \tau_4 \times \tau_5 \blacktriangleleft \ell_4) \langle v_4, v_5 \rangle))$

6371 9.1. $v_2 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{add-trace}(\bar{b}_2, (\text{mon}^?(\ell_3 \blacktriangleleft \tau_4 \blacktriangleleft \ell_4) v_4)))$

6372 by inversion \lesssim and $\tau_3 \leq \tau_2$

6373 9.2. $\delta_N(\text{unop}, v_0) = v_2$

6374 by definition δ_N

6375 9.3. $\text{unop}\{\tau?\} v_1$

6376 $\rightarrow_A \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{unop}\{\tau?\} (\text{trace}_v \bar{b}_2 (\text{mon}(\ell_3 \blacktriangleleft \tau_4 \times \tau_5 \blacktriangleleft \ell_4) \langle v_4, v_5 \rangle)))$

6377 $\rightarrow_A \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{trace} \bar{b}_2 (\text{stat}(\ell_3 \blacktriangleleft \tau_4 \blacktriangleleft \ell_4) (\text{unop}\{\tau?\} \langle v_4, v_5 \rangle)))$

6378 $\rightarrow_A \text{dyn}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{trace} \bar{b}_2 (\text{stat}(\ell_3 \blacktriangleleft \tau_4 \blacktriangleleft \ell_4) v_4))$

6379 by definition \rightarrow_A

6380 9.4. QED

6381 by lemma 8.19

6382 10. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}^+ \bar{b}_0 (\text{mon } b_1 (\text{mon } b_2 \langle v_4, v_5 \rangle))$

6383 10.1. CONTRADICTION:

6384 by lemma 7.9

6385 11. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}^+ \bar{b}_0 (\text{mon } b_1 (\text{trace}_v \bar{b}_2 (\text{mon } b_3 \langle v_4, v_5 \rangle)))$

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6397 11.1. CONTRADICTION:

6398 by lemma 7.9

6399 12. CASE $\lambda x_2. e_2 \lesssim \lambda x_2. e_3$

6400 12.1. CONTRADICTION:

6402 $\cdot \vdash_N \text{unop}\{\tau?\} v_0 : \tau_0$

6403 13. CASE $\lambda(x_2 : \tau_2). e_2 \lesssim \lambda(x_2 : \tau_2). e_3$

6404 13.1. CONTRADICTION:

6406 $\cdot \vdash_N \text{unop}\{\tau?\} v_0 : \tau_0$

6407 14. CASE $\text{mon}(\ell_0 \blacktriangleleft \tau_2 \Rightarrow \tau_3 \blacktriangleleft \ell_1) (\text{mon } b_2 v_2) \lesssim \text{trace}_v \bar{b}_3 \lambda x_3. e_3$

6408 14.1. CONTRADICTION:

6409 $\cdot \vdash_N \text{unop}\{\tau?\} v_0 : \tau_0$

6411 15. CASE $\text{mon}(\ell_0 \blacktriangleleft \tau_2 \Rightarrow \tau_3 \blacktriangleleft \ell_1) v_2 \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_4 \Rightarrow \tau_5 \blacktriangleleft \ell_1) v_3$

6412 15.1. CONTRADICTION:

6413 $\cdot \vdash_N \text{unop}\{\tau?\} v_0 : \tau_0$

6414

6415

□

6416

6417 **LEMMA 8.23.** *If v_0 and v_1 are reduced WF expressions and $\text{unop}\{\tau?\} v_0 \lesssim \text{unop}\{\tau?\} v_1$ and $\cdot \vdash_N \text{unop}\{\tau?\} v_0 : \mathcal{U}$ and*

6418 *$\cdot \vdash_A \text{unop}\{\tau?\} v_1 : \mathcal{U}$ then $\text{unop}\{\tau?\} v_0 \blacktriangleright_N e_2$ and $\text{unop}\{\tau?\} v_1 \rightarrow_A^* e_3$ and $e_2 \lesssim e_3$.*

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PROOF. By cases on $v_0 \lesssim v_1$. The cases for $\text{unop} = \text{fst}\{\tau\}$ and $\text{unop} = \text{snd}\{\tau\}$ are similar; we present only the fst cases.

6422

1. CASE $i_0 \lesssim i_0$

6423

1.1. QED

6424

$\text{unop}\{\tau?\} v_0 \blacktriangleright_N \text{TagErr} \bullet$ and $\text{unop}\{\tau?\} v_1 \blacktriangleright_A \text{TagErr} \bullet$

6425

2. CASE $i_0 \lesssim \text{trace}_v \bar{b}_0 i_0$

6426

2.1. QED

6428

$\text{unop}\{\tau?\} v_0 \blacktriangleright_N \text{TagErr} \bullet$ and $\text{unop}\{\tau?\} v_1 \blacktriangleright_A \text{TagErr} \bullet$

6429

3. CASE $\langle v_2, v_3 \rangle \lesssim \langle v_4, v_5 \rangle$

6430

3.1. $v_2 \lesssim v_4$

6431

by inversion \lesssim

6432

3.2. QED

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$\text{unop}\{\tau?\} v_0 \blacktriangleright_N v_2$ and $\text{unop}\{\tau?\} v_1 \blacktriangleright_A v_4$

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4. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1) \langle v_4, v_5 \rangle$

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4.1. $v_2 \lesssim \text{mon}^?(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_4$

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by inversion \lesssim

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4.2. $\text{unop}\{\tau?\} v_0 \blacktriangleright_N v_2$

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by definition \blacktriangleright_N

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4.3. $\text{unop}\{\tau?\} v_1$

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$\rightarrow_A \text{stat}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{unop}\{\tau?\} \langle v_4, v_5 \rangle)$

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$\rightarrow_A \text{stat}(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) v_4$

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by definition \rightarrow_A

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4.4. QED

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6449 by lemma 8.24
6450 5. CASE $\langle v_2, v_3 \rangle \lesssim \text{trace}_v \bar{b}_0 \langle v_4, v_5 \rangle$
6451 5.1. $v_2 \lesssim \text{add-trace}(\bar{b}_0, v_4)$
6452 by inversion \lesssim
6453 5.2. $\text{unop}\{\tau?\} v_0 \blacktriangleright_N v_2$
6454 by definition \blacktriangleright_N
6455 5.3. $\text{unop}\{\tau?\} v_1 \blacktriangleright_A \text{add-trace}(\bar{b}_0, v_4)$
6456 by definition \blacktriangleright_A
6457 5.4. QED
6458 by 5.1
6459 6. CASE $\langle v_2, v_3 \rangle \lesssim \text{trace}_v \bar{b}_0 (\text{mon}(\ell_1 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_2) \langle v_4, v_5 \rangle)$
6460 6.1. $v_2 \lesssim \text{add-trace}(\bar{b}_0, (\text{mon}^?(\ell_1 \blacktriangleleft \tau_2 \blacktriangleleft \ell_2) v_4))$
6461 by inversion \lesssim
6462 6.2. $\text{unop}\{\tau?\} v_0 \blacktriangleright_N v_2$
6463 by definition \blacktriangleright_N
6464 6.3. $\text{unop}\{\tau?\} v_1$
6465 $\rightarrow_A \text{trace} \bar{b}_0 (\text{stat}(\ell_1 \blacktriangleleft \tau_2 \blacktriangleleft \ell_2) (\text{unop}\{\tau?\} \langle v_4, v_5 \rangle))$
6466 $\rightarrow_A \text{trace} \bar{b}_0 (\text{stat}(\ell_1 \blacktriangleleft \tau_2 \blacktriangleleft \ell_2) v_4)$
6467 $\rightarrow_A \text{trace} \bar{b}_0 (\text{mon}^?(\ell_1 \blacktriangleleft \tau_2 \blacktriangleleft \ell_2) v_4)$
6468 $\rightarrow_A \text{add-trace}(\bar{b}_0, (\text{mon}^?(\ell_1 \blacktriangleleft \tau_2 \blacktriangleleft \ell_2) v_4))$
6469 by definition \rightarrow_A and lemma 8.24
6470 6.4. QED
6471 7. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1) (\text{trace}_v \bar{b}_2 \langle v_4, v_5 \rangle)$
6472 7.1. CONTRADICTION:
6473 $\cdot \vdash_A v_1 : \mathcal{U}$
6474 8. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1) (\text{mon}(\ell_2 \blacktriangleleft \tau_4 \times \tau_5 \blacktriangleleft \ell_3) \langle v_4, v_5 \rangle)$
6475 8.1. CONTRADICTION:
6476 by lemma 7.10
6477 9. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon} \ell_0 \tau_2 \times \tau_3 \ell_1 (\text{trace}_v \bar{b}_2 (\text{mon} b_3 \langle v_4, v_5 \rangle))$
6478 9.1. CONTRADICTION:
6479 by lemma 7.10
6480 10. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}^+ \bar{b}_0 (\text{mon} b_1 (\text{mon} b_2 \langle v_4, v_5 \rangle))$
6481 10.1. CONTRADICTION:
6482 by lemma 7.10
6483 11. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}^+ \bar{b}_0 (\text{mon} b_1 (\text{trace}_v \bar{b}_2 (\text{mon} b_3 \langle v_4, v_5 \rangle)))$
6484 11.1. CONTRADICTION:
6485 by lemma 7.10
6486 12. CASE $\lambda x_2. e_2 \lesssim \lambda x_2. e_3$
6487 12.1. QED
6488 $\text{unop}\{\tau?\} v_0 \blacktriangleright_N \text{TagErr} \bullet$ and $\text{unop}\{\tau?\} v_1 \blacktriangleright_A \text{TagErr} \bullet$
6489 13. CASE $\lambda(x_2 : \tau_2). e_2 \lesssim \lambda(x_2 : \tau_2). e_3$
6490 2019-10-03 17:26. Page 125 of 1–148.

6501 13.1. CONTRADICTION:

6502 $\cdot \vdash_N v_0 : \mathcal{U}$

6503 14. CASE $\text{mon}(\ell_0 \blacktriangleleft \tau_2 \Rightarrow \tau_3 \blacktriangleleft \ell_1) (\text{mon } b_2 v_2) \lesssim \text{trace}_v \bar{b}_3 \lambda x_3. e_3$

6504 14.1. QED

6505 $\text{unop}\{\tau?\} v_0 \blacktriangleright_N \text{TagErr} \bullet$ and $\text{unop}\{\tau?\} v_1 \blacktriangleright_A \text{TagErr} \bullet$

6506 15. CASE $\text{mon}(\ell_0 \blacktriangleleft \tau_2 \Rightarrow \tau_3 \blacktriangleleft \ell_1) v_2 \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_4 \Rightarrow \tau_5 \blacktriangleleft \ell_1) v_3$

6507 15.1. QED

6508 $\text{unop}\{\tau?\} v_0 \blacktriangleright_N \text{TagErr} \bullet$ and $\text{unop}\{\tau?\} v_1 \blacktriangleright_A \text{TagErr} \bullet$

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LEMMA 8.24. *If v_0 and v_1 are reduced WF expressions and $\cdot \vdash_N v_0 : \mathcal{U}$ and $\cdot \vdash_A v_1 : \tau_0$ and $v_0 \lesssim \text{mon}^2(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_1$ then $\text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_1 \rightarrow_A^* v_2$ and $v_0 \lesssim v_2$.*

PROOF. By case analysis of \lesssim .

1. CASE $i_0 \lesssim i_0$

1.1. QED

$\text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_1 \blacktriangleright_A v_1$

2. CASE $i_0 \lesssim \text{trace}_v \bar{b}_2 i_0$

2.1. QED

$\text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_1 \blacktriangleright_A v_0$

3. CASE $\langle v_3, v_4 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \langle v_5, v_6 \rangle$

3.1. QED

$\text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_1 \blacktriangleright_A \text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_1$

4. CASE $\langle v_3, v_4 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{trace}_v \bar{b}_2 \langle v_5, v_6 \rangle)$

4.1. CONTRADICTION:

$\cdot \vdash_A v_1 : \tau_0$

5. CASE $\langle v_3, v_4 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{mon } b_2 \langle v_5, v_6 \rangle)$

5.1. $v_3 \lesssim \text{mon}^2(\ell_0 \blacktriangleleft \text{fst}(\tau_0) \blacktriangleleft \ell_1) (\text{mon } \text{fst}(b_2) v_5)$

and $v_4 \lesssim \text{mon}^2(\ell_0 \blacktriangleleft \text{snd}(\tau_0) \blacktriangleleft \ell_1) (\text{mon } \text{snd}(b_2) v_6)$

by inversion \lesssim

5.2. $\text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{mon } b_2 \langle v_5, v_6 \rangle) \blacktriangleright_A \text{trace}_v(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) b_2 \langle v_5, v_6 \rangle$

by definition \blacktriangleright_A

6. CASE $\langle v_3, v_4 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{trace}_v \bar{b}_2 (\text{mon } b_3 \langle v_4, v_5 \rangle))$

6.1. CONTRADICTION:

$\cdot \vdash_A v_1 : \tau_0$

7. CASE $\langle v_3, v_4 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{mon } b_2 (\text{trace}_v^2 \bar{b}_3 (\text{mon } b_4 \langle v_4, v_5 \rangle)))$

7.1. $\text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_1 \blacktriangleright_A \text{trace}_v(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) b_2 \bar{b}_3 (\text{mon } b_4 \langle v_4, v_5 \rangle)$

by definition \blacktriangleright_A

7.2. QED

by lemma 8.25

8. CASE $\langle v_3, v_4 \rangle \lesssim \text{mon}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{mon } b_2 (\text{mon } b_3 (\text{trace}_v^2 \bar{b}_4 (\text{mon } b_5 \langle v_4, v_5 \rangle))))$

8.1. CONTRADICTION:

6553 by lemma 7.10

6554 9. CASE $\text{mon } b_2 (\lambda x_2. e_2) \lesssim \text{mon } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\lambda x_2. e_3)$

6555 9.1. CONTRADICTION:

6556 $\cdot \vdash_A v_1 : \tau_0$

6558 10. CASE $\text{mon } b_2 (\lambda(x_2 : \tau_2). e_2) \lesssim \text{mon } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\lambda(x_2 : \tau_2). e_3)$

6559 10.1. QED

6560 $\text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_1 \blacktriangleright_A \text{mon } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_1$

6561 11. CASE $\text{mon } b_2 (\text{mon } b_3 (\lambda x_2. e_2)) \lesssim \text{mon } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{mon } b_4 (\text{trace}_v^? \bar{b}_5 (\lambda x_2. e_3)))$

6562 11.1. $\text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_1 \blacktriangleright_N \text{add-trace}((\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) b_4 \bar{b}_5, (\lambda x_2. e_3))$

6563 by definition \blacktriangleright_N

6564 11.2. QED

6565 by lemma 8.25

6566 12. CASE $\text{mon } b_2 (\text{mon } b_3 (\text{mon } b_4 (\lambda x_2. e_2))) \lesssim \text{mon } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{mon } b_5 (\text{trace}_v^? \bar{b}_6 (\text{mon } b_7 (\lambda(x_2 : \tau_2). e_3))))$

6567 12.1. $\text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_1 \blacktriangleright_N \text{add-trace}((\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) b_4 b_5 \bar{b}_6, (\text{mon } b_7 (\lambda(x_2 : \tau_2). e_3)))$

6568 by definition \blacktriangleright_N

6569 12.2. QED

6570 by lemma 8.25

6571 13. CASE $\text{mon } b_2 (\text{mon } b_3 (\text{mon } b_4 (\text{mon } b_5 (\lambda x_2. e_2)))) \lesssim \text{mon } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{mon } b_5 (\text{mon } b_6 (\text{mon } b_7 v_2)))$

6572 13.1. CONTRADICTION:

6573 by lemma 7.10

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LEMMA 8.25. If v_0 and v_1 are reduced WF expressions and $\cdot \vdash_N v_0 : \mathcal{U}$ and $\cdot \vdash_A v_1 : \mathcal{U}$ and $v_0 \lesssim \text{mon}^? b_0 (\text{mon}^? b_1 v_1)$ then $v_0 \lesssim \text{add-trace}(b_0 b_1, v_1)$.

PROOF. By induction on v_0 via case analysis of \lesssim .

1. CASE $i_0 \lesssim i_0$

1.1. QED

$i_0 \lesssim \text{trace}_v b_0 b_1 i_0$

2. CASE $i_0 \lesssim \text{trace}_v \bar{b}_2 i_0$

2.1. QED

$i_0 \lesssim \text{trace}_v b_0 b_1 \bar{b}_2 i_0$

3. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon } b_0 (\text{mon } b_1 \langle v_4, v_5 \rangle)$

3.1. $v_2 \lesssim \text{mon}^? \text{fst}(b_0) (\text{mon}^? \text{fst}(b_1) v_4)$

and $v_3 \lesssim \text{mon}^? \text{snd}(b_0) (\text{mon}^? \text{snd}(b_1) v_5)$

by inversion \lesssim

3.2. $v_2 \lesssim \text{add-trace}(\text{fst}(b_0) \text{fst}(b_1), v_4)$

and $v_3 \lesssim \text{add-trace}(\text{snd}(b_0) \text{snd}(b_1), v_5)$

by the induction hypothesis

3.3. QED

4. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon } b_0 (\text{mon } b_1 (\text{trace}_v^? \bar{b}_2 (\text{mon } b_3 \langle v_4, v_5 \rangle)))$

- 6605 4.1. $v_2 \lesssim \text{mon}^? \text{fst}(b_0) (\text{mon}^? \text{fst}(b_1) (\text{add-trace}(\text{fst}(\bar{b}_2), (\text{mon } b_3 v_4))))$
 6606 and $v_3 \lesssim \text{mon}^? \text{snd}(b_0) (\text{mon}^? \text{snd}(b_1) (\text{add-trace}(\text{snd}(\bar{b}_2), (\text{mon } b_3 v_5))))$
 6607 by inversion \lesssim
 6608 4.2. $v_2 \lesssim \text{add-trace}(\text{fst}(b_0)\text{fst}(b_1), (\text{add-trace}(\text{fst}(\bar{b}_2), (\text{mon } b_3 v_4))))$
 6609 and $v_3 \lesssim \text{add-trace}(\text{snd}(b_0)\text{snd}(b_1), (\text{add-trace}(\text{snd}(\bar{b}_2), (\text{mon } b_3 v_5))))$
 6610 by the induction hypothesis
 6611 4.3. QED
 6612 5. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon } b_0 (\text{mon } b_1 (\text{mon } b_2 (\text{trace}_v^? \bar{b}_3 (\text{mon } b_4 \langle v_4, v_5 \rangle))))$
 6613 5.1. CONTRADICTION:
 6614 by lemma 7.10
 6615 6. CASE $\text{mon } b_2 (\text{mon } b_3 (\lambda x_2. e_2)) \lesssim \text{mon } b_0 (\text{mon } b_1 (\lambda x_2. e_3))$
 6616 6.1. $\lambda x_2. e_2 \lesssim \lambda x_2. e_3$
 6617 by inversion \lesssim
 6618 6.2. QED
 6619 by definition \lesssim
 6620 7. CASE $\text{mon } b_2 (\text{mon } b_3 (\text{mon } b_4 (\lambda(x_2:\tau_2). e_2))) \lesssim \text{mon } b_0 (\text{mon } b_1 (\text{mon } b_5 (\lambda x_2. e_3)))$
 6621 7.1. $\lambda(x_2:\tau_2). e_2 \lesssim \lambda(x_2:\tau_2). e_3$
 6622 by inversion \lesssim
 6623 7.2. QED
 6624 by definition \lesssim
 6625 8. CASE $\text{mon } b_2 (\text{mon } b_3 (\text{mon } b_4 (\text{mon } b_5 v_2))) \lesssim \text{mon } b_0 (\text{mon } b_1 (\text{mon } b_6 (\text{mon } b_7 v_3)))$
 6626 8.1. CONTRADICTION:
 6627 by lemma 7.10
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6636 LEMMA 8.26. *If $\text{app}\{\mathcal{U}\} v_0 v_1$ and $\text{app}\{\mathcal{U}\} (\text{trace}_v b_0 \bar{b}_1 v_2) v_3$ are reduced WF expressions and $\text{app}\{\mathcal{U}\} v_0 v_1 \lesssim$
 6637 $\text{app}\{\mathcal{U}\} (\text{trace}_v b_0 \bar{b}_1 v_2) v_3$ then one of the following holds:*
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- 6639 • $\text{app}\{\mathcal{U}\} v_0 v_1 \rightarrow_N^* E_0[\text{app}\{\tau?\} v_4 (\text{BndryErr}(b_3, v_5))]$
 6640 and $E_0 \lesssim \text{trace } b_0 \cdots b_i []$
 6641 and $v_4 \lesssim \text{add-trace}(b_i \cdots b_n, v_2)$
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- 6643 • $\text{app}\{\mathcal{U}\} v_0 v_1 \rightarrow_N^* E_0[\text{app}\{\tau?\} v_4 v_5]$
 6644 and $E_0 \lesssim \text{trace } b_0 \bar{b}_1 []$
 6645 and $v_4 \lesssim v_2$
 6646 and $v_5 \lesssim \text{add-trace}(\text{flip}(b_0 \bar{b}_1), v_3)$
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6648 PROOF. By induction on the length of \bar{b}_1 .
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- 6650 1. $\bar{b}_1 = \cdot$
 6651 1.1. CONTRADICTION:
 6652 by \lesssim
 6653 2. $\bar{b}_1 = b_1$
 6654 2.1. $v_0 = \text{mon } b_2 (\text{mon } b_3 v_4)$ and $b_2 \leq b_0$ and $b_3 \leq b_1$
 6655
 6656

6657 by inversion \lesssim
6658 2.2. $\text{app}\{\mathcal{U}\} v_0 v_1 \blacktriangleright_{\mathbb{N}} \text{stat cod}(b_2) (\text{app}\{\tau_1\} (\text{mon } b_3 v_4) (\text{dyn dom}(b_2) v_1))$
6659 by definition $\blacktriangleright_{\mathbb{N}}$
6660 2.3. ASSUME $\text{dyn dom}(b_2)v_1 = v_5$
6662 otherwise, end with a boundary error
6663 2.4. $\text{app}\{\tau_1\} (\text{mon } b_3 v_4) v_5 \triangleright_{\mathbb{N}} \text{dyn cod}(b_3) (\text{app}\{\mathcal{U}\} v_4 (\text{stat dom}(b_3) v_5))$ and $\text{stat dom}(b_3) v_5 \blacktriangleright_{\mathbb{N}} v_6$
6664 by definition $\triangleright_{\mathbb{N}}$ and $\blacktriangleright_{\mathbb{N}}$
6665 2.5. $v_6 \lesssim \text{add-trace}(\text{flip}(\text{dom}(b_0)\text{dom}(b_1)), v_3)$
6666 by lemma 8.27
6668 2.6. QED
6669 $\text{stat cod}(b_2) (\text{dyn cod}(b_3) (\text{app}\{\mathcal{U}\} v_4 v_6)) \lesssim \text{trace}_v b_0 b_1 (\text{app}\{\mathcal{U}\} v_2 \text{trace}_v \text{flip}(b_0 b_1) v_3)$
6670 3. $\bar{b}_1 = b_1 b_2 \bar{b}_3$
6671 3.1. $v_0 = \text{mon } b_3 (\text{mon } b_4 v_4)$ and $b_3 \leq b_1$ and $b_4 \leq b_2$
6672 by inversion \lesssim
6673 3.2. $\text{app}\{\mathcal{U}\} v_0 v_1 \rightarrow_{\mathbb{N}}^* \text{stat cod}(b_3) (\text{dyn dom}(b_4) (\text{app}\{\mathcal{U}\} v_4 v_6))$
6674 and $v_4 \lesssim \text{trace}_v \bar{b}_3 v_2$ and $v_6 \lesssim \text{trace}_v \text{flip}(b_0 b_1) v_3$
6675 by similar reasoning as the previous case
6676 3.3. $\text{app}\{\mathcal{U}\} v_4 v_6 \rightarrow_{\mathbb{N}}^* E_1[\text{app}\{\mathcal{U}\} v_7 v_8]$
6677 by the induction hypothesis, assuming no boundary errors
6678 3.4. QED
6679 $\text{stat } b_3 (\text{dyn } b_4 (E_1[\text{app}\{\mathcal{U}\} v_7 v_8])) \lesssim \text{add-trace}(b_0 b_1 b_2 \bar{b}_3, \text{app}\{\mathcal{U}\} v_2 (\text{trace}_v \text{flip}(b_0 b_1 \bar{b}_2) v_3))$
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6685 LEMMA 8.27. If v_0 and v_1 are reduced WF expressions and $\cdot \vdash_{\mathbb{N}} v_0 : \mathcal{U}$ and $\cdot \vdash_{\mathbb{A}} v_1 : \mathcal{U}$ and $v_0 \lesssim v_1$ and $b_0 \leq b_2$ and
6686 $b_1 \leq b_3$ then $\text{stat } b_0 (\text{dyn } b_1 v_0) \rightarrow_{\mathbb{N}}^* e_2$ and $e_2 \lesssim \text{add-trace}(b_2 b_3, v_1)$.
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6688 PROOF. By induction on v_0 via cases on $v_0 \lesssim v_1$.
6689 We assume below that every subexpression of the form $\text{dyn } b v$ steps to a value, because otherwise e_2 is a boundary
6690 error and the result is immediate.
6691

6692 1. CASE $i_0 \lesssim i_0$
6693 1.1. QED
6694 $\text{stat } b_0 (\text{dyn } b_1 i_0) \rightarrow_{\mathbb{N}}^* i_0 \lesssim \text{trace}_v b_2 b_3 i_0$
6695

6696 2. CASE $i_0 \lesssim \text{trace}_v \bar{b}_0 \text{trace}_v \bar{b}_4 i_0$
6697 2.1. QED
6698 $\text{stat } b_0 (\text{dyn } b_1 i_0) \rightarrow_{\mathbb{N}}^* i_0 \lesssim \text{trace}_v b_2 b_3 \bar{b}_4 i_0$
6699

6700 3. CASE $\langle v_2, v_3 \rangle \lesssim \langle v_4, v_5 \rangle$
6701 3.1. $v_2 \lesssim v_4$ and $v_3 \lesssim v_5$
6702 by inversion \lesssim
6703 3.2. $\text{stat } b_0 (\text{dyn } b_1 \langle v_2, v_3 \rangle) \rightarrow_{\mathbb{N}} \text{stat } b_0 \langle \text{dyn fst}(b_1) v_2, \text{dyn snd}(b_1) v_3 \rangle$
6704 by definition $\rightarrow_{\mathbb{N}}$
6705 3.3. $\text{stat } b_0 \langle \text{dyn fst}(b_1) v_2, \text{dyn snd}(b_1) v_3 \rangle \rightarrow_{\mathbb{N}}^* \langle v_6, v_7 \rangle$
6706 if and only if $\langle \text{stat fst}(b_0) (\text{dyn fst}(b_1) v_2), \text{stat snd}(b_0) (\text{dyn snd}(b_1) v_3) \rangle \rightarrow_{\mathbb{N}}^* \langle v_6, v_7 \rangle$
6707
6708

6709 by lemma 8.28
6710 3.4. $v_6 \lesssim \text{add-trace}(\text{fst}(b_2)\text{fst}(b_3), v_4)$
6711 and $v_7 \lesssim \text{add-trace}(\text{snd}(b_2)\text{snd}(b_3), v_4)$
6712 by the induction hypothesis
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6714 3.5. QED
6715 by definition \lesssim
6716 4. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon } b_4 \langle v_4, v_5 \rangle$
6717 4.1. $v_2 \lesssim \text{mon}^? \text{fst}(b_4) v_4$ and $v_3 \lesssim \text{mon}^? \text{snd}(b_4) v_5$
6718 by inversion \lesssim
6719 4.2. $\text{stat } b_0 (\text{dyn } b_1 \langle v_2, v_3 \rangle) \rightarrow_N^* \langle v_6, v_7 \rangle$
6720 and $v_6 \lesssim \text{add-trace}(\text{fst}(b_2)\text{fst}(b_3), \text{mon}^? \text{fst}(b_4) v_4)$
6721 and $v_7 \lesssim \text{add-trace}(\text{snd}(b_2)\text{snd}(b_3), \text{mon}^? \text{snd}(b_4) v_5)$
6722 by definition \rightarrow_N and lemma 8.28 and the induction hypothesis
6723
6724 4.3. QED
6725 by definition \lesssim
6726 5. CASE $\langle v_2, v_3 \rangle \lesssim \text{trace}_v \bar{b}_4 \langle v_4, v_5 \rangle$
6727 5.1. $v_2 \lesssim \text{add-trace}(\text{fst}(\bar{b}_4), v_4)$ and $v_3 \lesssim \text{add-trace}(\text{snd}(\bar{b}_4), v_5)$
6728 by inversion \lesssim
6729 5.2. $\text{stat } b_0 (\text{dyn } b_1 \langle v_2, v_3 \rangle) \rightarrow_N^* \langle v_6, v_7 \rangle$
6730 and $v_6 \lesssim \text{add-trace}(\text{fst}(b_2)\text{fst}(b_3), \text{add-trace}(\text{fst}(\bar{b}_4), v_4))$
6731 and $v_7 \lesssim \text{add-trace}(\text{snd}(b_2)\text{snd}(b_3), \text{add-trace}(\text{snd}(\bar{b}_4), v_5))$
6732 by definition \rightarrow_N and lemma 8.28 and the induction hypothesis
6733
6734 5.3. QED
6735 by definition \lesssim
6736 6. CASE $\langle v_2, v_3 \rangle \lesssim \text{trace}_v \bar{b}_4 (\text{mon } b_5 \langle v_4, v_5 \rangle)$
6737 6.1. $v_2 \lesssim \text{add-trace}(\text{fst}(\bar{b}_4), (\text{mon}^? \text{fst}(b_5) v_4))$ and $v_3 \lesssim \text{add-trace}(\text{snd}(\bar{b}_4), (\text{mon}^? \text{snd}(b_5) v_5))$
6738 by inversion \lesssim
6739 6.2. $\text{stat } b_0 (\text{dyn } b_1 \langle v_2, v_3 \rangle) \rightarrow_N^* \langle v_6, v_7 \rangle$
6740 and $v_6 \lesssim \text{add-trace}(\text{fst}(b_2)\text{fst}(b_3), \text{add-trace}(\text{fst}(\bar{b}_4), (\text{mon}^? \text{fst}(b_5) v_4)))$
6741 and $v_7 \lesssim \text{add-trace}(\text{snd}(b_2)\text{snd}(b_3), \text{add-trace}(\text{snd}(\bar{b}_4), (\text{mon}^? \text{snd}(b_5) v_5)))$
6742 by definition \rightarrow_N and lemma 8.28 and the induction hypothesis
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6744 6.3. QED
6745 by definition \lesssim
6746 7. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon } b_4 (\text{trace}_v \bar{b}_5 \langle v_4, v_5 \rangle)$
6747 7.1. CONTRADICTION:
6748 $\cdot \vdash_A v_1 : \mathcal{U}$
6749 8. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon } b_4 (\text{mon } b_5 \langle v_4, v_5 \rangle)$
6750 8.1. CONTRADICTION:
6751 by lemma 7.10
6752 9. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon } b_4 (\text{trace}_v \bar{b}_5 (\text{mon } b_6 \langle v_4, v_5 \rangle))$
6753 9.1. CONTRADICTION:
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6761 by lemma 7.10

6762 10. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}^+ \bar{b}_4 (\text{mon } b_5 (\text{mon } b_6 \langle v_4, v_5 \rangle))$

6763 xxx

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6765 11. CASE $\langle v_2, v_3 \rangle \lesssim \text{mon}^+ \bar{b}_4 (\text{mon } b_5 (\text{trace}_v b_6 (\text{mon } b_7 \langle v_4, v_5 \rangle)))$

6766 11.1. CONTRADICTION:

6767 by lemma 7.10

6768 12. CASE $\lambda x_2. e_2 \lesssim \lambda x_2. e_3$

6769 12.1. QED

6771 stat $b_0 (\text{dyn } b_1 v_0) \rightarrow_N^* \text{mon } b_0 (\text{mon } b_1 v_0) \lesssim \text{trace}_v b_2 b_3 v_1$

6772 13. CASE $\lambda(x_2 : \tau_2). e_2 \lesssim \lambda(x_2 : \tau_2). e_3$

6773 13.1. CONTRADICTION:

6774 $\cdot \vdash_A v_1 : \mathcal{U}$

6776 14. CASE $\text{mon } b_4 (\text{mon } b_5 v_3) \lesssim \text{trace}_v b_6 b_7 v_4$

6777 and $v_4 \in \lambda x. e \cup \text{mon } b (\lambda(x : \tau). e)$

6778 14.1. QED

6779 stat $b_0 (\text{dyn } b_1 v_0) \rightarrow_N^* \text{mon } b_0 (\text{mon } b_1 v_0) \lesssim \text{trace}_v b_2 b_3 b_6 b_7 v_4$

6780 15. CASE $\text{mon } b_4 (\text{mon } b_5 v_3) \lesssim \text{trace}_v b_6 b_7 \bar{b}_8 v_4$

6782 and $v_4 \in \lambda x. e \cup \text{mon } b (\lambda(x : \tau). e)$

6783 15.1. QED

6784 stat $b_0 (\text{dyn } b_1 v_0) \rightarrow_N^* \text{mon } b_0 (\text{mon } b_1 v_0) \lesssim \text{trace}_v b_2 b_3 b_6 b_7 \bar{b}_8 v_4$

6786 16. CASE $\text{mon } b_4 v_3 \lesssim \text{mon } b_5 v_4$

6787 16.1. $v_4 \in \lambda(x : \tau). e$

6788 by inversion \vdash_A and \lesssim and lemma 7.10

6789 16.2. stat $b_0 (\text{dyn } b_1 v_0) \rightarrow_N^* \text{mon } b_0 (\text{mon } b_1 v_0)$

6791 by definition \rightarrow_N

6792 16.3. QED

6793 by definition \lesssim

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LEMMA 8.28. stat $b_0 \langle \text{dyn } b_1 v_0, \text{dyn } b_2 v_1 \rangle \rightarrow_N^* \langle v_2, v_3 \rangle$ if and only if
 $\langle \text{stat fst } (b_0) (\text{dyn } b_1 v_0), \text{stat snd } (b_0) (\text{dyn } b_2 v_1) \rangle \rightarrow_N^* \langle v_2, v_3 \rangle$

PROOF. By definition of \triangleright_N and \blacktriangleright_N .

□

□

LEMMA 8.29. If $\text{binop}\{\tau?\} v_0 v_1 \lesssim \text{binop}\{\tau?\} v_2 v_3$ then $\text{binop}\{\tau?\} v_0 v_1 \rightarrow_N e_4$ if and only if $\text{binop}\{\tau?\} v_2 v_3 \rightarrow_N e_5$

PROOF. The binary operations are only defined for integers and \lesssim relates integers iff they are equal.

1. CASE $\text{binop}\{\tau?\} v_0 v_1 \blacktriangleright_N \text{TagErr} \bullet$

1.1. SCASE $v_0 \notin i$

1.1.1. $v_2 \notin i$

by inversion \lesssim

6813 1.1.2. QED
6814 $binop\{\tau?\} svalue_2 v_3 \blacktriangleright_A \text{TagErr} \bullet$
6815
6816 1.2. SCASE $v_1 \notin i$
6817 1.2.1. $v_3 \notin i$
6818 by inversion \lesssim
6819 1.2.2. QED
6820 $binop\{\tau?\} v_2 v_3 \blacktriangleright_A \text{TagErr} \bullet$
6821 2. CASE $binop\{\tau?\} v_2 v_3 \blacktriangleright_A \text{TagErr} \bullet$
6822 2.1. SCASE $rem\text{-}trace(v_2) \notin i$
6823 2.1.1. $v_0 \notin i$
6824 by inversion \lesssim
6825 2.1.2. QED
6826 $binop\{\tau?\} svalue_0 v_1 \blacktriangleright_N \text{TagErr} \bullet$
6827 2.2. SCASE $rem\text{-}trace(v_3) \notin i$
6828 2.2.1. $v_1 \notin i$
6829 by inversion \lesssim
6830 2.2.2. QED
6831 $binop\{\tau?\} v_0 v_1 \blacktriangleright_N \text{TagErr} \bullet$
6832 3. CASE $binop = \text{sum}$ and $v_0 = i_0$ and $v_1 = i_1$
6833 3.1. $binop\{\tau?\} v_0 v_1 \rightarrow_N v_0 + v_1$
6834 by definition $\triangleright_N, \blacktriangleright_N$, and δ_N
6835 3.2. $rem\text{-}trace(v_2) = v_0$ and $rem\text{-}trace(v_3) = v_1$
6836 by inversion \lesssim
6837 3.3. QED
6838 $binop\{\tau?\} v_2 v_3 \rightarrow_A v_0 + v_1$
6839 4. CASE $binop = \text{sum}$ and $rem\text{-}trace(v_2) = i_0$ and $rem\text{-}trace(v_3) = i_1$
6840 4.1. $binop\{\tau?\} v_2 v_3 \rightarrow_A i_0 + i_1$
6841 by definition $\triangleright_A, \blacktriangleright_A$, and δ_A
6842 4.2. $v_0 = i_0$ and $v_1 = i_1$
6843 by inversion \lesssim
6844 4.3. QED
6845 $binop\{\tau?\} v_0 v_1 \rightarrow_N v_2 + v_3$
6846 5. CASE $binop = \text{quotient}$ and $v_0 = i_0$ and $v_1 = i_1$ and $i_1 = 0$
6847 5.1. $binop\{\tau?\} v_0 v_1 \rightarrow_N \text{TagErr} \bullet$
6848 by definition $\triangleright_N, \blacktriangleright_N$, and δ_N
6849 5.2. $rem\text{-}trace(v_2) = i_0$ and $rem\text{-}trace(v_3) = 0$
6850 by inversion \lesssim
6851 5.3. QED
6852 $binop\{\tau?\} v_2 v_3 \rightarrow_A \text{TagErr} \bullet$
6853 6. CASE $binop = \text{quotient}$ and $rem\text{-}trace(v_2) = i_2$ and $rem\text{-}trace(v_3) = i_3$ and $i_3 = 0$
6854 6.1. $binop\{\tau?\} v_2 v_3 \rightarrow_A \text{TagErr} \bullet$
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6865 by definition \triangleright_{Λ} , $\blacktriangleright_{\Lambda}$, and δ_A

6866 6.2. $v_0 = i_2$ and $v_1 = 0$

6867 by inversion \lesssim

6868 6.3. QED

6870 $\text{binop}\{\tau?\} v_0 v_1 \rightarrow_{\mathbb{N}} \text{TagErr} \bullet$

6871 7. CASE $\text{binop} = \text{quotient}$ and $v_0 = i_0$ and $v_1 = i_1$ and $i_1 \neq 0$

6872 7.1. $\text{binop}\{\tau?\} v_0 v_1 \rightarrow_{\mathbb{N}} [i_0/i_1]$

6873 by definition $\triangleright_{\mathbb{N}}$, $\blacktriangleright_{\mathbb{N}}$, and $\delta_{\mathbb{N}}$

6874 7.2. $\text{rem-trace}(v_2) = i_0$ and $\text{rem-trace}(v_3) = i_1$

6875 by inversion \lesssim

6876 7.3. QED

6877 $\text{binop}\{\tau?\} v_2 v_3 \rightarrow_{\Lambda} [i_0/i_1]$

6878 8. CASE $\text{binop} = \text{quotient}$ and $\text{rem-trace}(v_2) = i_2$ and $\text{rem-trace}(v_3) = i_3$ and $i_3 \neq 0$

6880 8.1. $\text{binop}\{\tau?\} v_2 v_3 \rightarrow_{\Lambda} [i_2/i_3]$

6881 by definition \triangleright_{Λ} , $\blacktriangleright_{\Lambda}$, and δ_A

6882 8.2. $v_0 = i_0$ and $v_1 = i_1$

6883 by inversion \lesssim

6884 8.3. QED

6885 $\text{binop}\{\tau?\} v_0 v_1 \rightarrow_{\mathbb{N}} [i_0/i_1]$

6886 □

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6888 LEMMA 8.30. If $e_0 \lesssim e_1$ and $v_0 \lesssim v_1$ then $e_0[x_0 \leftarrow v_0] \lesssim e_1[x_0 \leftarrow v_1]$

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6890 PROOF. By induction on the structure of both expressions (or rather, the pair $\langle e_0, e_1 \rangle$) via case analysis of $e_0 \lesssim e_1$.

6891

6892 □

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6894 LEMMA 8.31. If $e_0 \lesssim e_1$ and $E_0 \lesssim E_1$ then $E_0[e_0] \lesssim E_1[e_1]$

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6896 PROOF. By induction on the structure of E_0 .

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6917 9 A/T SIMULATION

6918 This simulation uses the Transient notion of reduction from the paper and a modified Amnesic notion of reduction that
6919 inserts check expressions that are guaranteed (by type soundness) to be no-ops.
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$e \sim e; \mathcal{H}; \mathcal{B}$

$$\begin{array}{c}
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\end{array}$$

$$\begin{array}{c}
\frac{v_0 \sim v_1; \mathcal{H}_0; \mathcal{B}_0}{\text{trace } \bar{b}_0 v_0 \sim v_1; \mathcal{H}_0; \mathcal{B}_0} \quad \frac{v_0 \sim \mathcal{H}_0(p_0); \mathcal{H}_0; \mathcal{B}_0}{v_0 \sim p_0; \mathcal{H}_0; \mathcal{B}_0} \quad \frac{}{i_0 \sim i_0; \mathcal{H}_0; \mathcal{B}_0} \quad \frac{}{\text{trace}_v \bar{b}_0 i_0 \sim i_0; \mathcal{H}_0; \mathcal{B}_0} \\
\frac{v_0 \sim v_2; \mathcal{H}_0; \mathcal{B}_0 \quad v_1 \sim v_3; \mathcal{H}_0; \mathcal{B}_0}{\text{trace}_v^2 \bar{b}_0 \langle v_0, v_1 \rangle \sim \langle v_2, v_3 \rangle; \mathcal{H}_0; \mathcal{B}_0} \quad \frac{v_0 \sim v_2; \mathcal{H}_0; \mathcal{B}_0 \quad v_1 \sim v_3; \mathcal{H}_0; \mathcal{B}_0}{\text{trace}_v^2 \bar{b}_0 (\text{mon } b_0 (\text{trace}_v^2 \bar{b}_1 \langle v_0, v_1 \rangle)) \sim \langle v_2, v_3 \rangle; \mathcal{H}_0; \mathcal{B}_0} \\
\frac{v_0 \sim v_2; \mathcal{H}_0; \mathcal{B}_0 \quad v_1 \sim v_3; \mathcal{H}_0; \mathcal{B}_0}{\text{mon } b_0 (\text{trace}_v^2 \bar{b}_0 (\text{mon } b_1 \langle v_0, v_1 \rangle)) \sim \langle v_2, v_3 \rangle; \mathcal{H}_0; \mathcal{B}_0} \quad \frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\text{trace}_v^2 \bar{b}_0 (\lambda x_0. e_0) \sim \lambda x_0. e_1; \mathcal{H}_0; \mathcal{B}_0} \\
\frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\text{mon } b_0 (\text{trace}_v^2 \bar{b}_0 (\lambda x_0. e_0)) \sim \lambda x_0. e_1; \mathcal{H}_0; \mathcal{B}_0} \quad \frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\text{trace}_v^2 \bar{b}_0 (\lambda(x_0 : \tau_0). e_0) \sim \lambda(x_0 : \tau_0). e_1; \mathcal{H}_0; \mathcal{B}_0} \\
\frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\text{trace}_v^2 \bar{b}_0 (\text{mon } b_0 (\lambda(x_0 : \tau_0). e_0)) \sim \lambda(x_0 : \tau_0). e_1; \mathcal{H}_0; \mathcal{B}_0} \\
\frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\text{mon } b_0 (\text{trace}_v^2 \bar{b}_0 (\text{mon } b_1 (\lambda(x_0 : \tau_0). e_0))) \sim \lambda(x_0 : \tau_0). e_1; \mathcal{H}_0; \mathcal{B}_0} \quad \frac{}{x_0 \sim x_0; \mathcal{H}_0; \mathcal{B}_0} \\
\frac{e_0 \sim e_2; \mathcal{H}_0; \mathcal{B}_0 \quad e_1 \sim e_3; \mathcal{H}_0; \mathcal{B}_0}{\langle e_0, e_1 \rangle \sim \langle e_2, e_3 \rangle; \mathcal{H}_0; \mathcal{B}_0} \quad \frac{e_0 \sim e_2; \mathcal{H}_0; \mathcal{B}_0 \quad e_1 \sim e_3; \mathcal{H}_0; \mathcal{B}_0}{\text{app}\{\tau?\} e_0 e_1 \sim \text{app}\{\tau?\} e_2 e_3; \mathcal{H}_0; \mathcal{B}_0} \\
\frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\text{unop}\{\tau?\} e_0 \sim \text{unop}\{\tau?\} e_1; \mathcal{H}_0; \mathcal{B}_0} \quad \frac{e_0 \sim e_2; \mathcal{H}_0; \mathcal{B}_0 \quad e_1 \sim e_3; \mathcal{H}_0; \mathcal{B}_0}{\text{binop}\{\tau?\} e_0 e_1 \sim \text{binop}\{\tau?\} e_2 e_3; \mathcal{H}_0; \mathcal{B}_0} \quad \frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\text{dyn } b_0 e_0 \sim \text{dyn } b_0 e_1; \mathcal{H}_0; \mathcal{B}_0} \\
\frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\text{stat } b_0 e_0 \sim \text{stat } b_0 e_1; \mathcal{H}_0; \mathcal{B}_0} \quad \frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) e_0 \sim \text{check } \tau_0 e_1 p_0; \mathcal{H}_0; \mathcal{B}_0} \\
\frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) e_0 \sim \text{check } \mathcal{U} e_1 p_0; \mathcal{H}_0; \mathcal{B}_0} \quad \frac{\tau_1 \leq \tau_0 \quad e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{stat } (\ell_2 \blacktriangleleft \tau_1 \blacktriangleleft \ell_3) e_0) \sim \text{check } \tau_0 e_1 p_0; \mathcal{H}_0; \mathcal{B}_0} \\
\frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\text{check } \tau? e_0 \bullet \sim \text{check } \tau? e_1 p_0; \mathcal{H}_0; \mathcal{B}_0} \quad \frac{}{\text{TagErr } \circ \sim \text{TagErr } \circ; \mathcal{H}_0; \mathcal{B}_0} \quad \frac{}{\text{TagErr } \bullet \sim \text{TagErr } \bullet; \mathcal{H}_0; \mathcal{B}_0} \\
\frac{}{\text{DivErr } \sim \text{DivErr}; \mathcal{H}_0; \mathcal{B}_0} \quad \frac{v_0 \sim v_1; \mathcal{H}_0; \mathcal{B}_0}{\text{BndryErr } (b_0, v_0) \sim \text{BndryErr } (b_0, v_1); \mathcal{H}_0; \mathcal{B}_0}
\end{array}$$

7021	$E \sim E; \mathcal{H}; \mathcal{B}$		
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7023	$\frac{E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0}{\text{trace } \bar{b}_0 E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0}$	$\frac{}{[] \sim []; \mathcal{H}_0; \mathcal{B}_0}$	$\frac{E_0 \sim E_2; \mathcal{H}_0; \mathcal{B}_0 \quad e_1 \sim e_3; \mathcal{H}_0; \mathcal{B}_0}{\langle E_0, e_1 \rangle \sim \langle E_2, e_3 \rangle; \mathcal{H}_0; \mathcal{B}_0}$
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7025	$\frac{v_0 \sim v_2; \mathcal{H}_0; \mathcal{B}_0 \quad E_1 \sim E_3; \mathcal{H}_0; \mathcal{B}_0}{\langle v_0, E_1 \rangle \sim \langle v_2, E_3 \rangle; \mathcal{H}_0; \mathcal{B}_0}$		$\frac{E_0 \sim E_2; \mathcal{H}_0; \mathcal{B}_0 \quad e_1 \sim e_3; \mathcal{H}_0; \mathcal{B}_0}{\text{app}\{\tau?\} E_0 e_1 \sim \text{app}\{\tau?\} E_2 e_3; \mathcal{H}_0; \mathcal{B}_0}$
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7031	$\frac{v_0 \sim v_2; \mathcal{H}_0; \mathcal{B}_0 \quad E_1 \sim E_3; \mathcal{H}_0; \mathcal{B}_0}{\text{app}\{\tau?\} v_0 E_1 \sim \text{app}\{\tau?\} v_2 E_3; \mathcal{H}_0; \mathcal{B}_0}$		$\frac{E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0}{\text{unop}\{\tau?\} E_0 \sim \text{unop}\{\tau?\} E_1; \mathcal{H}_0; \mathcal{B}_0}$
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7035	$\frac{E_0 \sim E_2; \mathcal{H}_0; \mathcal{B}_0 \quad e_1 \sim e_3; \mathcal{H}_0; \mathcal{B}_0}{\text{binop}\{\tau?\} E_0 e_1 \sim \text{binop}\{\tau?\} E_2 e_3; \mathcal{H}_0; \mathcal{B}_0}$		$\frac{v_0 \sim v_2; \mathcal{H}_0; \mathcal{B}_0 \quad E_1 \sim E_3; \mathcal{H}_0; \mathcal{B}_0}{\text{binop}\{\tau?\} v_0 E_1 \sim \text{binop}\{\tau?\} v_2 E_3; \mathcal{H}_0; \mathcal{B}_0}$
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7039	$\frac{E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0}{\text{dyn } b_0 E_0 \sim \text{dyn } b_0 E_1; \mathcal{H}_0; \mathcal{B}_0}$		$\frac{E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0}{\text{stat } b_0 E_0 \sim \text{stat } b_0 E_1; \mathcal{H}_0; \mathcal{B}_0}$
7040			
7041			
7042	$\frac{E_0 \notin \text{stat } b E \quad E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0}{\text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) E_0 \sim \text{check } \tau_0 E_1 p_0; \mathcal{H}_0; \mathcal{B}_0}$		$\frac{E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0}{\text{stat } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) E_0 \sim \text{check } \mathcal{U} E_1 p_0; \mathcal{H}_0; \mathcal{B}_0}$
7043			
7044			
7045			
7046	$\frac{\tau_1 \leq \tau_0 \quad E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0}{\text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{stat } (\ell_2 \blacktriangleleft \tau_1 \blacktriangleleft \ell_3) E_0) \sim \text{check } \tau_0 E_1 p_0; \mathcal{H}_0; \mathcal{B}_0}$		$\frac{E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0}{\text{check } \tau? E_0 \bullet \sim \text{check } \tau? E_1 p_0; \mathcal{H}_0; \mathcal{B}_0}$
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7073 COROLLARY 9.1. $\text{eval}_A(\uparrow s_0) \in \text{Err}$ if and only if $\text{eval}_T(\uparrow s_0) \in \text{Err}$

7074
7075 PROOF. By lemma 9.2

7076 □

7078 LEMMA 9.2. If $\text{eval}_A(\uparrow s_0) = r_0$ and $\text{eval}_T(\uparrow s_0) = r_1; \mathcal{H}_0; \mathcal{B}_0$ then one of the following holds:

- 7080 • $r_0 = r_1 = \Omega$
7081 • $r_0 \neq \Omega$ and $r_1 \neq \Omega$ and $r_0 \sim r_1; \mathcal{H}_0; \mathcal{B}_0$

7082
7083 PROOF. By lemma 9.3 and lemma 9.4.

7084 □

7086 LEMMA 9.3. $\uparrow s_0 \sim \uparrow s_0; \emptyset; \emptyset$ for all $\uparrow s_0$

7087
7088 PROOF. By induction on the structure of s_0 .

7089 1. CASE $s_0 = x_0$

7090 1.1. QED

7091 $x_0 \sim x_0; \mathcal{H}_0; \mathcal{B}_0$

7092
7093 2. CASE $s_0 = i_0$

7094 2.1. QED

7095 $i_0 \sim i_0; \mathcal{H}_0; \mathcal{B}_0$

7096
7097 3. CASE $s_0 = \lambda x_0. s_0$

7098 3.1. QED

7099 by the induction hypothesis and the definition of \sim

7100 4. CASE $s_0 = \lambda(x_0 : \tau_0). s_1$

7101 4.1. QED

7102 by the induction hypothesis and the definition of \sim

7103
7104 5. CASE $s_0 = \langle s_1, s_2 \rangle$

7105 5.1. QED

7106 by the induction hypothesis

7107
7108 6. CASE $s_0 = \text{app}\{\tau_0\} s_1 s_2$

7109 6.1. QED

7110 by the induction hypothesis

7111
7112 7. CASE $s_0 = \text{unop}\{\tau?\} s_1$

7113 7.1. QED

7114 by the induction hypothesis

7115
7116 8. CASE $s_0 = \text{binop}\{\tau?\} s_1 s_2$

7117 8.1. QED

7118 by the induction hypothesis

7119
7120 9. CASE $s_0 = \text{dyn } \tau_0 s_1$

7121 9.1. QED

7122 by the induction hypothesis

7123 10. CASE $s_0 = \text{stat } \tau_0 s_1$

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7125 10.1. QED

7126 by the induction hypothesis

7127

7128 □

7129

7130 LEMMA 9.4. *If $e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0$ then:*

7131 • *if $e_0 \rightarrow_{\wedge} e_2$ then $e_2 \rightarrow_{\wedge}^* e_3$ and $e_1; \mathcal{H}_0; \mathcal{B}_0 \rightarrow_{\top}^* e_4; \mathcal{H}_1; \mathcal{B}_1$ and $e_3 \sim e_4; \mathcal{H}_1; \mathcal{B}_1$*

7132 • *if $e_1; \mathcal{H}_0; \mathcal{B}_0 \rightarrow_{\top} e_3; \mathcal{H}_1; \mathcal{B}_1$ then $e_0 \rightarrow_{\wedge}^* e_2$ and $e_2 \sim e_3; \mathcal{H}_1; \mathcal{B}_1$*

7133

7134 PROOF. By lemma 9.5 and lemma 9.6 and lemma 9.7 and lemma 9.8 and lemma 9.9.

7135

7136 □

7137

7138 LEMMA 9.5. *If $E_0[e_0] \sim e_1; \mathcal{H}_0; \mathcal{B}_0$ then $e_1 = E_1[e_2]$ and $E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0$ and $e_0 \sim e_2; \mathcal{H}_0; \mathcal{B}_0$*

7139

7140 PROOF. By induction on the structure of E_0 .

7141 1. CASE $E_0 = []$

7142 1.1. QED

7143 $E_1 = []$

7144

7145 2. CASE $E_0 = \langle E_2, e_2 \rangle$

7146 2.1. QED

7147 by the induction hypothesis

7148

7149 3. CASE $E_0 = \langle v_0, E_2 \rangle$

7150 3.1. QED

7151 by the induction hypothesis

7152

7153 4. CASE $E_0 = \text{app}\{\tau?\} E_2 e_2$

7154 4.1. QED

7155 by the induction hypothesis

7156

7157 5. CASE $E_0 = \text{app}\{\tau?\} v_0 E_2$

7158 5.1. QED

7159 by the induction hypothesis

7160

7161 6. CASE $E_0 = \text{unop}\{\tau?\} E_2$

7162 6.1. QED

7163 by the induction hypothesis

7164

7165 7. CASE $E_0 = \text{binop}\{\tau?\} E_2 e_2$

7166 7.1. QED

7167 by the induction hypothesis

7168

7169 8. CASE $E_0 = \text{binop}\{\tau?\} E_2 e_2$

7170 8.1. QED

7171 by the induction hypothesis

7172

7173 9. CASE $E_0 = \text{dyn } b_0 E_2$ and $E_2 \neq \text{check } \tau? E_3$ •

7174

7175 9.1. SCASE $E_1 = \text{dyn } b_0 E_3$

7176

7177 9.1.1. QED

7178 by the induction hypothesis

7179

- 7177 9.2. SCASE $E_1 = \text{check } \tau_0 E_3 p_0$
 7178 9.2.1. QED
 7179 by the induction hypothesis
 7180
 7181 10. CASE $E_0 = \text{dyn } b_0 (\text{stat } b_1 E_2)$
 7182 10.0.1. QED
 7183 by the induction hypothesis
 7184
 7185 11. CASE $E_0 = \text{stat } b_0 E_2$
 7186 11.1. SCASE $E_1 = \text{stat } b_0 E_3$
 7187 11.1.1. QED
 7188 by the induction hypothesis
 7189 11.2. SCASE $E_1 = \text{check } \mathcal{U} E_3 p_0$
 7190 11.2.1. QED
 7191 by the induction hypothesis
 7192
 7193 12. CASE $E_0 = \text{check } \tau? E_2 \bullet$
 7194 12.1. QED
 7195 by the induction hypothesis
 7196

□

7199 LEMMA 9.6. *If $e_0 \sim E_1[e_1]; \mathcal{H}_0; \mathcal{B}_0$ then $e_0 = E_0[e_2]$ and $E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0$ and $e_2 \sim e_1; \mathcal{H}_0; \mathcal{B}_0$*

7200 PROOF. By induction on the structure of E_1 and E_0 ; the latter is because E_0 may be a suffix context ($\text{trace } \bar{b} E$).

- 7202 1. CASE $E_1 = []$
 7203 1.1. SCASE $E_0 = []$
 7204 1.1.1. QED
 7205 1.2. SCASE $E_0 = \text{trace } \bar{b}_0 E_2$
 7206 1.2.1. QED
 7207 by the induction hypothesis
 7208
 7209 2. CASE $E_1 = \langle E_2, e_3 \rangle$
 7210 2.1. QED
 7211 by the induction hypothesis
 7212
 7213 3. CASE $E_1 = \langle v_0, E_2 \rangle$
 7214 3.1. QED
 7215 by the induction hypothesis
 7216
 7217 4. CASE $E_1 = \text{app}\{\tau?\} E_2 e_3$
 7218 4.1. QED
 7219 by the induction hypothesis
 7220
 7221 5. CASE $E_1 = \text{app}\{\tau?\} v_0 E_2$
 7222 5.1. QED
 7223 by the induction hypothesis
 7224
 7225 6. CASE $E_1 = \text{unop}\{\tau?\} E_2$
 7226 6.1. QED
 7227 by the induction hypothesis
 7228

- 7229 7. CASE $E_1 = \text{binop}\{\tau?\} E_2 e_3$
 7230 7.1. QED
 7231 by the induction hypothesis
 7232
 7233 8. CASE $E_1 = \text{binop}\{\tau?\} v_0 E_2$
 7234 8.1. QED
 7235 by the induction hypothesis
 7236
 7237 9. CASE $E_1 = \text{dyn } b_0 E_2$
 7238 9.1. QED
 7239 by the induction hypothesis
 7240
 7241 10. CASE $E_1 = \text{stat } b_0 E_2$
 7242 10.1. QED
 7243 by the induction hypothesis
 7244
 7245 11. CASE $E_1 = \text{check } \tau_0 E_2 p_0$
 7246 11.1. SCASE $E_0 = \text{dyn } b_0 E_3$ and $E_3 \notin \text{stat } b E$
 7247 by the induction hypothesis
 7248 11.2. SCASE $E_0 = \text{dyn } b_0 (\text{stat } b_1 E_3)$
 7249 by the induction hypothesis
 7250 11.3. SCASE $E_0 = \text{check } \tau_0 E_3 \bullet$
 7251 by the induction hypothesis
 7252
 7253 12. CASE $E_1 = \text{check } \mathcal{U} E_2 p_0$
 7254 12.1. SCASE $E_0 = \text{stat } b_0 E_3$
 7255 by the induction hypothesis
 7256 12.2. SCASE $E_0 = \text{check } \mathcal{U} E_3 \bullet$
 7257 by the induction hypothesis
 7258
 7259

□

7261 LEMMA 9.7. *If $E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0$ and $e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0$ then $E_0[e_0] \sim E_1[e_1]; \mathcal{H}_0; \mathcal{B}_0$*

7262 PROOF. By case analysis of $E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0$ and induction on the structure of E_0 and E_1 .

- 7263 1. CASE $\text{trace } \bar{b}_0 E_2 \sim E_1; \mathcal{H}_0; \mathcal{B}_0$
 7264 1.1. QED
 7265 by the induction hypothesis
 7266
 7267 2. CASE $[\] \sim [\]; \mathcal{H}_0; \mathcal{B}_0$
 7268 2.1. QED
 7269 by $e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0$
 7270
 7271 3. CASE $\langle E_2, e_2 \rangle \sim \langle E_3, e_3 \rangle; \mathcal{H}_0; \mathcal{B}_0$
 7272 3.1. QED
 7273 by the induction hypothesis
 7274
 7275 4. CASE $\langle v_0, E_2 \rangle \sim \langle v_1, E_3 \rangle; \mathcal{H}_0; \mathcal{B}_0$
 7276 4.1. QED
 7277 by the induction hypothesis
 7278
 7279 5. CASE $\text{app}\{\tau?\} E_2 e_2 \sim \text{app}\{\tau?\} E_3 e_3; \mathcal{H}_0; \mathcal{B}_0$
 7280

- 7281 5.1. QED
7282 by the induction hypothesis
7283 6. CASE $\text{app}\{\tau?\} v_0 E_2 \sim \text{app}\{\tau?\} v_1 E_3; \mathcal{H}_0; \mathcal{B}_0$
7284 6.1. QED
7285 by the induction hypothesis
7286 7. CASE $\text{unop}\{\tau?\} E_2 \sim \text{unop}\{\tau?\} E_3; \mathcal{H}_0; \mathcal{B}_0$
7287 7.1. QED
7288 by the induction hypothesis
7289 8. CASE $\text{binop}\{\tau?\} E_2 e_2 \sim \text{binop}\{\tau?\} E_3 e_3; \mathcal{H}_0; \mathcal{B}_0$
7290 8.1. QED
7291 by the induction hypothesis
7292 9. CASE $\text{binop}\{\tau?\} v_0 E_2 \sim \text{binop}\{\tau?\} v_1 E_3; \mathcal{H}_0; \mathcal{B}_0$
7293 9.1. QED
7294 by the induction hypothesis
7295 10. CASE $\text{dyn } b_0 E_2 \sim \text{dyn } b_0 E_3; \mathcal{H}_0; \mathcal{B}_0$
7296 10.1. QED
7297 by the induction hypothesis
7298 11. CASE $\text{stat } b_0 E_2 \sim \text{stat } b_0 E_3; \mathcal{H}_0; \mathcal{B}_0$
7299 11.1. QED
7300 by the induction hypothesis
7301 12. CASE $\text{dyn } b_0 E_2 \sim \text{check } \tau_0 E_3 p_0; \mathcal{H}_0; \mathcal{B}_0$
7302 12.1. QED
7303 by the induction hypothesis
7304 13. CASE $\text{stat } b_0 E_2 \sim \text{check } \mathcal{U} E_3 p_0; \mathcal{H}_0; \mathcal{B}_0$
7305 13.1. QED
7306 by the induction hypothesis
7307 14. CASE $\text{dyn } b_0 (\text{stat } b_1 E_2) \sim \text{check } \tau_0 E_3 p_0; \mathcal{H}_0; \mathcal{B}_0$
7308 14.1. QED
7309 by the induction hypothesis
7310 15. CASE $\text{check } \tau? E_2 \bullet \sim \text{check } \tau? E_3 p_0; \mathcal{H}_0; \mathcal{B}_0$
7311 15.1. QED
7312 by the induction hypothesis

□

7323 LEMMA 9.8. *If $e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0$ and $e_0 (\triangleright_{\mathcal{A}} \cup \blacktriangleright_{\mathcal{A}}) e_2$ then $e_2 \rightarrow_{\mathcal{A}}^* e_3$ and $e_1; \mathcal{H}_0; \mathcal{B}_0 \rightarrow_{\mathcal{T}}^* e_4; \mathcal{H}_1; \mathcal{B}_1$ and $e_3 \sim e_4; \mathcal{H}_1; \mathcal{B}_1$*
7324

7325 PROOF. By case analysis of $(\triangleright_{\mathcal{A}} \cup \blacktriangleright_{\mathcal{A}})$ and inversion on the \sim relation. In short, the shape of the Amnesic expression
7326 e_0 determines (and matches) the shape of the Transient expression.

7327 Expressions on the Transient side may be pre-values. This proof assumes that all pre-values have already been allocated
7328 to the heap by stutter steps (lemma 9.10).
7329

7330 1. CASE $\text{unop}\{\tau?\} \langle v_0, v_1 \rangle \triangleright_{\mathcal{A}} \delta(\text{unop}, \langle v_0, v_1 \rangle)$

7331 1.1. QED

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- 7333 by lemma 9.14
- 7334 2. CASE $\text{binop}\{\tau?\} v_0 v_1 \triangleright_{\mathbb{A}} \delta(\text{binop}, v_0, v_1)$
- 7335 2.1. QED
- 7336 by lemma 9.15
- 7337
- 7338 3. CASE $\text{fst}\{\tau_0\} (\text{mon } (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) v_0) \triangleright_{\mathbb{A}} \text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{fst}\{\mathcal{U}\} v_0)$
- 7339 3.1. CASE $v_0 = \langle v_1, v_2 \rangle$
- 7340 3.1.1. $\text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{fst}\{\mathcal{U}\} v_0) \rightarrow_{\mathbb{A}} \text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \delta(\text{fst}\{\mathcal{U}\}, v_0)$
- 7341 3.1.2. $e_1 = \text{unop}\{\tau?\} v_2$ and $e_1; \mathcal{H}_0; \mathcal{B}_0 \rightarrow_{\mathbb{T}}^* \text{check } \tau_0 \delta(\text{fst}\{\tau_0\}, v_2) v_2; \mathcal{H}_0; \mathcal{B}_0$
- 7342 3.1.3. QED
- 7343 by definition \sim
- 7344
- 7345 3.2. CASE $v_0 = \text{mon } (\ell_2 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3) \langle v_1, v_2 \rangle$
- 7346 3.2.1. $\text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{fst}\{\mathcal{U}\} v_0) \rightarrow_{\mathbb{A}} \text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{stat } (\ell_2 \blacktriangleleft \text{fst}(\tau_2) \blacktriangleleft \ell_3) \delta(\text{fst}\{\text{fst}(\tau_2)v_0\}))$
- 7347 3.2.2. $e_1 = \text{unop}\{\tau?\} v_2$ and $e_1; \mathcal{H}_0; \mathcal{B}_0 \rightarrow_{\mathbb{T}}^* \text{check } \tau_0 \delta(\text{fst}\{\tau_0\}, v_2) v_2; \mathcal{H}_0; \mathcal{B}_0$
- 7348 3.2.3. QED
- 7349 by definition \sim
- 7350
- 7351 4. CASE $\text{snd}\{\tau_0\} (\text{mon } b_0 v_0) \triangleright_{\mathbb{A}} \dots$
- 7352 4.1. QED
- 7353 similar to fst case
- 7354
- 7355 5. CASE $\text{app}\{\tau_0\} (\lambda(x_0 : \tau_0). e_2) v_1 \triangleright_{\mathbb{A}} \text{check } \tau_0 e_2[x_0 \leftarrow v_0] \bullet$
- 7356 5.1. QED
- 7357 by lemma 9.17
- 7358
- 7359 6. CASE $\text{app}\{\tau_0\} (\text{mon } (\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2) \blacktriangleleft \ell_1) v_0) v_1 \triangleright_{\mathbb{A}} \text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{app}\{\mathcal{U}\} v_0 (\text{stat } (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1))$
- 7360 6.1. $\text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{app}\{\mathcal{U}\} v_0 (\text{stat } (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1)) \rightarrow_{\mathbb{A}} \text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{app}\{\mathcal{U}\} v_0 v_2)$
- 7361 6.2. QED
- 7362 by lemma 9.12/lemma 9.13 for stat and the definition of \sim for the whole
- 7363
- 7364 7. CASE $\text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \triangleright_{\mathbb{A}} \text{mon } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0$
- 7365 7.1. QED
- 7366 by lemma 9.16 and lemma 9.13
- 7367
- 7368 8. CASE $\text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{trace}_{\mathbb{V}}^2 \bar{b}_0 i_0) \triangleright_{\mathbb{A}} i_0$
- 7369 8.1. QED
- 7370 by lemma 9.16
- 7371
- 7372 9. CASE $\text{dyn } (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0 \triangleright_{\mathbb{A}} \text{BndryErr}(\bar{b}_0, v_0)$
- 7373 9.1. QED
- 7374 by lemma 9.16
- 7375
- 7376 10. CASE $\text{check } \tau_0 v_0 \bullet \triangleright_{\mathbb{A}} v_0$
- 7377 10.1. QED
- 7378 by inversion \sim
- 7379
- 7380 11. CASE $\text{check } \tau_0 v_0 \bullet \triangleright_{\mathbb{A}} \text{BndryErr}(\bar{b}_0, v_0)$
- 7381 11.1. CONTRADICTION:
- 7382 by THEOREM 7.2
- 7383 12. CASE $\text{trace } \bar{b}_0 v_0 \triangleright_{\mathbb{A}} \text{add-trace}(\bar{b}_0, v_0)$
- 7384

- 7385 12.1. QED
7386 by lemma 9.12
7387
7388 13. CASE $unop\{\tau?\} v_0 \blacktriangleright_{\mathbb{A}} \text{TagErr} \bullet$
7389 13.1. QED
7390 by lemma 9.14
7391 14. CASE $unop\{\tau?\} v_0 \blacktriangleright_{\mathbb{A}} \text{check } \mathcal{U} \delta(unop, v_0) \bullet$
7392 14.1. QED
7393 by lemma 9.14
7394
7395 15. CASE $binop\{\tau?\} v_0 v_1 \blacktriangleright_{\mathbb{A}} \text{TagErr} \bullet$
7396 15.1. QED
7397 by lemma 9.15
7398
7399 16. CASE $binop\{\tau?\} v_0 v_1 \blacktriangleright_{\mathbb{A}} \delta(binop, v_0, v_1)$
7400 16.1. QED
7401 by lemma 9.15
7402
7403 17. CASE $\text{fst}\{\mathcal{U}\} (\text{trace}_{\check{v}}^? \bar{b}_0 (\text{mon} (\ell_0 \blacktriangleleft (\tau_0 \times \tau_1) \blacktriangleleft \ell_1) v_0)) \blacktriangleright_{\mathbb{A}} \text{trace } \bar{b}_0 (\text{stat} (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \text{fst}\{\tau_0\} v_0)$
7404 17.1. $\text{trace } \bar{b}_0 (\text{stat} (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \text{fst}\{\tau_0\} v_0) \rightarrow_{\mathbb{A}} \text{trace } \bar{b}_0 (\text{stat} (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \delta(\text{fst}\{\text{fst}(\tau_0)\}, v_0))$
7405 17.2. $e_1 = \text{fst}\{\mathcal{U}\} v_1$ and $e_1; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{check } \mathcal{U} \delta(\text{fst}\{\mathcal{U}\}, v_1) v_1; \mathcal{H}_0; \mathcal{B}_0$
7406 17.3. QED
7407 by definition \sim
7408
7409 18. CASE $\text{snd}\{\mathcal{U}\} \dots \blacktriangleright_{\mathbb{A}} \dots$
7410 18.1. QED
7411 similar to previous case
7412
7413 19. CASE $\text{app}\{\mathcal{U}\} (\text{trace}_{\check{v}}^? \bar{b}_0 (\lambda x_0. e_0)) v_0 \blacktriangleright_{\mathbb{A}} \text{trace } \bar{b}_0 \text{check } \mathcal{U} (e_0[x_0 \leftarrow v_0]) \bullet$
7414 19.1. QED
7415 by lemma 9.17
7416
7417 20. CASE $\text{app}\{\mathcal{U}\} (\text{trace}_{\check{v}}^? \bar{b}_0 (\text{mon} (\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2) \blacktriangleleft \ell_1) v_0)) v_1 \blacktriangleright_{\mathbb{A}}$
7418 $\text{trace } \bar{b}_0 (\text{stat} (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{app}\{\tau_2\} v_0 (\text{dyn} (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1)))$
7419 20.1. CASE $\text{trace } \bar{b}_0 (\text{stat} (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{app}\{\tau_2\} v_0 (\text{dyn} (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1))) \rightarrow_{\mathbb{A}}$
7420 $\text{trace } \bar{b}_0 (\text{stat} (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{app}\{\tau_2\} v_0 v_2))$
7421 20.1.1. QED
7422 by lemma 9.16
7423
7424 20.2. CASE $\text{trace } \bar{b}_0 (\text{stat} (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{app}\{\tau_2\} v_0 (\text{dyn} (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0) v_1))) \rightarrow_{\mathbb{A}}$
7425 $\text{trace } \bar{b}_0 (\text{stat} (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) (\text{app}\{\tau_2\} v_0 \text{BndryErr}(\bar{b}_0, v_1)))$
7426 20.2.1. QED
7427 by lemma 9.16
7428
7429 21. CASE $\text{stat } b_0 v_0 \blacktriangleright_{\mathbb{A}} \text{mon } b_0 v_0$
7430 21.1. QED
7431 by lemma 9.16 and lemma 9.13
7432
7433 22. CASE $\text{stat } b_0 (\text{mon } b_1 (\text{trace}_{\check{v}}^? \bar{b}_0 v_0)) \blacktriangleright_{\mathbb{A}} \text{trace } (b_0 b_1 \bar{b}_0) v_0$
7434 22.1. QED
7435 by lemma 9.16
7436

7437 23. CASE $\text{stat } b_0 i_0 \triangleright_A i_0$

7438 23.1. QED

7439 by lemma 9.16

7441 24. CASE $\text{check } \mathcal{U} v_0 p_0 \triangleright_A v_0$

7442 24.1. QED

7443 by inversion \sim

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LEMMA 9.9. *If $e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0$ and $e_1; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} e_3; \mathcal{H}_1; \mathcal{B}_1$ then $e_0 \rightarrow_{\Lambda}^* e_2$ and $e_2 \sim e_3; \mathcal{H}_1; \mathcal{B}_1$*

PROOF. By case analysis of \triangleright_{\top} .

Any Amnesic expression can have the form $(\text{trace } \bar{b} v)$; the proof assumes these have been reduced by stutter steps (lemma 9.11).

1. CASE $w_0; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} p_0; p_0 \mapsto w_0, \mathcal{H}_0; p_0 \mapsto \emptyset, \mathcal{B}_0$

1.1. QED

by inversion \sim and lemma 9.18

2. CASE $\text{unop}\{\tau?\} p_0; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{check } \tau? \delta(\text{unop}. v_0) p_0; \mathcal{H}_0; \mathcal{B}_0$

2.1. SCASE $e_0 \in \text{unop}\{\tau?\} \text{trace}_v^? \bar{b} \langle v, v \rangle$

2.1.1. QED

by lemma 9.14

2.2. SCASE $e_0 = \text{unop}\{\tau?\} \text{mon} (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{trace}_v^? \bar{b}_0 \langle v_2, v_3 \rangle)$

and $\tau? = \tau_0$

2.2.1. $e_0 \rightarrow_{\Lambda}^* \text{dyn} (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \delta(\text{unop}, \text{trace}_v^? \bar{b}_0 \langle v_2, v_3 \rangle)$

2.2.2. QED

2.3. SCASE $e_0 = \text{unop}\{\tau?\} \text{mon} (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) (\text{trace}_v^? \bar{b}_0 (\text{mon} (\ell_2 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_3) \langle v_2, v_3 \rangle))$

and $\tau? = \tau_0$

2.3.1. $e_0 \rightarrow_{\Lambda}^* \text{dyn} (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) (\text{stat} (\ell_2 \blacktriangleleft \tau_1 \blacktriangleleft \ell_3) \delta(\text{unop}, \text{trace}_v^? \bar{b}_0 \langle v_2, v_3 \rangle))$

2.3.2. QED

2.4. SCASE $e_0 = \text{unop}\{\tau?\} \text{trace}_v^? \bar{b}_0 (\text{mon} (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1) \langle v_2, v_3 \rangle)$

and $\tau? = \mathcal{U}$

2.4.1. $e_0 \rightarrow_{\Lambda}^* \text{stat} (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \delta(\text{unop}, \text{trace}_v^? \bar{b}_0 \langle v_2, v_3 \rangle)$

2.4.2. QED

3. CASE $\text{unop}\{\tau?\} v_0; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{TagErr } \bullet; \mathcal{H}_0; \mathcal{B}_0$

3.1. QED

by inversion \sim

4. CASE $\text{binop}\{\tau?\} i_0 i_1; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \delta(\text{binop}. v_0, v_1); \mathcal{H}_0; \mathcal{B}_0$

4.1. QED

by lemma 9.15

5. CASE $\text{binop}\{\tau?\} i_0 i_1; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{TagErr } \bullet; \mathcal{H}_0; \mathcal{B}_0$

5.1. QED

by lemma 9.15

6. CASE $\text{app}\{\tau?\} p_0 v_0; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{check } \tau? e_2[x_0 \leftarrow v_0] p_0; \mathcal{H}_0; \mathcal{B}_0[v_0 \cup \text{rev}(\mathcal{B}_0(p_0))]$

- 7489 6.1. QED
7490 by lemma 9.17
7491 7. CASE $\text{app}\{\tau?\} p_0 v_0; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{BndryErr}(\bar{b}_0, v_0); \mathcal{H}_0; \mathcal{B}_0$
7492 7.1. QED
7493 by lemma 9.16
7494 8. CASE $\text{app}\{\tau_0\} p_0 v_0; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{check } \tau_0 e_2[x_0 \leftarrow v_0] p_0; \mathcal{H}_0; \mathcal{B}_0[v_0 \cup \text{rev}(\mathcal{B}_0(p_0))]$
7495 8.1. QED
7496 by lemma 9.17
7497 9. CASE $\text{app}\{\mathcal{U}\} p_0 v_0; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} e_2[x_0 \leftarrow v_0]; \mathcal{H}_0; \mathcal{B}_0$
7498 9.1. QED
7499 similar to previous case
7500 10. CASE $\text{app}\{\mathcal{U}\} v_0 v_1; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{TagErr } \bullet; \mathcal{H}_0; \mathcal{B}_0$
7501 10.1. QED
7502 by lemma 9.16
7503 11. CASE $\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} v_0; \mathcal{H}_0; \mathcal{B}_0[v_0 \cup \{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\}]$
7504 11.1. QED
7505 by lemma 9.16
7506 12. CASE $\text{dyn}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{BndryErr}(\bar{b}_0, v_0); \mathcal{H}_0; \mathcal{B}_0$
7507 12.1. QED
7508 by lemma 9.16
7509 13. CASE $\text{stat}(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) v_0; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} v_0; \mathcal{H}_0; \mathcal{B}_0[v_0 \cup \{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\}]$
7510 13.1. QED
7511 by inversion \sim and lemma 9.12 and lemma 9.13
7512 14. CASE $\text{check } \mathcal{U} v_0 p_0; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} v_0; \mathcal{H}_0; \mathcal{B}_0$
7513 14.1. QED
7514 by inversion \sim
7515 15. CASE $\text{check } \tau_0 v_0 p_0; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} v_0; \mathcal{H}_0; \mathcal{B}_0[v_0 \cup \mathcal{B}_0(p_0)]$
7516 15.1. QED
7517 by lemma 9.16
7518 16. CASE $\text{check } \tau_0 v_0 p_0; \mathcal{H}_0; \mathcal{B}_0 \triangleright_{\top} \text{BndryErr}(\bar{b}_0, v_0); \mathcal{H}_0; \mathcal{B}_0$
7519 16.1. QED
7520 by lemma 9.16

□

7531 LEMMA 9.10. *If $v_0 \sim e_0; \mathcal{H}_0; \mathcal{B}_0$ then $e_0; \mathcal{H}_0; \mathcal{B}_0 \rightarrow_{\top}^* v_1; \mathcal{H}_1; \mathcal{B}_1$ and $v_0 \sim v_1; \mathcal{H}_1; \mathcal{B}_1$*

7532 PROOF. 1. CASE $e_0 \in v$

7533 1.1. QED

7534 2. CASE $e_0 \in w$

7535 2.1. $e_0; \mathcal{H}_0; \mathcal{B}_0 \rightarrow_{\top}^* p_0; p_0 \mapsto e_0, \mathcal{H}_0; p_0 \mapsto \emptyset, \mathcal{B}_0$

7536 2.2. QED

7537 by $v_0 \sim e_0; \mathcal{H}_0; \mathcal{B}_0$ and lemma 9.18

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□

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LEMMA 9.11. *If $e_0 \sim v_0; \mathcal{H}_0; \mathcal{B}_0$ then $e_0 \xrightarrow{*}_A v_1$ and $v_1 \sim v_0; \mathcal{H}_0; \mathcal{B}_0$*

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PROOF. 1. CASE $e_0 \in v$

7546

1.1. QED

7547

2. CASE $e_0 = \text{trace } \bar{b}_0 v_1$

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2.1. $e_0 \xrightarrow{*}_A \text{add-trace}(\bar{b}_0, v_1)$

7549

2.2. QED

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by lemma 9.12

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□

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LEMMA 9.12. *If $v_0 \sim v_1; \mathcal{H}_0; \mathcal{B}_0$ and $\Gamma_0 \vdash_A v_0 : \mathcal{U}$ then $\text{add-trace}(\bar{b}_0, v_0) \sim v_1; \mathcal{H}_0; \mathcal{B}_0$*

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PROOF. By case analysis of $v_0 \sim v_1; \mathcal{H}_0; \mathcal{B}_0$

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1. CASE $\text{add-trace}(\bar{b}_0, v_0) = v_0$

7558

1.1. QED

7559

2. CASE $\text{add-trace}(\bar{b}_0, v_0) = \text{trace}_v \bar{b}_1 v_2$

7560

2.1. SCASE $v_0 = \text{trace}_v^? \bar{b}_2 i_0$

7561

2.1.1. QED

7562

2.2. SCASE $v_0 = \text{trace}_v^? \bar{b}_2 \langle v_2, v_3 \rangle$

7563

2.2.1. QED

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2.3. SCASE $v_0 = \text{trace}_v^? \bar{b}_2 (\text{mon } b_0 \langle v_2, v_3 \rangle)$

7565

2.3.1. QED

7566

2.4. SCASE $v_0 = \text{trace}_v^? \bar{b}_2 (\text{mon } b_0 (\text{trace}_v^? \bar{b}_3 (\text{mon } b_1 \langle v_2, v_3 \rangle)))$

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2.4.1. CONTRADICTION:

7568

by lemma 7.10

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2.5. SCASE $v_0 = \text{trace}_v^? \bar{b}_2 (\lambda x_0. e_0)$

7570

2.5.1. QED

7571

2.6. SCASE $v_0 = \text{trace}_v^? \bar{b}_2 (\text{mon } b_0 (\lambda x_0. e_0))$

7572

2.6.1. CONTRADICTION:

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by $\Gamma_0 \Vdash_A v_0 : \mathcal{U}$

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2.7. SCASE $v_0 = \text{trace}_v^? \bar{b}_2 (\lambda(x_0 : \tau_0). e_0)$

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2.7.1. CONTRADICTION:

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by $\Gamma_0 \Vdash_A v_0 : \mathcal{U}$

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2.8. SCASE $v_0 = \text{trace}_v^? \bar{b}_2 (\text{mon } b_0 (\lambda(x_0 : \tau_0). e_0))$

7578

2.8.1. QED

7579

2.9. SCASE $v_0 = \text{trace}_v^? \bar{b}_2 (\text{mon } b_0 (\text{trace}_v^? \bar{b}_3 (\text{mon } b_1 (\lambda(x_0 : \tau_0). e_0))))$

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2.9.1. CONTRADICTION:

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by lemma 7.10

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LEMMA 9.13. *If $v_0 \sim v_1; \mathcal{H}_0; \mathcal{B}_0$ and $v_0 \in (\lambda(x : \tau). e) \cup \langle v, v \rangle$ then $\text{mon } b_0 v_0 \sim v_1; \mathcal{H}_0; \mathcal{B}_0$*

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PROOF. Immediate from the definition of \sim .

□

LEMMA 9.14. *If $\text{unop}\{\tau?\} v_0 \sim \text{unop}\{\tau?\} v_1; \mathcal{H}_0; \mathcal{B}_0$ and $v_0 \notin \text{mon } b \ v$ then $\delta(\text{unop}, v_0)$ is defined iff $\delta(\text{unop}, v_1)$ is defined. Furthermore, if both are defined, then $\delta(\text{unop}, v_0) \sim \delta(\text{unop}, v_1); \mathcal{H}_0; \mathcal{B}_0$*

PROOF. 1. $v_0 \in \text{trace}_v^? \bar{b} \langle v, v \rangle$ iff $v_1 \in \langle v, v \rangle$

by definition \sim

2. QED

by definition δ

□

LEMMA 9.15. *If $\text{binop}\{\tau?\} v_0 v_1 \sim \text{binop}\{\tau?\} v_2 v_3; \mathcal{H}_0; \mathcal{B}_0$ then $\delta(\text{binop}, v_0, v_1)$ is defined iff $\delta(\text{binop}, v_2, v_3)$ is defined. Furthermore, if both are defined, then $\delta(\text{binop}, v_0, v_1) \sim \delta(\text{binop}, v_2, v_3); \mathcal{H}_0; \mathcal{B}_0$*

PROOF. 1. $v_0 \in i$ iff $v_2 \in i$

and $v_1 \in i$ iff $v_3 \in i$

by definition \sim

2. QED

by definition δ

□

LEMMA 9.16. *If $v_0 \sim v_1; \mathcal{H}_0; \mathcal{B}_0$ then $\text{tag-match}(\lfloor \tau_0 \rfloor, v_0)$ iff $\text{tag-match}(\lfloor \tau_1 \rfloor, v_1)$*

PROOF. By case analysis of $v_0 \sim v_1$; and tag-match

1. CASE $\text{trace}_v^? \bar{b}_0 i_0 \sim i_0; \mathcal{H}_0; \mathcal{B}_0$

1.1. QED

by the definition of tag-match

2. CASE $\text{trace}_v^? \bar{b}_0 \langle v_2, v_3 \rangle \sim p_0; \mathcal{H}_0; \mathcal{B}_0$

and $\mathcal{H}_0(p_0) = \langle v_4, v_5 \rangle$

2.1. QED

by tag-match

3. CASE $\text{trace}_v^? \bar{b}_0 (\text{mon } b_0 \langle v_2, v_3 \rangle) \sim p_0; \mathcal{H}_0; \mathcal{B}_0$

and $\mathcal{H}_0(p_0) = \langle v_4, v_5 \rangle$

3.1. QED

by tag-match

4. CASE $\text{mon } b_0 (\text{trace}_v^? \bar{b}_0 (\text{mon } b_1 \langle v_2, v_3 \rangle)) \sim p_0; \mathcal{H}_0; \mathcal{B}_0$

and $\mathcal{H}_0(p_0) = \langle v_4, v_5 \rangle$

4.1. QED

by tag-match

5. CASE $\text{trace}_v^? \bar{b}_0 (\lambda x_0. e_0) \sim p_0; \mathcal{H}_0; \mathcal{B}_0$

and $\mathcal{H}_0(p_0) = \lambda x_0. e_1$

5.1. QED

7645 by *tag-match*

7646 6. CASE $\text{trace}_v^? \bar{b}_0 (\text{mon } b_0 (\lambda x_0. e_0)) \sim p_0; \mathcal{H}_0; \mathcal{B}_0$
 7647 and $\mathcal{H}_0(p_0) = \lambda x_0. e_1$
 7648

7649 6.1. QED

7650 by *tag-match*

7651 7. CASE $\text{trace}_v^? \bar{b}_0 (\lambda(x_0 : \tau_0). e_0) \sim p_0; \mathcal{H}_0; \mathcal{B}_0$
 7652 and $\mathcal{H}_0(p_0) = \lambda(x_0 : \tau_0). e_1$
 7653

7654 7.1. QED

7655 by *tag-match*

7656 8. CASE $\text{trace}_v^? \bar{b}_0 (\text{mon } b_0 (\lambda(x_0 : \tau_0). e_0)) \sim p_0; \mathcal{H}_0; \mathcal{B}_0$
 7657 and $\mathcal{H}_0(p_0) = \lambda(x_0 : \tau_0). e_1$
 7658

7659 8.1. QED

7660 by *tag-match*

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LEMMA 9.17. If $e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0$ and $v_0 \sim v_1; \mathcal{H}_0; \mathcal{B}_0$ then $e_0[x_0 \leftarrow v_0] \sim e_1[x_0 \leftarrow v_1]; \mathcal{H}_0; \mathcal{B}_0$

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PROOF. By case analysis of $e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0$ and induction on the structure of e_0 and e_1

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LEMMA 9.18. If $e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0$ and \mathcal{H}_1 adds bindings to \mathcal{H}_0 and \mathcal{B}_1 adds bindings and blame information to \mathcal{B}_0 then $e_0 \sim e_1; \mathcal{H}_1; \mathcal{B}_1$

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