Complete Monitoring for Gradual Types: Supplementary Material

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Key concepts: path-based ownership (N 2.2, T 3.2, A 4.2), heap-based ownership (T 3.3), type soundness (N 5.1, A 7.2, T 6.6), complete monitoring (N 5.4), blame soundness + completeness (N 5.5, A 7.5), heap-based blame soundness (T 6.4).

Contents

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1 COMMON DEFINITIONS

1.1 Surface Definitions

Surface Language

$$\begin{array}{lcl}
e & = & x \mid n \mid i \mid \lambda x.\, e \mid \lambda(x\!:\!\tau).\, e \mid \langle e, e\rangle \mid \mathsf{app}\{\tau?\}\; e\; e \mid \mathit{unop}\{\tau?\}\; e \mid \mathit{binop}\{\tau?\}\; e\; e \mid \mathsf{dyn}\; b\; e \mid \mathsf{stat}\; b\; e \\
\tau & = & \mathsf{Nat} \mid \mathsf{Int} \mid \tau\times\tau \mid \tau\Rightarrow\tau \\
\tau? & = & \tau \mid \mathcal{U} \\
b & = & (\ell \blacktriangleleft \tau \blacktriangleleft \ell) \\
\ell & = & \text{countable set of names} \\
\mathit{unop} & = & \mathsf{fst} \mid \mathsf{snd} \\
\mathit{binop} & = & \mathsf{sum} \mid \mathsf{quotient} \\
\Gamma & = & \cdot \mid (x\!:\!\tau?), \Gamma \\
L & = & \cdot \mid (x\!:\!\ell), L \\
n & = & \mathbb{N} \\
i & = & \mathbb{Z} \\
\overline{b} & = & \cdot \mid b, \overline{b}
\end{array}$$

$e : \tau?$ **wf**

$$\begin{array}{lll}
e_0 : \tau_0\ \textbf{wf} & \text{iff} & \exists\, \ell_0.\ \ell_0 \Vdash e_0 \text{ and } \vdash e_0 : \tau_0 \\
e_0 : \mathcal{U}\ \textbf{wf} & \text{iff} & \exists\, \ell_0.\ \ell_0 \Vdash e_0 \text{ and } \vdash e_0 : \mathcal{U}
\end{array}$$

$L; \ell \Vdash e$ well-named components

$$\frac{(x_0\!:\!\ell_0) \in L_0}{L_0; \ell_0 \Vdash x_0} \qquad \frac{}{L_0; \ell_0 \Vdash i_0} \qquad \frac{(x_0\!:\!\ell_0), L_0; \ell_0 \Vdash e_0}{L_0; \ell_0 \Vdash \lambda x_0.\, e_0} \qquad \frac{(x_0\!:\!\ell_0), L_0; \ell_0 \Vdash e_0}{L_0; \ell_0 \Vdash \lambda(x_0\!:\!\tau_0).\, e_0} \qquad \frac{L_0; \ell_0 \Vdash e_0 \quad L_0; \ell_0 \Vdash e_1}{L_0; \ell_0 \Vdash \langle e_0, e_1\rangle}$$

$$\frac{L_0; \ell_0 \Vdash e_0 \quad L_0; \ell_0 \Vdash e_1}{L_0; \ell_0 \Vdash \mathsf{app}\{\tau?\}\; e_0\; e_1} \qquad \frac{L_0; \ell_0 \Vdash e_0}{L_0; \ell_0 \Vdash \mathit{unop}\{\tau?\}\; e_0} \qquad \frac{L_0; \ell_0 \Vdash e_0 \quad L_0; \ell_0 \Vdash e_1}{L_0; \ell_0 \Vdash \mathit{binop}\{\tau?\}\; e_0\; e_1} \qquad \frac{L_0; \ell_1 \Vdash e_0}{L_0; \ell_0 \Vdash \mathsf{dyn}\; (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; e_0}$$

$$\frac{L_0; \ell_1 \Vdash e_0}{L_0; \ell_0 \Vdash \mathsf{stat}\; (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; e_0}$$

$\boxed{\Gamma \vdash e : \tau}$ static typing

$$\frac{(x_0 : \tau_0) \in \Gamma_0}{\Gamma_0 \vdash x_0 : \tau_0} \qquad \frac{}{\Gamma_0 \vdash n_0 : \mathsf{Nat}} \qquad \frac{}{\Gamma_0 \vdash i_0 : \mathsf{Int}} \qquad \frac{(x_0 : \tau_0), \Gamma_0 \vdash e_0 : \tau_1}{\Gamma_0 \vdash \lambda(x_0 : \tau_0).\, e_0 : \tau_0 \Rightarrow \tau_1} \qquad \frac{\Gamma_0 \vdash e_0 : \tau_0 \quad \Gamma_0 \vdash e_1 : \tau_1}{\Gamma_0 \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1}$$

$$\frac{\Gamma_0 \vdash e_0 : \tau_0 \Rightarrow \tau_1 \quad \Gamma_0 \vdash e_1 : \tau_0}{\Gamma_0 \vdash \mathsf{app}\{\tau_1\}\, e_0\, e_1 : \tau_1} \qquad \frac{\Gamma_0 \vdash e_0 : \tau_0 \quad \Delta(unop, \tau_0) = \tau_1 \quad \tau_1 \leqslant: \tau_2}{\Gamma_0 \vdash unop\{\tau_2\}\, e_0 : \tau_2} \qquad \frac{\Gamma_0 \vdash e_0 : \tau_0 \quad \Gamma_0 \vdash e_1 : \tau_1 \quad \Delta(binop, \tau_0, \tau_1) = \tau_2 \quad \tau_2 \leqslant: \tau_3}{\Gamma_0 \vdash binop\{\tau_3\}\, e_0\, e_1 : \tau_3} \qquad \frac{\Gamma_0 \vdash e_0 : \mathcal{U}}{\Gamma_0 \vdash \mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, e_0 : \tau_0}$$

$$\frac{\Gamma_0 \vdash e_0 : \tau_0 \quad \tau_0 \leqslant: \tau_1}{\Gamma_0 \vdash e_0 : \tau_1}$$

$\boxed{\Gamma \vdash e : \mathcal{U}}$ dynamic typing

$$\frac{x_0 \in \Gamma_0}{\Gamma_0 \vdash x_0 : \mathcal{U}} \qquad \frac{}{\Gamma_0 \vdash i_0 : \mathcal{U}} \qquad \frac{(x_0 : \mathcal{U}), \Gamma_0 \vdash e_0 : \mathcal{U}}{\Gamma_0 \vdash \lambda x_0.\, e_0 : \mathcal{U}} \qquad \frac{\Gamma_0 \vdash e_0 : \mathcal{U} \quad \Gamma_0 \vdash e_1 : \mathcal{U}}{\Gamma_0 \vdash \langle e_0, e_1 \rangle : \mathcal{U}} \qquad \frac{\Gamma_0 \vdash e_0 : \mathcal{U} \quad \Gamma_0 \vdash e_1 : \mathcal{U}}{\Gamma_0 \vdash \mathsf{app}\{\mathcal{U}\}\, e_0\, e_1 : \mathcal{U}}$$

$$\frac{\Gamma_0 \vdash e_0 : \mathcal{U}}{\Gamma_0 \vdash unop\{\mathcal{U}\}\, e_0 : \mathcal{U}} \qquad \frac{\Gamma_0 \vdash e_0 : \mathcal{U} \quad \Gamma_0 \vdash e_1 : \mathcal{U}}{\Gamma_0 \vdash binop\{\mathcal{U}\}\, e_0\, e_1 : \mathcal{U}} \qquad \frac{\Gamma_0 \vdash e_0 : \tau_0}{\Gamma_0 \vdash \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, e_0 : \mathcal{U}}$$

$\boxed{\tau \leqslant: \tau}$

$$\frac{}{\mathsf{Nat} \leqslant: \mathsf{Int}} \qquad \frac{\tau_0 \leqslant: \tau_2 \quad \tau_1 \leqslant: \tau_3}{\tau_0 \times \tau_1 \leqslant: \tau_2 \times \tau_3} \qquad \frac{\tau_2 \leqslant: \tau_0 \quad \tau_1 \leqslant: \tau_3}{\tau_0 \Rightarrow \tau_1 \leqslant: \tau_2 \Rightarrow \tau_3} \qquad \frac{}{\tau_0 \leqslant: \tau_0}$$

$\boxed{b \leqslant: b}$

$$\frac{\tau_0 \leqslant: \tau_1}{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \leqslant: (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)}$$

$\boxed{\Delta : unop \times \tau \longrightarrow \tau}$

$\Delta(\mathsf{fst}, \tau_0 \times \tau_1) = \tau_0$

$\Delta(\mathsf{snd}, \tau_0 \times \tau_1) = \tau_1$

$\boxed{\Delta : binop{\times}\tau{\times}\tau \longrightarrow \tau}$

$$
\begin{aligned}
&\Delta(\mathsf{sum}, \mathsf{Nat}, \mathsf{Nat}) &&= \mathsf{Nat}\\
&\Delta(\mathsf{sum}, \mathsf{Int}, \mathsf{Int}) &&= \mathsf{Int}\\
&\Delta(\mathsf{quotient}, \mathsf{Nat}, \mathsf{Nat}) &&= \mathsf{Nat}\\
&\Delta(\mathsf{quotient}, \mathsf{Int}, \mathsf{Int}) &&= \mathsf{Int}
\end{aligned}
$$

1.2 Evaluation Definitions

$\boxed{\text{Base Evaluation Language}}$

$$
\begin{aligned}
v &= n \mid i \mid \langle v, v\rangle \mid \lambda x.\, e \mid \lambda(x\!:\!\tau).\, e\\
E &= [\,] \mid \langle E, e\rangle \mid \langle v, E\rangle \mid unop\{\tau?\}\,E \mid \mathsf{app}\{\tau?\}\,E\,e \mid \mathsf{app}\{\tau?\}\,v\,E \mid binop\{\tau?\}\,E\,e \mid binop\{\tau?\}\,v\,E \mid \mathsf{dyn}\,b\,E \mid\\
&\quad\ \mathsf{stat}\,b\,E\\
\mathsf{Err} &= \mathsf{TagErr}\,\bullet \mid \mathsf{TagErr}\,\circ \mid \mathsf{DivErr} \mid \mathsf{BndryErr}\,(\overline{b}, v)\\
e &= \mathsf{Err} \mid x \mid n \mid i \mid \lambda x.\, e \mid \lambda(x\!:\!\tau).\, e \mid \langle e, e\rangle \mid \mathsf{app}\{\tau?\}\,e\,e \mid unop\{\tau?\}\,e \mid binop\{\tau?\}\,e\,e \mid \mathsf{dyn}\,b\,e \mid \mathsf{stat}\,b\,e\\
K &= \mathsf{Nat} \mid \mathsf{Int} \mid \mathsf{Pair} \mid \mathsf{Fun}\\
\tau &= \mathsf{Nat} \mid \mathsf{Int} \mid \tau{\times}\tau \mid \tau \Rightarrow \tau\\
\tau? &= \tau \mid \mathcal{U}\\
b &= (\ell \blacktriangleleft \tau \blacktriangleleft \ell)\\
\ell &= \text{countable set of names}\\
unop &= \mathsf{fst} \mid \mathsf{snd}\\
binop &= \mathsf{sum} \mid \mathsf{quotient}\\
\Gamma &= \cdot \mid (x\!:\!\tau?), \Gamma\\
L &= \cdot \mid (x\!:\!\ell), L\\
n &= \mathbb{N}\\
i &= \mathbb{Z}\\
\overline{b} &= \cdot \mid b, \overline{b}\\
b^* &= \mathcal{P}(b)\\
\ell^* &= \mathcal{P}(\ell)
\end{aligned}
$$

$\boxed{\mathit{tag\text{-}match} : K \times v \longrightarrow \mathcal{B}}$

$$\mathit{tag\text{-}match}(K_0, v_0) = \begin{cases} \mathsf{True} \\ \quad \text{if } K_0 = \mathsf{Nat} \text{ and } v_0 \in n \\ \quad \text{or } K_0 = \mathsf{Int} \text{ and } v_0 \in i \\ \quad \text{or } K_0 = \mathsf{Pair} \text{ and} \\ \qquad v_0 \in \langle v, v \rangle \cup \\ \qquad\qquad (\mathsf{mon}\ (\ell \blacktriangleleft (\tau \times \tau) \blacktriangleleft \ell)\ v) \\ \quad \text{or } K_0 = \mathsf{Fun} \text{ and} \\ \qquad v_0 \in (\lambda x.\, e) \cup (\lambda(x : \tau).\, e) \cup \\ \qquad\qquad (\mathsf{mon}\ (\ell \blacktriangleleft (\tau \Rightarrow \tau) \blacktriangleleft \ell)\ v) \\ \mathit{tag\text{-}match}(K_0, v_1) \\ \quad \text{if } v_0 = \mathsf{trace}_{\mathsf{v}}\ \overline{b}_0\ v_1 \\ \mathsf{False} \\ \quad \text{otherwise} \end{cases}$$

$\boxed{\lfloor \cdot \rfloor : \tau \longrightarrow K}$

$$\lfloor \tau_0 \rfloor = \begin{cases} \mathsf{Nat} & \text{if } \tau_0 = \mathsf{Nat} \\ \mathsf{Int} & \text{if } \tau_0 = \mathsf{Int} \\ \mathsf{Pair} & \text{if } \tau_0 \in \tau \times \tau \\ \mathsf{Fun} & \text{if } \tau_0 \in \tau \Rightarrow \tau \end{cases}$$

$\boxed{\mathit{rev} : \overline{b} \longrightarrow \overline{b}}$

$$\mathit{rev}((\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1), \cdots, (\ell_{2n} \blacktriangleleft \tau_n \blacktriangleleft \ell_{2n+1})) = (\ell_{2n+1} \blacktriangleleft \tau_n \blacktriangleleft \ell_{2n}), \cdots, (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0)$$

$\boxed{\delta : \mathit{unop} \times v \longrightarrow e}$

$$\delta(\mathit{unop}, \langle v_0, v_1 \rangle) = \begin{cases} v_0 \\ \quad \text{if } \mathit{unop} = \mathsf{fst}\{\tau?\} \\ v_1 \\ \quad \text{if } \mathit{unop} = \mathsf{snd}\{\tau?\} \end{cases}$$

$\boxed{\delta : \mathit{binop} \times v \times v \longrightarrow e}$

$$\delta(\mathit{binop}, i_0, i_1) = \begin{cases} i_0 + i_1 \\ \quad \text{if } \mathit{binop} = \mathsf{sum}\{\tau?\} \\ \mathsf{DivErr} \\ \quad \text{if } \mathit{binop} = \mathsf{quotient}\{\tau?\} \\ \quad \text{and } i_1 = 0 \\ \lfloor i_0 / i_1 \rfloor \\ \quad \text{if } \mathit{binop} = \mathsf{quotient}\{\tau?\} \\ \quad \text{and } i_1 \neq 0 \end{cases}$$

1.3 Ownership Evaluation Definitions

Ownership Evaluation Language

ℓ = countable set of labels, with 1-1 correspondence to names

$\ell_\bullet$ = label for the "top" of an expression

$\overline{\ell}$ = $\cdot \mid \overline{\ell}, \ell$

v = $n \mid i \mid \langle v, v\rangle \mid \lambda x.\, e \mid \lambda(x\!:\!\tau).\, e$

E = $[\,] \mid (E)^{\ell} \mid \langle E, e\rangle \mid \langle v, E\rangle \mid \mathit{unop}\{\tau?\}\, E \mid \mathsf{app}\{\tau?\}\, E\, e \mid \mathsf{app}\{\tau?\}\, v\, E \mid \mathit{binop}\{\tau?\}\, E\, e \mid \mathit{binop}\{\tau?\}\, v\, E \mid$ $\mathsf{dyn}\, b\, E \mid \mathsf{stat}\, b\, E$

Err = $\mathsf{TagErr}\,\bullet \mid \mathsf{TagErr}\,\circ \mid \mathsf{DivErr} \mid \mathsf{BndryErr}\,(\overline{b}, v)$

e = $(e)^{\ell} \mid \mathsf{Err} \mid x \mid n \mid i \mid \lambda x.\, e \mid \lambda(x\!:\!\tau).\, e \mid \langle e, e\rangle \mid \mathsf{app}\{\tau?\}\, e\, e \mid \mathit{unop}\{\tau?\}\, e \mid \mathit{binop}\{\tau?\}\, e\, e \mid \mathsf{dyn}\, b\, e \mid \mathsf{stat}\, b\, e$

K = $\mathsf{Nat} \mid \mathsf{Int} \mid \mathsf{Pair} \mid \mathsf{Fun}$

τ = $\mathsf{Nat} \mid \mathsf{Int} \mid \tau\times\tau \mid \tau\Rightarrow\tau$

$\tau?$ = $\tau \mid \mathcal{U}$

b = $(\ell \blacktriangleleft \tau \blacktriangleleft \ell)$

ℓ = countable set of names

unop = $\mathsf{fst} \mid \mathsf{snd}$

binop = $\mathsf{sum} \mid \mathsf{quotient}$

Γ = $\cdot \mid (x\!:\!\tau?), \Gamma$

L = $\cdot \mid (x\!:\!\ell), L$

n = $\mathbb{N}$

i = $\mathbb{Z}$

$\overline{b}$ = $\cdot \mid b, \overline{b}$

b^* = $\mathcal{P}(b)$

ℓ^* = $\mathcal{P}(\ell)$

$e\!:\!\tau?\ \overline{\mathbf{wf}}$ well-formed expression

$(e_0)^{\ell_0}\!:\!\tau_0\ \overline{\mathbf{wf}}$ iff $\ell_0 \Vdash (e_0)^{\ell_0}$ and $\vDash (e_0)^{\ell_0} : \tau_0$

$(e_0)^{\ell_0}\!:\!\mathcal{U}\ \overline{\mathbf{wf}}$ iff $\ell_0 \Vdash (e_0)^{\ell_0}$ and $\vDash (e_0)^{\ell_0} : \mathcal{U}$

$\boxed{L; \ell \Vdash e}$ (selected rules)

$$\frac{L_0; \ell_0 \Vdash e_0}{L_0; \ell_0 \Vdash (e_0)^{\ell_0}} \qquad \frac{(x_0 \!:\! \ell_0) \in L_0}{L_0; \ell_0 \Vdash x_0} \qquad \frac{}{L_0; \ell_0 \Vdash i_0} \qquad \frac{(x_0 \!:\! \ell_0), L_0; \ell_0 \Vdash e_0}{L_0; \ell_0 \Vdash \lambda x_0.\, e_0} \qquad \frac{(x_0 \!:\! \ell_0), L_0; \ell_0 \Vdash e_0}{L_0; \ell_0 \Vdash \lambda (x_0 \!:\! \tau_0).\, e_0}$$

$$\frac{\begin{array}{c} L_0; \ell_0 \Vdash e_0 \\ L_0; \ell_0 \Vdash e_1 \end{array}}{L_0; \ell_0 \Vdash \langle e_0, e_1 \rangle} \qquad \frac{\begin{array}{c} L_0; \ell_0 \Vdash e_0 \\ L_0; \ell_0 \Vdash e_1 \end{array}}{L_0; \ell_0 \Vdash \mathsf{app}\{\tau?\}\, e_0\, e_1} \qquad \frac{L_0; \ell_0 \Vdash e_0}{L_0; \ell_0 \Vdash \mathit{unop}\{\tau?\}\, e_0} \qquad \frac{\begin{array}{c} L_0; \ell_0 \Vdash e_0 \\ L_0; \ell_0 \Vdash e_1 \end{array}}{L_0; \ell_0 \Vdash \mathit{binop}\{\tau?\}\, e_0\, e_1}$$

$$\frac{L_0; \ell_1 \Vdash e_0}{L_0; \ell_0 \Vdash \mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (e_0)^{\ell_1}} \qquad \frac{L_0; \ell_1 \Vdash e_0}{L_0; \ell_0 \Vdash \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (e_0)^{\ell_1}} \qquad \frac{L_0; \ell_1 \Vdash v_0}{L_0; \ell_0 \Vdash \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (v_0)^{\ell_1}}$$

$$\frac{L_0; \ell_0 \Vdash v_0}{L_0; \ell_0 \Vdash \mathsf{trace}_{\mathsf{v}}\ \overline{b}_0\ v_0} \qquad \frac{L_0; \ell_0 \Vdash e_0}{L_0; \ell_0 \Vdash \mathsf{trace}\ \overline{b}_0\ e_0}$$

$\boxed{\Gamma \vdash e : \tau}$

$$\frac{\Gamma_0 \vdash e_0 : \tau_0}{\Gamma_0 \vdash (e_0)^{\ell_0} : \tau_0} \qquad \frac{(x_0 \!:\! \tau_0) \in \Gamma_0}{\Gamma_0 \vdash x_0 : \tau_0} \qquad \frac{}{\Gamma_0 \vdash n_0 : \mathsf{Nat}} \qquad \frac{}{\Gamma_0 \vdash i_0 : \mathsf{Int}} \qquad \frac{(x_0 \!:\! \tau_0), \Gamma_0 \vdash e_0 : \tau_1}{\Gamma_0 \vdash \lambda (x_0 \!:\! \tau_0).\, e_0 : \tau_0 \Rightarrow \tau_1}$$

$$\frac{\begin{array}{c} \Gamma_0 \vdash e_0 : \tau_0 \\ \Gamma_0 \vdash e_1 : \tau_1 \end{array}}{\Gamma_0 \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1} \qquad \frac{\begin{array}{c} \Gamma_0 \vdash e_0 : \tau_0 \Rightarrow \tau_1 \\ \Gamma_0 \vdash e_1 : \tau_0 \end{array}}{\Gamma_0 \vdash \mathsf{app}\{\tau_1\}\, e_0\, e_1 : \tau_1} \qquad \frac{\begin{array}{c} \Gamma_0 \vdash e_0 : \tau_0 \\ \Delta(\mathit{unop}, \tau_0) = \tau_1 \end{array}}{\Gamma_0 \vdash \mathit{unop}\{\tau_1\}\, e_0 : \tau_1} \qquad \frac{\begin{array}{c} \Gamma_0 \vdash e_0 : \tau_0 \\ \Gamma_0 \vdash e_1 : \tau_1 \\ \Delta(\mathit{binop}, \tau_0, \tau_1) = \tau_2 \end{array}}{\Gamma_0 \vdash \mathit{binop}\{\tau_2\}\, e_0\, e_1 : \tau_2}$$

$$\frac{\Gamma_0 \vdash e_0 : \mathcal{U}}{\Gamma_0 \vdash \mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ e_0 : \tau_0} \qquad \frac{\begin{array}{c} \Gamma_0 \vdash e_0 : \tau_0 \\ \tau_0 \leqslant: \tau_1 \end{array}}{\Gamma_0 \vdash e_0 : \tau_1}$$

$\boxed{\Gamma \vdash e : \mathcal{U}}$

$$\frac{\Gamma_0 \vdash e_0 : \mathcal{U}}{\Gamma_0 \vdash (e_0)^{\ell_0} : \mathcal{U}} \qquad \frac{x_0 \in \Gamma_0}{\Gamma_0 \vdash x_0 : \mathcal{U}} \qquad \frac{}{\Gamma_0 \vdash i_0 : \mathcal{U}} \qquad \frac{(x_0 \!:\! \mathcal{U}), \Gamma_0 \vdash e_0 : \mathcal{U}}{\Gamma_0 \vdash \lambda x_0.\, e_0 : \mathcal{U}} \qquad \frac{\begin{array}{c} \Gamma_0 \vdash e_0 : \mathcal{U} \\ \Gamma_0 \vdash e_1 : \mathcal{U} \end{array}}{\Gamma_0 \vdash \langle e_0, e_1 \rangle : \mathcal{U}}$$

$$\frac{\begin{array}{c} \Gamma_0 \vdash e_0 : \mathcal{U} \\ \Gamma_0 \vdash e_1 : \mathcal{U} \end{array}}{\Gamma_0 \vdash \mathsf{app}\{\mathcal{U}\}\, e_0\, e_1 : \mathcal{U}} \qquad \frac{\Gamma_0 \vdash e_0 : \mathcal{U}}{\Gamma_0 \vdash \mathit{unop}\{\mathcal{U}\}\, e_0 : \mathcal{U}} \qquad \frac{\begin{array}{c} \Gamma_0 \vdash e_0 : \mathcal{U} \\ \Gamma_0 \vdash e_1 : \mathcal{U} \end{array}}{\Gamma_0 \vdash \mathit{binop}\{\mathcal{U}\}\, e_0\, e_1 : \mathcal{U}} \qquad \frac{\Gamma_0 \vdash e_0 : \tau_0}{\Gamma_0 \vdash \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ e_0 : \mathcal{U}}$$

$\boxed{\mathit{fst} : \tau \longrightarrow \tau}$

$\mathit{fst}(\tau_0 \times \tau_1) = \tau_0$

$\boxed{snd : \tau \longrightarrow \tau}$

$snd(\tau_0 \times \tau_1) = \tau_1$

$\boxed{dom : \tau \longrightarrow \tau}$

$dom(\tau_0 \Rightarrow \tau_1) = \tau_0$

$\boxed{cod : \tau \longrightarrow \tau}$

$cod(\tau_0 \Rightarrow \tau_1) = \tau_1$

$\boxed{\cdot\cdot : \overline{\ell} \times \overline{\ell} \longrightarrow \overline{\ell}}$ append labels, but remove consecutive duplicates

$(\ell_0, \cdots, \ell_n)(\ell_m, \cdots, \ell_k) = \ell_0, \cdots, \ell_n, \ell_m, \cdots, \ell_k$
  if $\ell_n \neq \ell_m$

$(\ell_0, \cdots, \ell_n)(\ell_n, \cdots, \ell_k) = \ell_0, \cdots, \ell_n, \cdots, \ell_k$

$\boxed{\cdot\cdot : \overline{b} \times \overline{b} \longrightarrow \overline{b}}$ append boundaries

$\cdot\overline{b}_1 = \overline{b}_1$

$(b_0, \overline{b}_0)\overline{b}_1 = b_0, (\overline{b}_0\overline{b}_1)$

$\boxed{rev : \overline{\ell} \longrightarrow \overline{\ell}}$ reverse a list of ownership labels

$rev(\ell_0, \cdots, \ell_n) = \ell_n, \cdots, \ell_0$

$\boxed{\overline{b} \simeq \overline{\ell}}$

$$\frac{}{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1), \cdot \simeq \cdot, \ell_1, \ell_0} \qquad \frac{\overline{b}_1 \simeq \overline{\ell}_2, \ell_1}{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1), \overline{b}_1 \simeq \overline{\ell}_2, \ell_1, \ell_0}$$

$\boxed{\textit{hole-owner} : E \longrightarrow \ell}$ return the ownership label that surrounds the hole

$\textit{hole-owner}(E_0) = \textit{h-own}(\ell_\bullet, E_0)$

$\boxed{\textit{h-own} : \ell \times E \longrightarrow \ell}$

$\textit{h-own}(\ell_0, [\,]) = \ell_0$

$\textit{h-own}(\ell_0, \langle E_0, e_1 \rangle) = \textit{h-own}(\ell_0, E_0)$

$\textit{h-own}(\ell_0, \langle v_0, E_1 \rangle) = \textit{h-own}(\ell_0, E_1)$

$\textit{h-own}(\ell_0, \mathsf{app}\{\tau?\}\, E_0\; e_1) = \textit{h-own}(\ell_0, E_0)$

$\textit{h-own}(\ell_0, \mathsf{app}\{\tau?\}\, v_0\; E_1) = \textit{h-own}(\ell_0, E_1)$

$\textit{h-own}(\ell_0, \textit{unop}\{\tau?\}\, E_0) = \textit{h-own}(\ell_0, E_0)$

$\textit{h-own}(\ell_0, \textit{binop}\{\tau?\}\, E_0\; e_1) = \textit{h-own}(\ell_0, E_0)$

$\textit{h-own}(\ell_0, \textit{binop}\{\tau?\}\, v_0\; E_1) = \textit{h-own}(\ell_0, E_1)$

$\textit{h-own}(\ell_0, \mathsf{dyn}\,(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\; E_0) = \textit{h-own}(\ell_0, E_0)$

$\textit{h-own}(\ell_0, \mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\; E_0) = \textit{h-own}(\ell_0, E_0)$

$\textit{h-own}(\ell_0, (E_0)^{\ell_1}) = \textit{h-own}(\ell_1, E_0)$

$\boxed{\mathit{has\text{-}boundary}(e, b)}$ check if a boundary appears in an expression

$$\frac{\mathit{has\text{-}boundary}(e_0, b_0)}{\mathit{has\text{-}boundary}(\lambda x_0.\, e_0, b_0)} \qquad \frac{\mathit{has\text{-}boundary}(e_0, b_0)}{\mathit{has\text{-}boundary}(\lambda(x_0 : \tau_0).\, e_0, b_0)} \qquad \frac{\mathit{has\text{-}boundary}(e_0, b_0)}{\mathit{has\text{-}boundary}(\langle e_0, e_1\rangle, b_0)}$$

$$\frac{\mathit{has\text{-}boundary}(e_1, b_0)}{\mathit{has\text{-}boundary}(\langle e_0, e_1\rangle, b_0)} \qquad \frac{\mathit{has\text{-}boundary}(e_0, b_0)}{\mathit{has\text{-}boundary}(\mathsf{app}\{\tau?\}\; e_0\; e_1, b_0)} \qquad \frac{\mathit{has\text{-}boundary}(e_1, b_0)}{\mathit{has\text{-}boundary}(\mathsf{app}\{\tau?\}\; e_0\; e_1, b_0)}$$

$$\frac{\mathit{has\text{-}boundary}(e_0, b_0)}{\mathit{has\text{-}boundary}(\mathit{unop}\{\tau?\}\; e_0, b_0)} \qquad \frac{\mathit{has\text{-}boundary}(e_0, b_0)}{\mathit{has\text{-}boundary}(\mathit{binop}\{\tau?\}\; e_0\; e_1, b_0)} \qquad \frac{\mathit{has\text{-}boundary}(e_1, b_0)}{\mathit{has\text{-}boundary}(\mathit{binop}\{\tau?\}\; e_0\; e_1, b_0)}$$

$$\frac{}{\mathit{has\text{-}boundary}(\mathsf{dyn}\; b_0\; e_0, b_0)} \qquad \frac{\mathit{has\text{-}boundary}(e_0, b_0)}{\mathit{has\text{-}boundary}(\mathsf{dyn}\; b_1\; e_0, b_0)} \qquad \frac{}{\mathit{has\text{-}boundary}(\mathsf{stat}\; b_0\; e_0, b_0)}$$

$$\frac{\mathit{has\text{-}boundary}(e_0, b_0)}{\mathit{has\text{-}boundary}(\mathsf{stat}\; b_1\; e_0, b_0)}$$

$\boxed{\mathit{add\text{-}trace} : \overline{b} \times v \longrightarrow v}$ extend existing trace (if any), otherwise start a new one

$\mathit{add\text{-}trace}(\cdot, v_1) = v_1$

$\mathit{add\text{-}trace}(\overline{b}_0, (\!(\mathsf{trace_v}\; \overline{b}_1\; v_1)\!)^{\overline{\ell}_2}) = \mathsf{trace_v}\; \overline{b}_0 \overline{b}_1\; (\!(v_1)\!)^{\overline{\ell}_2}$
  if $\overline{b}_0 \neq \cdot$

$\mathit{add\text{-}trace}(\overline{b}_0, v_1) = \mathsf{trace_v}\; \overline{b}_0\; v_1$
  if $\overline{b}_0 \neq \cdot$ and $v_1 \notin (\!(\mathsf{trace_v}\; \overline{b}\; v)\!)^{\overline{\ell}}$

$\boxed{\mathit{get\text{-}trace} : v \longrightarrow \overline{b}}$ get trace (if any) from a value

$\mathit{get\text{-}trace}(\mathsf{trace_v}\; \overline{b}_0\; (\!(v_0)\!)^{\overline{\ell}_1}) = \overline{b}_0$

$\mathit{get\text{-}trace}(v_0) = \cdot$
  if $v_0 \notin \mathsf{trace_v}\; \overline{b}\; (\!(v)\!)^{\overline{\ell}}$

$\boxed{\mathit{rem\text{-}trace} : v \longrightarrow v}$ remove trace (if any) from a value

$\mathit{rem\text{-}trace}((\!(\mathsf{trace_v}\; \overline{b}_0\; (\!(v_0)\!)^{\overline{\ell}_1})\!)^{\overline{\ell}_2}) = (\!(v_0)\!)^{\overline{\ell}_1 \overline{\ell}_2}$

$\mathit{rem\text{-}trace}(v_0) = v_0$
  if $v_0 \notin (\!(\mathsf{trace_v}\; \overline{b}\; (\!(v)\!)^{\overline{\ell}})\!)^{\overline{\ell}}$

$\boxed{\mathit{owners} : v \longrightarrow \overline{\ell}}$

$\mathit{owners}(v_0) = \{\ell_0\} \cup \mathit{owners}(v_1)$
  if $v_0 = (v_1)^{\ell_0}$

$\mathit{owners}(v_0) = \mathit{owners}(v_1)$
  if $v_0 = \mathsf{trace_v}\; \overline{b}_0\; v_1$

$\mathit{owners}(v_0) = \{\}$
  otherwise

$\boxed{senders : \overline{b} \longrightarrow \overline{\ell}}$

$$\begin{aligned}
&senders(\cdot) = \{\}\\
&senders(\overline{b}_0) = \{\ell_1\} \cup senders(\overline{b}_1)\\
&\quad \text{if } \overline{b}_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1), \overline{b}_1
\end{aligned}$$

$\boxed{forget : e \longrightarrow e}$

$$\begin{aligned}
&forget((e_0)^{\ell_0}) &&= forget(e_0)\\
&forget(x_0) &&= x_0\\
&forget(i_0) &&= i_0\\
&forget(\lambda x_0.\, e_0) &&= \lambda x_0.\, forget(e_0)\\
&forget(\lambda(x_0 : \tau_0).\, e_0) &&= \lambda(x_0 : \tau_0).\, forget(e_0)\\
&forget(\langle e_0, e_1\rangle) &&= \langle forget(e_0), forget(e_1)\rangle\\
&forget(\mathsf{app}\{\tau?\}\; e_0\; e_1) &&= \mathsf{app}\{\tau?\}\; forget(e_0)\; forget(e_1)\\
&forget(unop\{\tau?\}\; e_0) &&= unop\{\tau?\}\; forget(e_0)\\
&forget(binop\{\tau?\}\; e_0\; e_1) &&= binop\{\tau?\}\; forget(e_0)\; forget(e_1)\\
&forget(\mathsf{dyn}\; b_0\; e_0) &&= \mathsf{dyn}\; b_0\; forget(e_0)\\
&forget(\mathsf{stat}\; b_0\; e_0) &&= \mathsf{stat}\; b_0\; forget(e_0)
\end{aligned}$$

$\boxed{mon\text{-}depth : v \longrightarrow n}$ count monitors around a value

$$\begin{aligned}
&mon\text{-}depth(i) &&= 0\\
&mon\text{-}depth(\langle v_0, v_1\rangle) &&= 0\\
&mon\text{-}depth(\lambda x_0.\, e_0) &&= 0\\
&mon\text{-}depth(\lambda(x_0 : \tau_0).\, e_0) &&= 0\\
&mon\text{-}depth(\mathsf{trace}_{\mathsf{v}}\; \overline{b}_0\; v_0) &&= mon\text{-}depth(v_0)\\
&mon\text{-}depth(\mathsf{mon}\; b_0\; v_0) &&= *1 + mon\text{-}depth(v_0)
\end{aligned}$$

$\boxed{last : \overline{\ell} \longrightarrow \ell}$ count monitors around a value

$$last(\ell_0 \cdots \ell_n) = \ell_n$$

1.4 Abbreviations

$$\begin{aligned}
&e_0 = (\!(e_1)\!)^{\overline{\ell}_0} \iff e_0 = (\cdots(e_1)^{\ell_n}\cdots)^{\ell_1}\\
&(\mathsf{trace}^?_{\mathsf{v}}\; \overline{b}_0\; v_1) = v_0 \iff rem\text{-}trace(v_0) = v_1 \text{ and } get\text{-}trace(v_0) = \overline{b}_0\\
&(\mathsf{mon}^?\; b_0\; v_1) = v_0 \iff \text{if } v_0 = \mathsf{mon}\; b_0\; v_1 \text{ or } v_0 \notin \mathsf{mon}\; b\; v\\
&(\mathsf{mon}^{+?}\; b_0 \ldots b_n\; v_1) = v_0 \iff v_0 = \mathsf{mon}\; b_0\; (\ldots \mathsf{mon}\; b_n\; v_0 \ldots)\\
&E_0[e_0]^{\ell_0} \iff hole\text{-}owner(E_0) = \ell_0
\end{aligned}$$

2 Natural

2.1 **Normal** Natural

$\boxed{\text{Natural Language}}$ extends Base Evaluation Language

$v \;=\; n \mid i \mid \langle v, v\rangle \mid \lambda x.\, e \mid \lambda(x\!:\!\tau).\, e \mid \mathsf{mon}\,(\ell \blacktriangleleft \tau \Rightarrow \tau \blacktriangleleft \ell)\; v$

$\boxed{\Gamma \vdash_{\mathsf{N}} e : \tau}$ extends $\Gamma \vdash e : \tau$

$$\frac{\Gamma \vdash_{\mathsf{N}} v_1 : \mathcal{U}}{\Gamma \vdash_{\mathsf{N}} \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_1 : \tau_0} \qquad \frac{}{\Gamma \vdash_{\mathsf{N}} \mathsf{Err} : \tau_0}$$

$\boxed{\Gamma \vdash_{\mathsf{N}} e : \mathcal{U}}$ extends $\Gamma \vdash e : \mathcal{U}$

$$\frac{\Gamma \vdash_{\mathsf{N}} v_1 : \tau_0}{\Gamma \vdash_{\mathsf{N}} \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_1 : \mathcal{U}} \qquad \frac{}{\Gamma \vdash_{\mathsf{N}} \mathsf{Err} : \mathcal{U}}$$

$\boxed{\rightarrow^*_{\mathsf{N}}}$ reflexive-transitive closure of $\rightarrow_{\mathsf{N}}$

$\boxed{\rightarrow_{\mathsf{N}}}$ compatible closure of $\rhd_{\mathsf{N}} \cup \blacktriangleright_{\mathsf{N}}$

$\boxed{e \rhd_{\mathsf{N}} e}$

$unop\{\tau_0\}\; v_0 \quad \rhd_{\mathsf{N}} \quad \mathsf{TagErr}\circ$
  if $\delta(unop, v_0)$ is undefined

$unop\{\tau_0\}\; v_0 \quad \rhd_{\mathsf{N}} \quad \delta(unop, v_0)$
  if $\delta(unop, v_0)$ is defined

$binop\{\tau_0\}\; v_0\; v_1 \quad \rhd_{\mathsf{N}} \quad \mathsf{TagErr}\circ$
  if $\delta(binop, v_0, v_1)$ is undefined

$binop\{\tau_0\}\; v_0\; v_1 \quad \rhd_{\mathsf{N}} \quad \delta(binop, v_0, v_1)$
  if $\delta(binop, v_0, v_1)$ is defined

$\mathsf{app}\{\tau_0\}\; v_0\; v_1 \quad \rhd_{\mathsf{N}} \quad \mathsf{TagErr}\circ$
  if $v_0 \notin (\lambda x.\, e) \cup (\mathsf{mon}\; b\; v)$

$\mathsf{app}\{\tau_0\}\; (\lambda(x_0\!:\!\tau_1).\, e_0)\; v_1 \quad \rhd_{\mathsf{N}} \quad e_0[x_0 \leftarrow v_1]$

$\mathsf{app}\{\tau_0\}\; (\mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\; v_0)\; v_1 \quad \rhd_{\mathsf{N}} \quad \mathsf{dyn}\; b_0\; (\mathsf{app}\{\mathcal{U}\}\; v_0\; (\mathsf{stat}\; b_1\; v_1))$
  where $b_0 = (\ell_0 \blacktriangleleft cod(\tau_1) \blacktriangleleft \ell_1)$ and $b_1 = (\ell_1 \blacktriangleleft dom(\tau_1) \blacktriangleleft \ell_0)$

$\mathsf{dyn}\,(\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1)\; v_0 \quad \rhd_{\mathsf{N}} \quad \mathsf{mon}\,(\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1)\; v_0$
  if $tag\text{-}match(\lfloor \tau_0 \Rightarrow \tau_1 \rfloor, v_0)$

$\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; \langle v_0, v_1\rangle \quad \rhd_{\mathsf{N}} \quad \langle \mathsf{dyn}\; b_0\; v_0, \mathsf{dyn}\; b_1\; v_1\rangle$
  if $tag\text{-}match(\lfloor \tau_0 \rfloor, \langle v_0, v_1\rangle)$ and $b_0 = (\ell_0 \blacktriangleleft fst(\tau_0) \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft snd(\tau_0) \blacktriangleleft \ell_1)$

$\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; i_0 \quad \rhd_{\mathsf{N}} \quad i_0$
  if $tag\text{-}match(\lfloor \tau_0 \rfloor, i_0)$

$\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_0 \quad \rhd_{\mathsf{N}} \quad \mathsf{BndryErr}\,((\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1), v_0)$
  if $\neg tag\text{-}match(\lfloor \tau_0 \rfloor, v_0)$

$\boxed{e \blacktriangleright_{\mathsf{N}} e}$

$unop\{\mathcal{U}\}\ v_0 \quad \blacktriangleright_{\mathsf{N}} \quad \mathsf{TagErr}\ \bullet$
  if $\delta(unop, v_0)$ is undefined

$unop\{\mathcal{U}\}\ v_0 \quad \blacktriangleright_{\mathsf{N}} \quad \delta(unop, v_0)$
  if $\delta(unop, v_0)$ is defined

$binop\{\mathcal{U}\}\ v_0\ v_1 \quad \blacktriangleright_{\mathsf{N}} \quad \mathsf{TagErr}\ \bullet$
  if $\delta(binop, v_0, v_1)$ is undefined

$binop\{\mathcal{U}\}\ v_0\ v_1 \quad \blacktriangleright_{\mathsf{N}} \quad \delta(binop, v_0, v_1)$
  if $\delta(binop, v_0, v_1)$ is defined

$\mathsf{app}\{\mathcal{U}\}\ v_0\ v_1 \quad \blacktriangleright_{\mathsf{N}} \quad \mathsf{TagErr}\ \bullet$
  if $v_0 \notin (\lambda x.\, e) \cup (\mathsf{mon}\ b\ v)$

$\mathsf{app}\{\mathcal{U}\}\ (\lambda x_0.\, e_0)\ v_1 \quad \blacktriangleright_{\mathsf{N}} \quad e_0[x_0 \leftarrow v_1]$

$\mathsf{app}\{\mathcal{U}\}\ (\mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0)\ v_1 \quad \blacktriangleright_{\mathsf{N}} \quad \mathsf{stat}\ b_0\ (\mathsf{app}\{\tau_1\}\ v_0\ (\mathsf{dyn}\ b_1\ v_1))$
  where $b_0 = (\ell_0 \blacktriangleleft cod(\tau_0) \blacktriangleleft \ell_1)$ and $b_1 = (\ell_1 \blacktriangleleft dom(\tau_0) \blacktriangleleft \ell_0)$ and $\tau_1 = cod(\tau_0)$

$\mathsf{stat}\ (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1)\ v_0 \quad \blacktriangleright_{\mathsf{N}} \quad \mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1)\ v_0$
  if $tag\text{-}match(\lfloor \tau_0 \Rightarrow \tau_1 \rfloor, v_0)$ and $v_0 \in (\lambda(x\!:\!\tau).\, e) \cup (\mathsf{mon}\ b\ v)$

$\mathsf{stat}\ (\ell_0 \blacktriangleleft (\tau_0 \times \tau_1) \blacktriangleleft \ell_1)\ \langle v_0, v_1 \rangle \quad \blacktriangleright_{\mathsf{N}} \quad \langle \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0, \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ v_1 \rangle$

$\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ i_0 \quad \blacktriangleright_{\mathsf{N}} \quad i_0$
  if $tag\text{-}match(\lfloor \tau_0 \rfloor, i_0)$

$\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0 \quad \blacktriangleright_{\mathsf{N}} \quad \mathsf{TagErr}\ \circ$
  if $\neg tag\text{-}match(\lfloor \tau_0 \rfloor, v_0)$

2.2 Natural **Ownership Lifting**

$\boxed{\text{Natural Ownership Language}}$ extends Ownership Evaluation Language

$v \ = \ \ldots \mid \mathsf{mon}\ (\ell \blacktriangleleft \tau \Rightarrow \tau \blacktriangleleft \ell)\ v$

$\boxed{\longrightarrow^{*}_{\overline{\mathsf{N}}}}$ reflexive-transitive closure of $\longrightarrow_{\overline{\mathsf{N}}}$

$\boxed{\longrightarrow_{\overline{\mathsf{N}}}}$ compatible closure of $\rhd_{\overline{\mathsf{N}}} \cup \blacktriangleright_{\overline{\mathsf{N}}}$

$$\boxed{(e)^{\ell} \rhd_{\overline{\mathsf{N}}} (e)^{\ell}}$$

$$(\mathit{unop}\{\tau_0\}\ \llparenthesis v_0 \rrparenthesis^{\overline{\ell}_0})^{\ell_0} \quad \rhd_{\overline{\mathsf{N}}} \quad (\mathsf{TagErr}\ \circ)^{\ell_0}$$

if $v_0 \notin (v)^{\ell}$ and $\delta(\mathit{unop}, v_0)$ is undefined

$$(\mathit{unop}\{\tau_0\}\ \llparenthesis v_0 \rrparenthesis^{\overline{\ell}_0})^{\ell_0} \quad \rhd_{\overline{\mathsf{N}}} \quad (\delta(\mathit{unop}, v_0))^{\overline{\ell}_0 \ell_0}$$

if $\delta(\mathit{unop}, v_0$ is defined

$$(\mathit{binop}\{\tau_0\}\ \llparenthesis v_0 \rrparenthesis^{\overline{\ell}_0}\ \llparenthesis v_1 \rrparenthesis^{\overline{\ell}_1})^{\ell_0} \quad \rhd_{\overline{\mathsf{N}}} \quad (\mathsf{TagErr}\ \circ)^{\ell_0}$$

if $v_0 \notin (v)^{\ell}$ and $v_1 \notin (v)^{\ell}$ and $\delta(\mathit{binop}, v_0, v_1)$ is undefined

$$(\mathit{binop}\{\tau_0\}\ \llparenthesis v_0 \rrparenthesis^{\overline{\ell}_0}\ \llparenthesis v_1 \rrparenthesis^{\overline{\ell}_1})^{\ell_0} \quad \rhd_{\overline{\mathsf{N}}} \quad (\delta(\mathit{binop}, v_0, v_1))^{\ell_0}$$

if $\delta(\mathit{binop}, v_0, v_1)$ is defined

$$(\mathsf{app}\{\tau_0\}\ \llparenthesis v_0 \rrparenthesis^{\overline{\ell}_0}\ v_1)^{\ell_0} \quad \rhd_{\overline{\mathsf{N}}} \quad (\mathsf{TagErr}\ \circ)^{\ell_0}$$

if $v_0 \notin (v)^{\ell} \cup (\lambda x.\, e) \cup (\mathsf{mon}\ b\ v)$

$$(\mathsf{app}\{\tau_0\}\ \llparenthesis \lambda(x_0 : \tau_1).\, e_0 \rrparenthesis^{\overline{\ell}_0}\ v_1)^{\ell_0} \quad \rhd_{\overline{\mathsf{N}}} \quad \llparenthesis e_0[x_0 \leftarrow \llparenthesis v_1 \rrparenthesis^{\ell_0\, \mathit{rev}(\overline{\ell}_0)}] \rrparenthesis^{\overline{\ell}_0 \ell_0}$$

$$(\mathsf{app}\{\tau_0\}\ \llparenthesis \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (v_0)^{\ell_2} \rrparenthesis^{\overline{\ell}_0}\ v_1)^{\ell_3} \quad \rhd_{\overline{\mathsf{N}}} \quad \llparenthesis \mathsf{dyn}\ b_0\ (\mathsf{app}\{\mathcal{U}\}\ v_0\ (\mathsf{stat}\ b_1\ \llparenthesis v_1 \rrparenthesis^{\ell_3\, \mathit{rev}(\overline{\ell}_0) \ell_2}))^{\ell_2} \rrparenthesis^{\overline{\ell}_0 \ell_3}$$

where $b_0 = (\ell_0 \blacktriangleleft \mathit{cod}(\tau_1) \blacktriangleleft \ell_1)$ and $b_1 = (\ell_1 \blacktriangleleft \mathit{dom}(\tau_1) \blacktriangleleft \ell_0)$

$$(\mathsf{dyn}\ (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1)\ \llparenthesis v_0 \rrparenthesis^{\overline{\ell}_0})^{\ell_2} \quad \rhd_{\overline{\mathsf{N}}} \quad (\mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1)\ \llparenthesis v_0 \rrparenthesis^{\overline{\ell}_0})^{\ell_2}$$

if $\mathit{tag\text{-}match}(\lfloor \tau_0 \Rightarrow \tau_1 \rfloor, v_0)$ and $v_0 \in (\lambda x.\, e) \cup (\mathsf{mon}\ b\ v)$

$$(\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ \llparenthesis \langle v_0, v_1 \rangle \rrparenthesis^{\overline{\ell}_0})^{\ell_2} \quad \rhd_{\overline{\mathsf{N}}} \quad (\langle \mathsf{dyn}\ b_0\ \llparenthesis v_0 \rrparenthesis^{\overline{\ell}_0}, \mathsf{dyn}\ b_1\ \llparenthesis v_1 \rrparenthesis^{\overline{\ell}_0} \rangle)^{\ell_2}$$

if $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, \langle v_0, v_1 \rangle)$ and $b_0 = (\ell_0 \blacktriangleleft \mathit{fst}(\tau_0) \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \mathit{snd}(\tau_0) \blacktriangleleft \ell_1)$

$$(\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ \llparenthesis i_0 \rrparenthesis^{\overline{\ell}_0})^{\ell_2} \quad \rhd_{\overline{\mathsf{N}}} \quad (i_0)^{\ell_2}$$

if $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, i_0)$

$$(\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ \llparenthesis v_0 \rrparenthesis^{\overline{\ell}_0})^{\ell_2} \quad \rhd_{\overline{\mathsf{N}}} \quad (\mathsf{BndryErr}\ ((\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1), \llparenthesis v_0 \rrparenthesis^{\overline{\ell}_0}))^{\ell_2}$$

if $\neg \mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

$$\boxed{(e)^{\ell} \blacktriangleright_{\overline{\mathsf{N}}} (e)^{\ell}}$$

$$(\mathit{unop}\{\mathcal{U}\}\ \llparenthesis v_0 \rrparenthesis^{\overline{\ell}_0})^{\ell_0} \quad \blacktriangleright_{\overline{\mathsf{N}}} \quad (\mathsf{TagErr}\,\bullet)^{\ell_0}$$

if $v_0 \notin (v)^{\ell}$ and $\delta(\mathit{unop}, v_0)$ is undefined

$$(\mathit{unop}\{\mathcal{U}\}\ \llparenthesis v_0 \rrparenthesis^{\overline{\ell}_0})^{\ell_0} \quad \blacktriangleright_{\overline{\mathsf{N}}} \quad (\delta(\mathit{unop}, v_0))^{\overline{\ell}_0 \ell_0}$$

if $\delta(\mathit{unop}, v_0)$ is defined

$$(\mathit{binop}\{\mathcal{U}\}\ \llparenthesis v_0 \rrparenthesis^{\overline{\ell}_0}\ \llparenthesis v_1 \rrparenthesis^{\overline{\ell}_1})^{\ell_0} \quad \blacktriangleright_{\overline{\mathsf{N}}} \quad (\mathsf{TagErr}\,\bullet)^{\ell_0}$$

if $v_0 \notin (v)^{\ell}$ and $v_1 \notin (v)^{\ell}$ and $\delta(\mathit{binop}, v_0, v_1)$ is undefined

$$(\mathit{binop}\{\mathcal{U}\}\ \llparenthesis v_0 \rrparenthesis^{\overline{\ell}_0}\ \llparenthesis v_1 \rrparenthesis^{\overline{\ell}_1})^{\ell_0} \quad \blacktriangleright_{\overline{\mathsf{N}}} \quad (\delta(\mathit{binop}, v_0, v_1))^{\ell_0}$$

if $\delta(\mathit{binop}, v_0, v_1)$ is defined

$$(\mathsf{app}\{\mathcal{U}\}\ \llparenthesis v_0 \rrparenthesis^{\overline{\ell}_0}\ v_1)^{\ell_0} \quad \blacktriangleright_{\overline{\mathsf{N}}} \quad (\mathsf{TagErr}\,\bullet)^{\ell_0}$$

if $v_0 \notin (v)^{\ell} \cup (\lambda x.\, e) \cup (\mathsf{mon}\ b\ v)$

$$(\mathsf{app}\{\mathcal{U}\}\ \llparenthesis \lambda x_0.\, e_0 \rrparenthesis^{\overline{\ell}_0}\ v_1)^{\ell_0} \quad \blacktriangleright_{\overline{\mathsf{N}}} \quad \llparenthesis e_0[x_0 \leftarrow \llparenthesis v_1 \rrparenthesis^{\ell_0 \mathit{rev}(\overline{\ell}_0)}] \rrparenthesis^{\overline{\ell}_0 \ell_0}$$

$$(\mathsf{app}\{\mathcal{U}\}\ \llparenthesis \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (v_0)^{\ell_2} \rrparenthesis^{\overline{\ell}_0}\ v_1)^{\ell_3} \quad \blacktriangleright_{\overline{\mathsf{N}}} \quad \llparenthesis \mathsf{stat}\ b_0\ (\mathsf{app}\{\tau_1\}\ v_0\ (\mathsf{dyn}\ b_1\ \llparenthesis v_1 \rrparenthesis^{\ell_3 \mathit{rev}(\overline{\ell}_0) \ell_2}))^{\ell_2} \rrparenthesis^{\overline{\ell}_0 \ell_3}$$

where $b_0 = (\ell_0 \blacktriangleleft \mathit{cod}(\tau_0) \blacktriangleleft \ell_1)$ and $b_1 = (\ell_1 \blacktriangleleft \mathit{dom}(\tau_0) \blacktriangleleft \ell_0)$ and $\tau_1 = \mathit{cod}(\tau_0)$

$$(\mathsf{stat}\ (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1)\ \llparenthesis v_0 \rrparenthesis^{\overline{\ell}_0})^{\ell_2} \quad \blacktriangleright_{\overline{\mathsf{N}}} \quad (\mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1)\ \llparenthesis v_0 \rrparenthesis^{\overline{\ell}_0})^{\ell_2}$$

if $\mathit{tag\text{-}match}(\lfloor \tau_0 \Rightarrow \tau_1 \rfloor, v_0)$ and $v_0 \in (\lambda x.\, e) \cup (\mathsf{mon}\ b\ v)$

$$(\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ \llparenthesis \langle v_0, v_1 \rangle \rrparenthesis^{\overline{\ell}_0})^{\ell_2} \quad \blacktriangleright_{\overline{\mathsf{N}}} \quad (\langle \mathsf{dyn}\ b_0\ \llparenthesis v_0 \rrparenthesis^{\overline{\ell}_0}, \mathsf{dyn}\ b_1\ \llparenthesis v_1 \rrparenthesis^{\overline{\ell}_0} \rangle)^{\ell_2}$$

if $\mathit{tag\text{-}match}(\langle v_0, v_1 \rangle, \tau_0)$ and $b_0 = (\ell_0 \blacktriangleleft \mathit{fst}(\tau_0) \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \mathit{snd}(\tau_0) \blacktriangleleft \ell_1)$

$$(\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ \llparenthesis i_0 \rrparenthesis^{\overline{\ell}_2})^{\ell_3} \quad \blacktriangleright_{\overline{\mathsf{N}}} \quad (i_0)^{\ell_3}$$

if $\mathit{tag\text{-}match}(i_0, \tau_0)$

$$(\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ \llparenthesis v_0 \rrparenthesis^{\overline{\ell}_2})^{\ell_2} \quad \blacktriangleright_{\overline{\mathsf{N}}} \quad (\mathsf{TagErr}\,\circ)^{\ell_2}$$

if $\neg \mathit{tag\text{-}match}(v_0, \tau_0)$

3 Transient

3.1 **Normal** Transient

Transient Language extends Base Evaluation Language

$$
\begin{array}{rcl}
\mathrm{p} & = & \text{countable set of heap locations} \\
v & = & i \mid n \mid \mathrm{p} \\
\mathrm{w} & = & \lambda x.\, e \mid \lambda(x\!:\!\tau).\, e \mid \langle v, v\rangle \\
e & = & \ldots \mid \mathrm{p} \mid \mathsf{check}\,\tau?\,e\,\mathrm{p} \\
E & = & \ldots \mid \mathsf{check}\,\tau?\,E\,\mathrm{p} \\
\mathcal{H} & = & \mathcal{P}((\mathrm{p} \mapsto \mathrm{w})) \\
\mathcal{B} & = & \mathcal{P}((\mathrm{p} \mapsto b^*)) \\
\mathcal{T} & = & \cdot \mid (\mathrm{p}\!:\!\tau?), \mathcal{T}
\end{array}
$$

$\cdot(\cdot) : \mathcal{H} \times v \longrightarrow v$ heap dereference

$$
\mathcal{H}_0(v_0) = \begin{cases} \mathrm{w}_0 & \text{if } v_0 \in \mathrm{p} \text{ and } (v_0 \mapsto \mathrm{w}_0) \in \mathcal{H}_0 \\ v_0 & \text{if } v_0 \notin \mathrm{p} \end{cases}
$$

$\cdot(\cdot) : \mathcal{B} \times v \longrightarrow b^*$ blame map dereference

$$
\mathcal{B}_0(v_0) = \begin{cases} b_0^* & \text{if } v_0 \in \mathrm{p} \text{ and } (v_0 \mapsto b_0^*) \in \mathcal{B}_0 \\ \emptyset & \text{otherwise} \end{cases}
$$

$\cdot[\cdot \mapsto \cdot] : \mathcal{B} \times v \times b^* \longrightarrow \mathcal{B}$ blame map replace

$$
\mathcal{B}_0[v_0 \mapsto b_0^*] = \begin{cases} \{v_0 \mapsto b_0^*\} \cup (\mathcal{B}_0 \setminus (v_0 \mapsto b_1^*)) \\ \qquad \text{if } v_0 \in \mathrm{p} \text{ and } (v_0 \mapsto b_1^*) \in \mathcal{B}_0 \\ \mathcal{B}_0 \quad \text{otherwise} \end{cases}
$$

$\cdot[\cdot \cup \cdot] : \mathcal{B} \times v \longrightarrow b^*$ blame map update

$$
\mathcal{B}_0[v_0 \cup b_0^*] = \quad \mathcal{B}_0[v_0 \mapsto b_0^* \cup \mathcal{B}_0(v_0)]
$$

$\boxed{\mathcal{T}; \Gamma \vdash_{\mathsf{T}} e : K}$

$$\frac{(\mathsf{p}_0 : \tau_0) \in \mathcal{T}_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \mathsf{p}_0 : \lfloor \tau_0 \rfloor} \qquad \frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_0 : K_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \mathsf{check}\, \tau_0\, e_0\, \mathsf{p}_0 : \lfloor \tau_0 \rfloor} \qquad \frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_0 : \mathcal{U}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \mathsf{check}\, \tau_0\, e_0\, \mathsf{p}_0 : \lfloor \tau_0 \rfloor} \qquad \frac{(x_0 : \tau_0) \in \Gamma_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} x_0 : \lfloor \tau_0 \rfloor}$$

$$\frac{}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} n_0 : \mathsf{Nat}} \qquad \frac{}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} i_0 : \mathsf{Int}} \qquad \frac{\mathcal{T}_0; (x_0 : \tau_0), \Gamma_0 \vdash_{\mathsf{T}} e_0 : K_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \lambda(x_0 : \tau_0).\, e_0 : \mathsf{Fun}} \qquad \frac{\mathcal{T}_0; (x_0 : \mathcal{U}), \Gamma_0 \vdash_{\mathsf{T}} e_0 : \mathcal{U}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \lambda x_0.\, e_0 : \mathsf{Fun}}$$

$$\frac{\begin{array}{c}\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_0 : K_0 \\ \mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_1 : K_1\end{array}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \langle e_0, e_1 \rangle : \mathsf{Pair}} \qquad \frac{\begin{array}{c}\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_0 : \mathsf{Fun} \\ \mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_1 : K_0\end{array}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \mathsf{app}\{\tau_1\}\, e_0\, e_1 : \lfloor \tau_1 \rfloor} \qquad \frac{\begin{array}{c}\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_0 : K_0 \\ \Delta(\mathit{unop}, K_0) = K_1 \\ K_1 \leqslant: \lfloor \tau_2 \rfloor\end{array}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \mathit{unop}\{\tau_2\}\, e_0 : \lfloor \tau_2 \rfloor}$$

$$\frac{\begin{array}{c}\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_0 : K_0 \\ \mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_1 : K_1 \\ \Delta(\mathit{binop}, K_0, K_1) = K_2 \\ K_2 \leqslant: \lfloor \tau_3 \rfloor\end{array}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \mathit{binop}\{\tau_3\}\, e_0\, e_1 : \lfloor \tau_3 \rfloor} \qquad \frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_0 : \mathcal{U}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, e_0 : \lfloor \tau_0 \rfloor} \qquad \frac{\begin{array}{c}\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_0 : K_0 \\ K_0 \leqslant: K_1\end{array}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_0 : K_1}$$

$\boxed{\mathcal{T}; \Gamma \vdash_{\mathsf{T}} e : \mathcal{U}}$

$$\frac{(\mathsf{p}_0 : \mathcal{U}) \in \mathcal{T}_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \mathsf{p}_0 : \mathcal{U}} \qquad \frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_0 : \mathcal{U}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \mathsf{check}\, \mathcal{U}\, e_0\, \mathsf{p}_0 : \mathcal{U}} \qquad \frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_0 : K_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \mathsf{check}\, \mathcal{U}\, e_0\, \mathsf{p}_0 : \mathcal{U}} \qquad \frac{(x_0 : \mathcal{U}) \in \Gamma_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} x_0 : \mathcal{U}}$$

$$\frac{}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} n_0 : \mathcal{U}} \qquad \frac{}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} i_0 : \mathcal{U}} \qquad \frac{\mathcal{T}_0; (x_0 : \tau_0), \Gamma_0 \vdash_{\mathsf{T}} e_0 : K_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \lambda(x_0 : \tau_0).\, e_0 : \mathcal{U}} \qquad \frac{\mathcal{T}_0; (x_0 : \mathcal{U}), \Gamma_0 \vdash_{\mathsf{T}} e_0 : \mathcal{U}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \lambda x_0.\, e_0 : \mathcal{U}}$$

$$\frac{\begin{array}{c}\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_0 : \mathcal{U} \\ \mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_1 : \mathcal{U}\end{array}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \langle e_0, e_1 \rangle : \mathcal{U}} \qquad \frac{\begin{array}{c}\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_0 : \mathcal{U} \\ \mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_1 : \mathcal{U}\end{array}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \mathsf{app}\{\tau_1\}\, e_0\, e_1 : \mathcal{U}} \qquad \frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_0 : \mathcal{U}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \mathit{unop}\{\tau_2\}\, e_0 : \mathcal{U}}$$

$$\frac{\begin{array}{c}\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_0 : \mathcal{U} \\ \mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_1 : \mathcal{U}\end{array}}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \mathit{binop}\{\tau_3\}\, e_0\, e_1 : \mathcal{U}} \qquad \frac{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} e_0 : K_0}{\mathcal{T}_0; \Gamma_0 \vdash_{\mathsf{T}} \mathsf{stat}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, e_0 : \mathcal{U}}$$

$\boxed{\mathcal{T} \vdash_{\mathsf{T}} \mathcal{H}}$

$$\frac{\forall (\mathsf{p}_0 \mapsto v_0) \in \mathcal{H}_0\, .\, \mathcal{T}_0; \cdot \vdash_{\mathsf{T}} v_0 : \mathcal{T}_0(\mathsf{p}_0)}{\mathcal{T}_0 \vdash_{\mathsf{T}} \mathcal{H}_0}$$

$\boxed{K \leqslant: K}$

$$\frac{}{\mathsf{Nat} \leqslant: \mathsf{Int}} \qquad \frac{}{K_0 \leqslant: K_0}$$

$\boxed{\rightarrow_{\mathsf{T}}^{*}}$ reflexive-transitive closure of $\rightarrow_{\mathsf{T}}$

$\boxed{e; \mathcal{H}; \mathcal{B} \rightarrow_{\mathsf{T}} e; \mathcal{H}; \mathcal{B}}$

$E[e_0]; \mathcal{H}_0; \mathcal{B}_0 \quad \rightarrow_{\mathsf{T}} \; E[e_1]; \mathcal{H}_1; \mathcal{B}_1$
  if $e_0; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} e_1; \mathcal{H}_1; \mathcal{B}_1$

$E[\mathsf{Err}]; \mathcal{H}_0; \mathcal{B}_0 \;\rightarrow_{\mathsf{T}}\; \mathsf{Err}; \mathcal{H}_0; \mathcal{B}_0$

$\boxed{e; \mathcal{H}; \mathcal{B} \rhd_{\mathsf{T}} e; \mathcal{H}; \mathcal{B}}$

$\mathsf{w}_0; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{p}_0; (\{\mathsf{p}_0 \mapsto \mathsf{w}_0\} \cup \mathcal{H}_0); (\{\mathsf{p}_0 \mapsto \emptyset\} \cup \mathcal{B}_0)$
  where p_0 fresh in $\mathcal{H}_0$ and $\mathcal{B}_0$

$(unop\{\tau_0\}\ v_0); \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{TagErr}\,\circ; \mathcal{H}_0; \mathcal{B}_0$
  if $\delta(unop, \mathcal{H}_0(v_0))$ is undefined

$(unop\{\mathcal{U}\}\ v_0); \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{TagErr}\,\bullet; \mathcal{H}_0; \mathcal{B}_0$
  if $\delta(unop, \mathcal{H}_0(v_0))$ is undefined

$(unop\{\tau?\}\ \mathsf{p}_0); \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} (\mathsf{check}\ \tau?\ \delta(unop, \mathcal{H}_0(\mathsf{p}_0))\ \mathsf{p}_0); \mathcal{H}_0; \mathcal{B}_0$
  if $\delta(unop, \mathcal{H}_0(\mathsf{p}_0))$ is defined

$(binop\{\tau_0\}\ v_0\ v_1); \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{TagErr}\,\circ; \mathcal{H}_0; \mathcal{B}_0$
  if $\delta(binop, v_0, v_1)$ is undefined

$(binop\{\mathcal{U}\}\ v_0\ v_1); \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{TagErr}\,\bullet; \mathcal{H}_0; \mathcal{B}_0$
  if $\delta(binop, v_0, v_1)$ is undefined

$(binop\{\tau?\}\ i_0\ i_1); \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \delta(binop, i_0, i_1); \mathcal{H}_0; \mathcal{B}_0$
  if $\delta(binop, i_0, i_1)$ is defined

$(\mathsf{app}\{\tau_0\}\ v_0\ v_1); \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{TagErr}\,\circ; \mathcal{H}_0; \mathcal{B}_0$
  if $\mathcal{H}_0(v_0) \notin (\lambda x.\, e) \cup (\lambda(x\!:\!\tau).\, e)$

$(\mathsf{app}\{\mathcal{U}\}\ v_0\ v_1); \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{TagErr}\,\bullet; \mathcal{H}_0; \mathcal{B}_0$
  if $\mathcal{H}_0(v_0) \notin (\lambda x.\, e) \cup (\lambda(x\!:\!\tau).\, e)$

$(\mathsf{app}\{\tau?\}\ \mathsf{p}_0\ v_0); \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} (\mathsf{check}\ \tau?\ e_0[x_0 \leftarrow v_0]\ \mathsf{p}_0); \mathcal{H}_0; \mathcal{B}_0[v_0 \cup rev(\mathcal{B}_0(\mathsf{p}_0))]$
  if $\mathcal{H}_0(\mathsf{p}_0) = \lambda(x_0\!:\!\tau_0).\, e_0$ and $tag\text{-}match(\lfloor\tau_0\rfloor, \mathcal{H}_0(v_0))$

$(\mathsf{app}\{\tau?\}\ \mathsf{p}_0\ v_0); \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{BndryErr}\,(\mathcal{B}_0(\mathsf{p}_0), v_0); \mathcal{H}_0; \mathcal{B}_0$
  if $\mathcal{H}_0(\mathsf{p}_0) = \lambda(x_0\!:\!\tau_0).\, e_0$ and $\neg tag\text{-}match(\lfloor\tau_0\rfloor, \mathcal{H}_0(v_0))$

$(\mathsf{app}\{\tau_0\}\ \mathsf{p}_0\ v_0); \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} (\mathsf{check}\ \tau_0\ e_0[x_0 \leftarrow v_0]\ \mathsf{p}_0); \mathcal{H}_0; \mathcal{B}_0[v_0 \cup rev(\mathcal{B}_0(\mathsf{p}_0))]$
  if $\mathcal{H}_0(\mathsf{p}_0) = \lambda x_0.\, e_0$

$(\mathsf{app}\{\mathcal{U}\}\ \mathsf{p}_0\ v_0); \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} (e_0[x_0 \leftarrow v_0]); \mathcal{H}_0; \mathcal{B}_0$
  if $\mathcal{H}_0(\mathsf{p}_0) = \lambda x_0.\, e_0$

$(\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0); \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} v_0; \mathcal{H}_0; (\mathcal{B}_0[v_0 \cup \{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\}])$
  if $tag\text{-}match(\lfloor\tau_0\rfloor, \mathcal{H}_0(v_0))$

$(\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0); \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{BndryErr}\,(\{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\}, v_0); \mathcal{H}_0; \mathcal{B}_0$
  if $\neg tag\text{-}match(\lfloor\tau_0\rfloor, \mathcal{H}_0(v_0))$

$(\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0); \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} v_0; \mathcal{H}_0; (\mathcal{B}_0[v_0 \cup \{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\}])$
  if $tag\text{-}match(\lfloor\tau_0\rfloor, \mathcal{H}_0(v_0))$

$(\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0); \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{TagErr}\,\circ; \mathcal{H}_0; \mathcal{B}_0$
  if $\neg tag\text{-}match(\lfloor\tau_0\rfloor, \mathcal{H}_0(v_0))$

$(\mathsf{check}\ \mathcal{U}\ v_0\ \mathsf{p}_0); \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} v_0; \mathcal{H}_0; \mathcal{B}_0$

$(\mathsf{check}\ \tau_0\ v_0\ \mathsf{p}_0); \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} v_0; \mathcal{H}_0; (\mathcal{B}_0[v_0 \cup \mathcal{B}_0(\mathsf{p}_0)])$
  if $tag\text{-}match(\lfloor\tau_0\rfloor, \mathcal{H}_0(v_0))$

$(\mathsf{check}\ \tau_0\ v_0\ \mathsf{p}_0); \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{BndryErr}\,(\mathcal{B}_0(v_0) \cup \mathcal{B}_0(\mathsf{p}_0), v_0); \mathcal{H}_0; \mathcal{B}_0$
  if $\neg tag\text{-}match(\lfloor\tau_0\rfloor, \mathcal{H}_0(v_0))$

3.2 Transient **(Path-Based) Ownership Lifting**

Transient Language extends Ownership Evaluation Language

$$\begin{array}{lcl} \mathrm{p} & = & \text{countable set of heap locations} \\ v & = & i \mid n \mid \mathrm{p} \mid (v)^{\ell} \\ \mathrm{w} & = & \lambda x.\, e \mid \lambda(x\!:\!\tau).\, e \mid \langle v, v\rangle \\ e & = & \ldots \mid \mathrm{p} \mid \mathsf{check}\,\tau?\,e\,\mathrm{p} \mid (e)^{\ell} \\ E & = & \ldots \mid \mathsf{check}\,\tau?\,E\,\mathrm{p} \mid (E)^{\ell} \\ \mathcal{H} & = & \mathcal{P}((\mathrm{p} \mapsto \mathrm{w})) \\ \mathcal{B} & = & \mathcal{P}((\mathrm{p} \mapsto b^{*})) \\ O & = & \mathcal{P}((\mathrm{p} \mapsto \bar{\ell})) \\ \mathcal{T} & = & \cdot \mid (\mathrm{p}\!:\!\tau?), \mathcal{T} \end{array}$$

$\cdot[\cdot \mapsto \cdot] : O \times v \times \ell^{*} \longrightarrow O$ ownership map replace

$$O_0[v_0 \mapsto \ell_0^{*}] = \begin{cases} \{v_0 \mapsto \ell_0^{*}\} \cup (O_0 \setminus (v_0 \mapsto \ell_1^{*})) & \\ \quad \text{if } v_0 \in \mathrm{p} \text{ and } (v_0 \mapsto \ell_1^{*}) \in O_0 & \\ O_0[v_1 \mapsto \ell_0^{*}] & \\ \quad \text{if } v_0 = (\!(v_1)\!)^{\bar{\ell}_0} & \\ O_0 \quad \text{otherwise} & \end{cases}$$

$\cdot[\cdot \cup \cdot] : O \times v \longrightarrow \ell^{*}$ ownership map update

$$O_0[v_0 \cup \ell_0^{*}] = \quad O_0[v_0 \mapsto \ell_0^{*} \cup O_0(v_0)]$$

$\mathcal{T};\Gamma \vdash_{\mathsf{T}} e : K$ extends $\vdash_{\mathsf{T}}$

$$\frac{\mathcal{T}_0;\Gamma_0 \vdash_{\mathsf{T}} e_0 : K_0}{\mathcal{T}_0;\Gamma_0 \vdash_{\mathsf{T}} (e_0)^{\ell_0} : K_0}$$

$\mathcal{T};\Gamma \vdash_{\mathsf{T}} e : \mathcal{U}$ extends $\vdash_{\mathsf{T}}$

$$\frac{\mathcal{T}_0;\Gamma_0 \vdash_{\mathsf{T}} e_0 : \mathcal{U}}{\mathcal{T}_0;\Gamma_0 \vdash_{\mathsf{T}} (e_0)^{\ell_0} : \mathcal{U}}$$

$O; L; \ell \Vdash_{\mathsf{T}} e$ lifts and extends $\Vdash$ and enforces ownership consistency

$$\frac{O_0; L_0; \ell_0 \Vdash_{\mathsf{T}} e_0}{O_0; L_0; \ell_0 \Vdash_{\mathsf{T}} (e_0)^{\ell_0}} \qquad \frac{O_0; L_0; \ell_0 \Vdash_{\mathsf{T}} e_0 \quad \mathrm{p}_0 \mapsto \ell_0 \in O_0}{O_0; L_0; \ell_0 \Vdash_{\mathsf{T}} \mathsf{check}\,\tau?\,e_0\,\mathrm{p}_0}$$

$\longrightarrow^{*}_{\overline{\mathsf{T}}}$ reflexive-transitive closure of $\longrightarrow_{\overline{\mathsf{T}}}$

$$\boxed{e;\mathcal{H};\mathcal{B} \longrightarrow_{\overline{\mathsf{T}}} e;\mathcal{H};\mathcal{B}}$$

$$E[e_0]^{\ell_0};\mathcal{H}_0;\mathcal{B}_0 \;\longrightarrow_{\overline{\mathsf{T}}}\; E[e_1]^{\ell_0};\mathcal{H}_1;\mathcal{B}_1$$
$$\quad \text{if } (e_0)^{\ell_0};\mathcal{H}_0;\mathcal{B}_0 \rhd_{\overline{\mathsf{T}}} (e_1)^{\ell_0};\mathcal{H}_1;\mathcal{B}_1$$

$$E[\mathsf{Err}];\mathcal{H}_0;\mathcal{B}_0 \;\longrightarrow_{\overline{\mathsf{T}}}\; \mathsf{Err};\mathcal{H}_0;\mathcal{B}_0$$

$$\boxed{(e)^{\ell};\mathcal{H};\mathcal{B} \rhd_{\overline{\mathsf{T}}} (e)^{\ell};\mathcal{H};\mathcal{B}}$$

$$(\mathsf{w}_0)^{\ell_0};\mathcal{H}_0;\mathcal{B}_0 \;\rhd_{\overline{\mathsf{T}}}\; (\mathsf{p}_0)^{\ell_0};(\{\mathsf{p}_0 \mapsto \mathsf{w}_0\} \cup \mathcal{H}_0);(\{\mathsf{p}_0 \mapsto \emptyset\} \cup \mathcal{B}_0)$$
$$\quad \text{where } \mathsf{p}_0 \text{ fresh in } \mathcal{H}_0 \text{ and } \mathcal{B}_0$$

$$(\mathit{unop}\{\tau_0\}\; (\!(v_0)\!)^{\overline{\ell_0}})^{\ell_1};\mathcal{H}_0;\mathcal{B}_0 \;\rhd_{\overline{\mathsf{T}}}\; (\mathsf{TagErr}\circ)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0$$
$$\quad \text{if } \delta(\mathit{unop},\mathcal{H}_0(v_0)) \text{ is undefined}$$

$$(\mathit{unop}\{\mathcal{U}\}\; (\!(v_0)\!)^{\overline{\ell_0}})^{\ell_1};\mathcal{H}_0;\mathcal{B}_0 \;\rhd_{\overline{\mathsf{T}}}\; (\mathsf{TagErr}\bullet)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0$$
$$\quad \text{if } \delta(\mathit{unop},\mathcal{H}_0(v_0)) \text{ is undefined}$$

$$(\mathit{unop}\{\tau?\}\; (\!(\mathsf{p}_0)\!)^{\overline{\ell_0}})^{\ell_1};\mathcal{H}_0;\mathcal{B}_0 \;\rhd_{\overline{\mathsf{T}}}\; (\mathsf{check}\;\tau?\;\delta(\mathit{unop},\mathcal{H}_0(\mathsf{p}_0))\;\mathsf{p}_0)^{\overline{\ell_0}\ell_1};\mathcal{H}_0;\mathcal{B}_0$$
$$\quad \text{if } \delta(\mathit{unop},\mathcal{H}_0(\mathsf{p}_0)) \text{ is defined}$$

$$(\mathit{binop}\{\tau_0\}\; (\!(v_0)\!)^{\overline{\ell_0}}\; (\!(v_1)\!)^{\overline{\ell_1}})^{\ell_2};\mathcal{H}_0;\mathcal{B}_0 \;\rhd_{\overline{\mathsf{T}}}\; (\mathsf{TagErr}\circ)^{\ell_2};\mathcal{H}_0;\mathcal{B}_0$$
$$\quad \text{if } \delta(\mathit{binop},v_0,v_1) \text{ is undefined}$$

$$(\mathit{binop}\{\mathcal{U}\}\; (\!(v_0)\!)^{\overline{\ell_0}}\; (\!(v_1)\!)^{\overline{\ell_1}})^{\ell_2};\mathcal{H}_0;\mathcal{B}_0 \;\rhd_{\overline{\mathsf{T}}}\; (\mathsf{TagErr}\bullet)^{\ell_2};\mathcal{H}_0;\mathcal{B}_0$$
$$\quad \text{if } \delta(\mathit{binop},v_0,v_1) \text{ is undefined}$$

$$(\mathit{binop}\{\tau?\}\; (\!(i_0)\!)^{\overline{\ell_0}}\; (\!(i_1)\!)^{\overline{\ell_1}})^{\ell_2};\mathcal{H}_0;\mathcal{B}_0 \;\rhd_{\overline{\mathsf{T}}}\; (\delta(\mathit{binop},i_0,i_1))^{\ell_2};\mathcal{H}_0;\mathcal{B}_0$$
$$\quad \text{if } \delta(\mathit{binop},i_0,i_1) \text{ is defined}$$

$$(\mathsf{app}\{\tau_0\}\; (\!(v_0)\!)^{\overline{\ell_0}}\; v_1)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0 \;\rhd_{\overline{\mathsf{T}}}\; (\mathsf{TagErr}\circ)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0$$
$$\quad \text{if } \mathcal{H}_0(v_0) \notin (\lambda x.\,e) \cup (\lambda(x\!:\!\tau).\,e)$$

$$(\mathsf{app}\{\mathcal{U}\}\; (\!(v_0)\!)^{\overline{\ell_0}}\; v_1)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0 \;\rhd_{\overline{\mathsf{T}}}\; (\mathsf{TagErr}\bullet)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0$$
$$\quad \text{if } \mathcal{H}_0(v_0) \notin (\lambda x.\,e) \cup (\lambda(x\!:\!\tau).\,e)$$

$$(\mathsf{app}\{\tau?\}\; (\!(\mathsf{p}_0)\!)^{\overline{\ell_0}}\; v_0)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0 \;\rhd_{\overline{\mathsf{T}}}\; (\mathsf{check}\;\tau?\;e_0[x_0 \leftarrow (\!(v_0)\!)^{\overline{\ell_1}\,\mathit{rev}(\overline{\ell_0})}]\;\mathsf{p}_0)^{\overline{\ell_0}\ell_1};\mathcal{H}_0;\mathcal{B}_0[v_0 \cup \mathit{rev}(\mathcal{B}_0(\mathsf{p}_0))]$$
$$\quad \text{if } \mathcal{H}_0(\mathsf{p}_0) = \lambda(x_0\!:\!\tau_0).\,e_0 \text{ and } \mathit{tag\text{-}match}(\lfloor\tau_0\rfloor,\mathcal{H}_0(v_0))$$

$$(\mathsf{app}\{\tau?\}\; (\!(\mathsf{p}_0)\!)^{\overline{\ell_0}}\; v_0)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0 \;\rhd_{\overline{\mathsf{T}}}\; (\mathsf{BndryErr}\;((\!(\mathcal{B}_0(\mathsf{p}_0))\!)^{\overline{\ell_0}},v_0))^{\ell_1};\mathcal{H}_0;\mathcal{B}_0$$
$$\quad \text{if } \mathcal{H}_0(\mathsf{p}_0) = \lambda(x_0\!:\!\tau_0).\,e_0 \text{ and } \neg\mathit{tag\text{-}match}(\lfloor\tau_0\rfloor,\mathcal{H}_0(v_0))$$

$$(\mathsf{app}\{\tau_0\}\; (\!(\mathsf{p}_0)\!)^{\overline{\ell_0}}\; v_0)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0 \;\rhd_{\overline{\mathsf{T}}}\; (\mathsf{check}\;\tau_0\;e_0[x_0 \leftarrow (\!(v_0)\!)^{\overline{\ell_1}\,\mathit{rev}(\overline{\ell_0})}]\;\mathsf{p}_0)^{\overline{\ell_0}\ell_1};\mathcal{H}_0;\mathcal{B}_0[v_0 \cup \mathit{rev}(\mathcal{B}_0(\mathsf{p}_0))]$$
$$\quad \text{if } \mathcal{H}_0(\mathsf{p}_0) = \lambda x_0.\,e_0$$

$$(\mathsf{app}\{\mathcal{U}\}\; (\!(\mathsf{p}_0)\!)^{\overline{\ell_0}}\; v_0)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0 \;\rhd_{\overline{\mathsf{T}}}\; (e_0[x_0 \leftarrow (\!(v_0)\!)^{\overline{\ell_1}\,\mathit{rev}(\overline{\ell_0})}])^{\overline{\ell_0}\ell_1};\mathcal{H}_0;\mathcal{B}_0$$
$$\quad \text{if } \mathcal{H}_0(\mathsf{p}_0) = \lambda x_0.\,e_0$$

$$(\mathsf{dyn}\;(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; (\!(v_0)\!)^{\overline{\ell_0}})^{\ell_2};\mathcal{H}_0;\mathcal{B}_0 \;\rhd_{\overline{\mathsf{T}}}\; (v_0)^{\overline{\ell_0}\ell_2};\mathcal{H}_0;(\mathcal{B}_0[v_0 \cup \{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\}])$$
$$\quad \text{if } \mathit{tag\text{-}match}(\lfloor\tau_0\rfloor,\mathcal{H}_0(v_0))$$

$$(\mathsf{dyn}\;(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; (\!(v_0)\!)^{\overline{\ell_0}})^{\ell_2};\mathcal{H}_0;\mathcal{B}_0 \;\rhd_{\overline{\mathsf{T}}}\; (\mathsf{BndryErr}\;(\{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\},(\!(v_0)\!)^{\overline{\ell_0}}))^{\ell_2};\mathcal{H}_0;\mathcal{B}_0$$
$$\quad \text{if } \neg\mathit{tag\text{-}match}(\lfloor\tau_0\rfloor,\mathcal{H}_0(v_0))$$

$$(\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (\!(v_0)\!)^{\overline{\ell_0}})^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0 \ \rhd_{\overline{\mathsf{T}}}\ (v_0)^{\overline{\ell_0}\ell_2}; \mathcal{H}_0; (\mathcal{B}_0[v_0 \cup \{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\}])$$
$$\text{if } \mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$$
$$(\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (\!(v_0)\!)^{\overline{\ell_0}})^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0 \ \rhd_{\overline{\mathsf{T}}}\ (\mathsf{TagErr}\circ)^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0$$
$$\text{if } \neg\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$$
$$(\mathsf{check}\ \mathcal{U}\ (\!(v_0)\!)^{\overline{\ell_0}}\ \mathsf{p}_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0 \ \rhd_{\overline{\mathsf{T}}}\ (v_0)^{\overline{\ell_0}\ell_1}; \mathcal{H}_0; \mathcal{B}_0$$
$$(\mathsf{check}\ \tau_0\ (\!(v_0)\!)^{\overline{\ell_0}}\ \mathsf{p}_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0 \ \rhd_{\overline{\mathsf{T}}}\ (\!(v_0)\!)^{\overline{\ell_0}\ell_1}; \mathcal{H}_0; (\mathcal{B}_0[v_0 \cup \mathcal{B}_0(\mathsf{p}_0)])$$
$$\text{if } \mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$$
$$(\mathsf{check}\ \tau_0\ (\!(v_0)\!)^{\overline{\ell_0}}\ \mathsf{p}_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0 \ \rhd_{\overline{\mathsf{T}}}\ (\mathsf{BndryErr}\,(\mathcal{B}_0(v_0) \cup \mathcal{B}_0(\mathsf{p}_0), (\!(v_0)\!)^{\overline{\ell_0}}))^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0$$
$$\text{if } \neg\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$$

3.3 Transient **Heap-Based Ownership Lifting**

Transient Language extends Transient Ownership Evaluation Language

$$O = \mathcal{P}((\mathsf{p} \mapsto \ell^*))$$

$O \Vdash_{\mathsf{T}} \mathcal{H}$ enforces ownership consistency for the heap

$$\frac{\forall (\mathsf{p}_0 \mapsto v_0) \in \mathcal{H}_0\ .\ O_0; \cdot; O(\mathsf{p}_0) \Vdash_{\mathsf{T}} v_0}{O_0 \Vdash_{\mathsf{T}} \mathcal{H}_0}$$

$\longrightarrow^*_{\overline{\mathsf{T}}2}$ reflexive-transitive closure of $\longrightarrow_{\overline{\mathsf{T}}2}$

$e; \mathcal{H}; \mathcal{B}; O \longrightarrow_{\overline{\mathsf{T}}2} e; \mathcal{H}; \mathcal{B}; O$

$$E[e_0]^{\ell_0}; \mathcal{H}_0; \mathcal{B}_0; O_0 \ \longrightarrow_{\overline{\mathsf{T}}2}\ E[e_1]^{\ell_0}; \mathcal{H}_1; \mathcal{B}_1; O_0$$
$$\text{if } (e_0)^{\ell_0}; \mathcal{H}_0; \mathcal{B}_0; O_0 \rhd_{\overline{\mathsf{T}}2} (e_1)^{\ell_0}; \mathcal{H}_1; \mathcal{B}_1; O_0$$
$$E[\mathsf{Err}]; \mathcal{H}_0; \mathcal{B}_0; O_0 \ \longrightarrow_{\overline{\mathsf{T}}2}\ \mathsf{Err}; \mathcal{H}_0; \mathcal{B}_0; O_0$$

$$\boxed{(e)^{\ell};\mathcal{H};\mathcal{B};O \rhd_{\overline{\mathrm{T2}}} (e)^{\ell};\mathcal{H};\mathcal{B};O}$$

$$(\mathrm{w}_0)^{\ell_0};\mathcal{H}_0;\mathcal{B}_0;O_0 \quad \rhd_{\overline{\mathrm{T2}}} \quad (\mathrm{p}_0)^{\ell_0};(\{\mathrm{p}_0 \mapsto \mathrm{w}_0\}\cup\mathcal{H}_0);(\{\mathrm{p}_0\mapsto\emptyset\}\cup\mathcal{B}_0);(\{\mathrm{p}_0\mapsto\ell_0\}\cup O_0)$$
where p_0 fresh in $\mathcal{H}_0$ and $\mathcal{B}_0$ and O_0

$$(\mathit{unop}\{\tau_0\}\,\llparenthesis v_0\rrparenthesis^{\overline{\ell_0}})^{\ell_1};\mathcal{H}_0;\mathcal{B}_0;O_0 \quad \rhd_{\overline{\mathrm{T2}}} \quad (\mathsf{TagErr}\,\circ)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0;O_0$$
if $\delta(\mathit{unop},\mathcal{H}_0(v_0))$ is undefined

$$(\mathit{unop}\{\mathcal{U}\}\,\llparenthesis v_0\rrparenthesis^{\overline{\ell_0}})^{\ell_1};\mathcal{H}_0;\mathcal{B}_0;O_0 \quad \rhd_{\overline{\mathrm{T2}}} \quad (\mathsf{TagErr}\,\bullet)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0;O_0$$
if $\delta(\mathit{unop},\mathcal{H}_0(v_0))$ is undefined

$$(\mathit{unop}\{\tau?\}\,\llparenthesis \mathrm{p}_0\rrparenthesis^{\overline{\ell_0}})^{\ell_1};\mathcal{H}_0;\mathcal{B}_0;O_0 \quad \rhd_{\overline{\mathrm{T2}}} \quad (\mathsf{check}\,\tau?\,\delta(\mathit{unop},\mathcal{H}_0(\mathrm{p}_0))\,\mathrm{p}_0)^{\overline{\ell_0}\ell_1};\mathcal{H}_0;\mathcal{B}_0;O_0$$
if $\delta(\mathit{unop},\mathcal{H}_0(\mathrm{p}_0))$ is defined

$$(\mathit{binop}\{\tau_0\}\,\llparenthesis v_0\rrparenthesis^{\overline{\ell_0}}\,\llparenthesis v_1\rrparenthesis^{\overline{\ell_1}})^{\ell_2};\mathcal{H}_0;\mathcal{B}_0;O_0 \quad \rhd_{\overline{\mathrm{T2}}} \quad (\mathsf{TagErr}\,\circ)^{\ell_2};\mathcal{H}_0;\mathcal{B}_0;O_0$$
if $\delta(\mathit{binop},v_0,v_1)$ is undefined

$$(\mathit{binop}\{\mathcal{U}\}\,\llparenthesis v_0\rrparenthesis^{\overline{\ell_0}}\,\llparenthesis v_1\rrparenthesis^{\overline{\ell_1}})^{\ell_2};\mathcal{H}_0;\mathcal{B}_0;O_0 \quad \rhd_{\overline{\mathrm{T2}}} \quad (\mathsf{TagErr}\,\bullet)^{\ell_2};\mathcal{H}_0;\mathcal{B}_0;O_0$$
if $\delta(\mathit{binop},v_0,v_1)$ is undefined

$$(\mathit{binop}\{\tau?\}\,\llparenthesis i_0\rrparenthesis^{\overline{\ell_0}}\,\llparenthesis i_1\rrparenthesis^{\overline{\ell_1}})^{\ell_2};\mathcal{H}_0;\mathcal{B}_0;O_0 \quad \rhd_{\overline{\mathrm{T2}}} \quad (\delta(\mathit{binop},i_0,i_1))^{\ell_2};\mathcal{H}_0;\mathcal{B}_0;O_0$$
if $\delta(\mathit{binop},i_0,i_1)$ is defined

$$(\mathsf{app}\{\tau_0\}\,\llparenthesis v_0\rrparenthesis^{\overline{\ell_0}}\,v_1)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0;O_0 \quad \rhd_{\overline{\mathrm{T2}}} \quad (\mathsf{TagErr}\,\circ)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0;O_0$$
if $\mathcal{H}_0(v_0)\notin(\lambda x.\,e)\cup(\lambda(x\!:\!\tau).\,e)$

$$(\mathsf{app}\{\mathcal{U}\}\,\llparenthesis v_0\rrparenthesis^{\overline{\ell_0}}\,v_1)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0;O_0 \quad \rhd_{\overline{\mathrm{T2}}} \quad (\mathsf{TagErr}\,\bullet)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0;O_0$$
if $\mathcal{H}_0(v_0)\notin(\lambda x.\,e)\cup(\lambda(x\!:\!\tau).\,e)$

$$(\mathsf{app}\{\tau?\}\,\llparenthesis \mathrm{p}_0\rrparenthesis^{\overline{\ell_0}}\,v_0)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0;O_0 \quad \rhd_{\overline{\mathrm{T2}}}$$
$$(\mathsf{check}\,\tau?\,e_0[x_0\leftarrow\llparenthesis v_0\rrparenthesis^{\ell_1\,\mathit{rev}(\overline{\ell_0})}]\,\mathrm{p}_0)^{\overline{\ell_0}\ell_1};\mathcal{H}_0;\mathcal{B}_0[v_0\cup\mathit{rev}(\mathcal{B}_0(\mathrm{p}_0))];O_0[v_1\cup O_0(\mathrm{p}_0)\cup\{\ell_1\}]$$
if $\mathcal{H}_0(\mathrm{p}_0)=\lambda(x_0\!:\!\tau_0).\,e_0$ and $\mathit{tag\text{-}match}(\lfloor\tau_0\rfloor,\mathcal{H}_0(v_0))$

$$(\mathsf{app}\{\tau?\}\,\llparenthesis \mathrm{p}_0\rrparenthesis^{\overline{\ell_0}}\,v_0)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0;O_0 \quad \rhd_{\overline{\mathrm{T2}}}$$
$$(\mathsf{BndryErr}\,(\llparenthesis\mathcal{B}_0(\mathrm{p}_0)\rrparenthesis^{\overline{\ell_0}},\llparenthesis v_0\rrparenthesis^{\ell_1\,\mathit{rev}(\overline{\ell_0})}))^{\ell_1};\mathcal{H}_0;\mathcal{B}_0;O_0[v_0\cup O_0(\mathrm{p}_0)\cup\ell_1]$$
if $\mathcal{H}_0(\mathrm{p}_0)=\lambda(x_0\!:\!\tau_0).\,e_0$ and $\neg\mathit{tag\text{-}match}(\lfloor\tau_0\rfloor,\mathcal{H}_0(v_0))$

$$(\mathsf{app}\{\tau_0\}\,\llparenthesis \mathrm{p}_0\rrparenthesis^{\overline{\ell_0}}\,v_0)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0;O_0 \quad \rhd_{\overline{\mathrm{T2}}}$$
$$(\mathsf{check}\,\tau_0\,e_0[x_0\leftarrow\llparenthesis v_0\rrparenthesis^{\ell_1\,\mathit{rev}(\overline{\ell_0})}]\,\mathrm{p}_0)^{\overline{\ell_0}\ell_1};\mathcal{H}_0;\mathcal{B}_0[v_0\cup\mathit{rev}(\mathcal{B}_0(\mathrm{p}_0))];O_0[v_0\cup O_0(\mathrm{p}_0)\cup\{\ell_0\}]$$
if $\mathcal{H}_0(\mathrm{p}_0)=\lambda x_0.\,e_0$

$$(\mathsf{app}\{\mathcal{U}\}\,\llparenthesis \mathrm{p}_0\rrparenthesis^{\overline{\ell_0}}\,v_0)^{\ell_1};\mathcal{H}_0;\mathcal{B}_0;O_0 \quad \rhd_{\overline{\mathrm{T2}}} \quad (e_0[x_0\leftarrow\llparenthesis v_0\rrparenthesis^{\ell_1\,\mathit{rev}(\overline{\ell_0})}])^{\overline{\ell_0}\ell_1};\mathcal{H}_0;\mathcal{B}_0;O_0[v_0\cup O_0(\mathrm{p}_0)\cup\{\ell_0\}]$$
if $\mathcal{H}_0(\mathrm{p}_0)=\lambda x_0.\,e_0$

$$(\mathsf{dyn}\,(\ell_0\blacktriangleleft\tau_0\blacktriangleleft\ell_1)\,\llparenthesis v_0\rrparenthesis^{\overline{\ell_0}})^{\ell_2};\mathcal{H}_0;\mathcal{B}_0;O_0 \quad \rhd_{\overline{\mathrm{T2}}} \quad (v_0)^{\overline{\ell_0}\ell_2};\mathcal{H}_0;(\mathcal{B}_0[v_0\cup\{(\ell_0\blacktriangleleft\tau_0\blacktriangleleft\ell_1)\}]);O_0[v_0\cup\ell_2]$$
if $\mathit{tag\text{-}match}(\lfloor\tau_0\rfloor,\mathcal{H}_0(v_0))$

$$(\mathsf{dyn}\,(\ell_0\blacktriangleleft\tau_0\blacktriangleleft\ell_1)\,\llparenthesis v_0\rrparenthesis^{\overline{\ell_0}})^{\ell_2};\mathcal{H}_0;\mathcal{B}_0;O_0 \quad \rhd_{\overline{\mathrm{T2}}} \quad (\mathsf{BndryErr}\,(\{(\ell_0\blacktriangleleft\tau_0\blacktriangleleft\ell_1)\},\llparenthesis v_0\rrparenthesis^{\overline{\ell_0}}))^{\ell_1};\mathcal{H}_0;\mathcal{B}_0;O_0$$
if $\neg\mathit{tag\text{-}match}(\lfloor\tau_0\rfloor,\mathcal{H}_0(v_0))$

$$(\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\,(\!(v_0)\!)^{\overline{\ell_0}})^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0; O_0 \;\rhd_{\overline{\mathsf{T}}2}\; (v_0)^{\overline{\ell_0}\ell_2}; \mathcal{H}_0; (\mathcal{B}_0[v_0 \cup \{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\}]); O_0[v_0 \cup \{\ell_2\}]$$
if $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$

$$(\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\,(\!(v_0)\!)^{\overline{\ell_0}})^{\ell_2}; \mathcal{H}_0; \mathcal{B}_0; O_0 \;\rhd_{\overline{\mathsf{T}}2}\; (\mathsf{TagErr}\,\circ)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0$$
if $\neg \mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$

$$(\mathsf{check}\,\mathcal{U}\,(\!(v_0)\!)^{\overline{\ell_0}}\,\mathsf{p}_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0 \;\rhd_{\overline{\mathsf{T}}2}\; (v_0)^{\overline{\ell_0}\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0[v_0 \cup O_0(\mathsf{p}_0)]$$

$$(\mathsf{check}\,\tau_0\,(\!(v_0)\!)^{\overline{\ell_0}}\,\mathsf{p}_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0 \;\rhd_{\overline{\mathsf{T}}2}\; (v_0)^{\overline{\ell_0}\ell_1}; \mathcal{H}_0; (\mathcal{B}_0[v_0 \cup \mathcal{B}_0(\mathsf{p}_0)]); O_0[v_0 \cup O_0(\mathsf{p}_0)]$$
if $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$

$$(\mathsf{check}\,\tau_0\,(\!(v_0)\!)^{\overline{\ell_0}}\,\mathsf{p}_0)^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0 \;\rhd_{\overline{\mathsf{T}}2}\; (\mathsf{BndryErr}\,(\mathcal{B}_0(v_0) \cup \mathcal{B}_0(\mathsf{p}_0), (\!(v_0)\!)^{\overline{\ell_0}}))^{\ell_1}; \mathcal{H}_0; \mathcal{B}_0; O_0[v_0 \cup O_0(\mathsf{p}_0)]$$
if $\neg \mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, \mathcal{H}_0(v_0))$

4 Amnesic

4.1 **Normal** Amnesic

Amnesic Language extends Base Evaluation Language

$v \;=\; \ldots \mid \mathsf{trace_v}\,\overline{b}\,v \mid \mathsf{mon}\,(\ell \blacktriangleleft \tau \Rightarrow \tau \blacktriangleleft \ell)\,(v)^{\ell} \mid \mathsf{mon}\,(\ell \blacktriangleleft \tau \times \tau \blacktriangleleft \ell)\,(v)^{\ell}$

$e \;=\; \ldots \mid \mathsf{trace}\,\overline{b}\,e$

$E \;=\; \ldots \mid \mathsf{trace}\,\overline{b}\,E$

$\Gamma \vdash_{\mathsf{A}} e : \tau$ extends $\Gamma \vdash e : \tau$

$$\frac{\Gamma \vdash_{\mathsf{A}} v_0 : \mathcal{U}}{\Gamma \vdash_{\mathsf{A}} \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_0 : \tau_0} \qquad \frac{}{\Gamma \vdash_{\mathsf{A}} \mathsf{Err} : \tau_0}$$

$\Gamma \vdash_{\mathsf{A}} e : \mathcal{U}$ extends $\Gamma \vdash e : \mathcal{U}$

$$\frac{\Gamma \vdash_{\mathsf{A}} v_0 : \tau_0}{\Gamma \vdash_{\mathsf{A}} \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_0 : \mathcal{U}} \qquad \frac{\Gamma \vdash_{\mathsf{A}} e_0 : \mathcal{U}}{\Gamma \vdash_{\mathsf{A}} \mathsf{trace}\,\overline{b}_0\; e_0 : \mathcal{U}} \qquad \frac{\Gamma \vdash_{\mathsf{A}} e_0 : \mathcal{U}}{\Gamma \vdash_{\mathsf{A}} \mathsf{trace_v}\,\overline{b}_0\; e_0 : \mathcal{U}} \qquad \frac{}{\Gamma \vdash_{\mathsf{A}} \mathsf{Err} : \mathcal{U}}$$

$\mathsf{trace}^{?}_{\mathsf{v}}\,\overline{b}_0\; v_0$ short for a v_1 such that $\textit{get-trace}(v_1) = \overline{b}_0$ and $\textit{rem-trace}(v_1) = v_0$

$\rightarrow^{*}_{\mathsf{A}}$ reflexive-transitive closure of $\rightarrow_{\mathsf{A}}$

$\rightarrow_{\mathsf{N}}$ compatible closure of $\rhd_{\mathsf{A}} \cup \blacktriangleright_{\mathsf{A}}$

$\boxed{e \rhd_{\mathsf{A}} e}$

$unop\{\tau_0\}\ v_0 \quad \rhd_{\mathsf{A}} \quad \mathsf{TagErr}\ \circ$
 if $v_0 \notin (\mathsf{mon}\ (\ell \blacktriangleleft (\tau \times \tau) \blacktriangleleft \ell)\ v)$ and $\delta(unop, v_0)$ is undefined

$unop\{\tau_0\}\ v_0 \quad \rhd_{\mathsf{A}} \quad \delta(unop, v_0)$
 if $\delta(unop, v_0)$ is defined

$\mathsf{fst}\{\tau_0\}\ (\mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ v_0) \quad \rhd_{\mathsf{A}} \quad \mathsf{dyn}\ b_0\ (\mathsf{fst}\{\mathcal{U}\}\ v_0)$
 where $\tau_2 = fst(\tau_1)$ and $b_0 = (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)$

$\mathsf{snd}\{\tau_0\}\ (\mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ v_0 \quad \rhd_{\mathsf{A}} \quad \mathsf{dyn}\ b_0\ (\mathsf{snd}\{\mathcal{U}\}\ v_0)$
 where $\tau_2 = snd(\tau_1)$ and $b_0 = (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)$

$binop\{\tau_0\}\ v_0\ v_1 \quad \rhd_{\mathsf{A}} \quad \mathsf{TagErr}\ \circ$
 if $\delta(binop, v_0, v_1)$ is undefined

$binop\{\tau_0\}\ v_0\ v_1 \quad \rhd_{\mathsf{A}} \quad \delta(binop, v_0, v_1)$
 if $\delta(binop, v_0, v_1)$ is defined

$\mathsf{app}\{\tau_0\}\ v_0\ v_1 \quad \rhd_{\mathsf{A}} \quad \mathsf{TagErr}\ \circ$
 if $v_0 \notin (\lambda(x:\tau).\, e) \cup (\mathsf{mon}\ (\ell \blacktriangleleft (\tau \Rightarrow \tau) \blacktriangleleft \ell)\ v)$

$\mathsf{app}\{\tau_0\}\ (\lambda(x_0:\tau_1).\, e_0)\ v_1 \quad \rhd_{\mathsf{A}} \quad e_0[x_0 \leftarrow v_1]$

$\mathsf{app}\{\tau_0\}\ (\mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ v_0)\ v_1 \quad \rhd_{\mathsf{A}} \quad \mathsf{dyn}\ b_0\ (\mathsf{app}\{\mathcal{U}\}\ v_0\ (\mathsf{stat}\ b_1\ v_1))$
 where $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_1 \blacktriangleleft dom(\tau_1) \blacktriangleleft \ell_0)$

$\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0 \quad \rhd_{\mathsf{A}} \quad \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0$
 if $tag\text{-}match(\lfloor \tau_0 \rfloor, v_0)$ and $v_0 \in (\mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b}\ (\lambda(x:\tau).\, e)) \cup (\mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b}\ (\mathsf{mon}\ (\ell \blacktriangleleft (\tau \Rightarrow \tau) \blacktriangleleft \ell)\ v))$

$\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (\mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b}_0\ i_0) \quad \rhd_{\mathsf{A}} \quad i_0$
 if $tag\text{-}match(\lfloor \tau_0 \rfloor, i_0)$

$\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0 \quad \rhd_{\mathsf{A}} \quad \mathsf{BndryErr}\ ((\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\overline{b}_0, v_0)$
 if $\neg tag\text{-}match(\lfloor \tau_0 \rfloor, v_0)$ and $\overline{b}_0 = get\text{-}trace(v_0)$

$\boxed{e \blacktriangleright_{\mathsf{A}} e}$

$\mathit{unop}\{\mathcal{U}\}\ v_0 \quad \blacktriangleright_{\mathsf{A}} \quad \mathsf{TagErr}\,\bullet$
 if $v_1 = \mathit{rem\text{-}trace}(v_0)$ and $v_1 \notin \mathsf{mon}\ (\ell \blacktriangleleft (\tau{\times}\tau) \blacktriangleleft \ell)\ v$ and $\delta(\mathit{unop}, v_1)$ is undefined

$\mathit{unop}\{\mathcal{U}\}\ v_0 \quad \blacktriangleright_{\mathsf{A}} \quad \mathit{add\text{-}trace}(\mathit{get\text{-}trace}(v_0), \delta(\mathit{unop}, v_1))$
 if $v_1 = \mathit{rem\text{-}trace}(v_0)$ and $\delta(\mathit{unop}, v_1)$ is defined

$\mathsf{fst}\{\mathcal{U}\}\ (\mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b_0}\ (\mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0)) \quad \blacktriangleright_{\mathsf{A}} \quad \mathsf{trace}\ \overline{b_0}\ (\mathsf{stat}\ b_0\ (\mathsf{fst}\{\tau_1\}\ v_0))$
 where $\tau_1 = \mathit{fst}(\tau_0)$ and $b_0 = (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)$

$\mathsf{snd}\{\mathcal{U}\}\ (\mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b_0}\ (\mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0)) \quad \blacktriangleright_{\mathsf{A}} \quad \mathsf{trace}\ \overline{b_0}\ (\mathsf{stat}\ b_0\ (\mathsf{snd}\{\tau_1\}\ v_0))$
 where $\tau_1 = \mathit{snd}(\tau_0)$ and $b_0 = (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)$

$\mathit{binop}\{\mathcal{U}\}\ v_0\ v_1 \quad \blacktriangleright_{\mathsf{A}} \quad \mathsf{TagErr}\,\bullet$
 if $v_2 = \mathit{rem\text{-}trace}(v_0)$ and $v_3 = \mathit{rem\text{-}trace}(v_1)$ and $\delta(\mathit{binop}, v_2, v_3)$ is undefined

$\mathit{binop}\{\mathcal{U}\}\ v_0\ v_1 \quad \blacktriangleright_{\mathsf{A}} \quad \delta(\mathit{binop}, v_2, v_3)$
 if $v_2 = \mathit{rem\text{-}trace}(v_0)$ and $v_3 = \mathit{rem\text{-}trace}(v_1)$ and $\delta(\mathit{binop}, v_2, v_3)$ is defined

$\mathsf{app}\{\mathcal{U}\}\ v_0\ v_1 \quad \blacktriangleright_{\mathsf{A}} \quad \mathsf{TagErr}\,\bullet$
 if $v_0 \notin (\mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b}\,(\lambda x.\, e)) \cup (\mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b}\,(\mathsf{mon}\ (\ell \blacktriangleleft (\tau \Rightarrow \tau) \blacktriangleleft \ell)\ v))$

$\mathsf{app}\{\mathcal{U}\}\ (\mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b_0}\ (\lambda x_0.\, e_0))\ v_0 \quad \blacktriangleright_{\mathsf{A}} \quad \mathsf{trace}\ \overline{b_0}\ (e_0[x_0 \leftarrow v_1])$
 where $v_1 = \mathit{add\text{-}trace}(\mathit{rev}(\overline{b_0}), v_0)$

$\mathsf{app}\{\mathcal{U}\}\ (\mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b_0}\ (\mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0))\ v_1 \quad \blacktriangleright_{\mathsf{A}} \quad \mathsf{trace}\ \overline{b_0}\ (\mathsf{stat}\ b_0\ (\mathsf{app}\{\tau_2\}\ v_0\ (\mathsf{dyn}\ b_1\ v_2)))$
 where $\tau_2 = \mathit{cod}(\tau_0)$ and $b_0 = (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_1 \blacktriangleleft \mathit{dom}(\tau_0) \blacktriangleleft \ell_0)$
 and $v_2 = \mathit{add\text{-}trace}(\mathit{rev}(\overline{b_0}), v_1)$

$\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0 \quad \blacktriangleright_{\mathsf{A}} \quad \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0$
 if $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$ and $v_0 \in (\lambda(x\,{:}\,\tau).\, e) \cup \langle v, v \rangle$

$\mathsf{stat}\ b_0\ (\mathsf{mon}\ b_1\ (\mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b_0}\ v_0)) \quad \blacktriangleright_{\mathsf{A}} \quad \mathsf{trace}\ (b_0 b_1 \overline{b_0})\ v_0$
 if $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ and $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$ and $v_0 \in (\lambda x.\, e) \cup (\langle v, v \rangle) \cup (\mathsf{mon}\ b\ (\lambda(x\,{:}\,\tau).\, e)) \cup (\mathsf{mon}\ b\ \langle v, v \rangle)$

$\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ i_0 \quad \blacktriangleright_{\mathsf{A}} \quad i_0$
 if $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor i_0,)$ and $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$

$\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0 \quad \blacktriangleright_{\mathsf{A}} \quad \mathsf{TagErr}\,\circ$
 if $\neg\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor v_0,)$

$\mathsf{trace}\ \overline{b_0}\ v_0 \quad \blacktriangleright_{\mathsf{A}} \quad v_1$
 where $v_1 = \mathit{add\text{-}trace}(\overline{b_0}, v_0)$

4.2 Amnesic **Ownership Lifting**

$\boxed{\text{Amnesic Ownership Language}}$ extends Ownership Evaluation Language

$v \ = \ \ldots \mid \mathsf{trace}_{\mathsf{v}}\ \overline{b}\ v \mid \mathsf{mon}\ (\ell \blacktriangleleft \tau \Rightarrow \tau \blacktriangleleft \ell)\ v \mid \mathsf{mon}\ (\ell \blacktriangleleft \tau{\times}\tau \blacktriangleleft \ell)\ v$

$e \ = \ \ldots \mid \mathsf{trace}\ \overline{b}\ e$

$E \ = \ \ldots \mid \mathsf{trace}\ \overline{b}\ E$

$\boxed{L; \ell \Vdash_{\mathsf{A}} e}$ extends $L; \ell \Vdash e$ enforces sound and complete blame

$$\frac{L_0; \ell_1 \Vdash_{\mathsf{A}} v_0}{L_0; \ell_0 \Vdash_{\mathsf{A}} \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (v_0)^{\ell_1}} \qquad \frac{\overline{b_0} \simeq (\ell_n \cdots \ell_0) \qquad L_0; \ell_n \Vdash_{\mathsf{A}} e_0}{L_0; \ell_0 \Vdash_{\mathsf{A}} \mathsf{trace}\ \overline{b_0}\ (\!(e_0)\!)^{(\ell_n \cdots \ell_0)}} \qquad \frac{\overline{b_0} \simeq (\ell_n \cdots \ell_0) \qquad L_0; \ell_n \Vdash_{\mathsf{A}} e_0}{L_0; \ell_0 \Vdash_{\mathsf{A}} \mathsf{trace}_{\mathsf{v}}\ \overline{b_0}\ (\!(e_0)\!)^{(\ell_n \cdots \ell_0)}}$$

$\boxed{\longrightarrow^{*}_{\overline{\mathsf{A}}}}$ reflexive-transitive closure of $\longrightarrow_{\overline{\mathsf{A}}}$

$\boxed{e \longrightarrow_{\overline{\mathsf{A}}} e}$ reflexive-transitive closure of $\rhd_{\overline{\mathsf{N}}} \cup \blacktriangleright_{\overline{\mathsf{N}}}$

$\boxed{(e)^{\ell} \rhd_{\mathsf{A}} (e)^{\ell}}$

$(\mathit{unop}\{\tau_0\}\ v_0)^{\ell_0} \quad \rhd_{\mathsf{A}} \quad (\mathsf{TagErr} \circ)^{\ell_0}$
  if $v_0 \notin (\mathsf{mon}\ (\ell \blacktriangleleft (\tau \times \tau) \blacktriangleleft \ell)\ v)$ and $\delta(\mathit{unop}, v_0)$ is undefined

$(\mathit{unop}\{\tau_0\}\ v_0)^{\ell_0} \quad \rhd_{\mathsf{A}} \quad (\delta(\mathit{unop}, v_0))^{\ell_0}$
  if $\delta(\mathit{unop}, v_0)$ is defined

$(\mathsf{fst}\{\tau_0\}\ (\!(\mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (v_0)^{\ell_2})\!)^{\ell_3^*})^{\ell_4} \quad \rhd_{\mathsf{A}} \quad (\!(\mathsf{dyn}\ b_0\ (\mathsf{fst}\{\mathcal{U}\}\ v_0)^{\ell_2})\!)^{\ell_3^* \ell_4}$
  where $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$

$(\mathsf{snd}\{\tau_0\}\ (\!(\mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (v_0)^{\ell_2})\!)^{\ell_3^*})^{\ell_4} \quad \rhd_{\mathsf{A}} \quad (\!(\mathsf{dyn}\ b_0\ (\mathsf{snd}\{\mathcal{U}\}\ v_0)^{\ell_2})\!)^{\ell_3^* \ell_4}$
  where $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$

$(\mathit{binop}\{\tau_0\}\ v_0\ v_1)^{\ell_0} \quad \rhd_{\mathsf{A}} \quad (\mathsf{TagErr} \circ)^{\ell_0}$
  if $\delta(\mathit{binop}, v_0, v_1)$ is undefined

$(\mathit{binop}\{\tau_0\}\ v_0\ v_1)^{\ell_0} \quad \rhd_{\mathsf{A}} \quad (\delta(\mathit{binop}, v_0, v_1))^{\ell_0}$
  if $\delta(\mathit{binop}, v_0, v_1)$ is defined

$(\mathsf{app}\{\tau_0\}\ v_0\ v_1)^{\ell_0} \quad \rhd_{\mathsf{A}} \quad (\mathsf{TagErr} \circ)^{\ell_0}$
  if $v_0 \notin (\!(\lambda(x\!:\!\tau).\ e)\!)^{\ell^*} \cup (\!(\mathsf{mon}\ (\ell \blacktriangleleft (\tau \Rightarrow \tau) \blacktriangleleft \ell)\ v)\!)^{\ell^*}$

$(\mathsf{app}\{\tau_0\}\ (\!(\lambda(x_0\!:\!\tau_1).\ e_0)\!)^{\ell_0^*}\ v_1)^{\ell_1} \quad \rhd_{\mathsf{A}} \quad (\!(e_0[x_0 \leftarrow (\!(v_1)\!)^{\ell_1\, \mathit{rev}(\ell_0^*)}])\!)^{\ell_0^* \ell_1}$

$(\mathsf{app}\{\tau_0\}\ (\!(\mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (v_0)^{\ell_2})\!)^{\ell_3^*}\ v_1)^{\ell_4} \quad \rhd_{\mathsf{A}} \quad (\!(\mathsf{dyn}\ b_0\ (\mathsf{app}\{\mathcal{U}\}\ v_0\ (\mathsf{stat}\ b_1\ (\!(v_1)\!)^{\ell_4\, \mathit{rev}(\ell_3^*)\ell_2}))^{\ell_2})\!)^{\ell_3^* \ell_4}$
  where $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_1 \blacktriangleleft \mathit{dom}(\tau_1) \blacktriangleleft \ell_0)$

$(\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (\!(v_0)\!)^{\ell_2^*})^{\ell_3} \quad \rhd_{\mathsf{A}} \quad (\mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (\!(v_0)\!)^{\ell_2^*})^{\ell_3}$
  if $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$ and $\mathit{rem\text{-}trace}(v_0) \in (\!(\lambda x.\ e)\!)^{\ell^*} \cup (\!(\langle v, v\rangle)\!)^{\ell^*} \cup (\!(\mathsf{mon}\ b\ v)\!)^{\ell^*}$

$(\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (\!(\mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b}_0\ (\!(i_0)\!)^{\ell_2^*})\!)^{\ell_3^*})^{\ell_4} \quad \rhd_{\mathsf{A}} \quad (i_0)^{\ell_4}$
  if $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, i_0)$

$(\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (\!(v_0)\!)^{\ell_2^*})^{\ell_3} \quad \rhd_{\mathsf{A}} \quad (\mathsf{BndryErr}\ ((\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\overline{b}_0, (\!(v_0)\!)^{\ell_2^*}))^{\ell_3}$
  if $\neg \mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$ and $\overline{b}_0 = \mathit{get\text{-}trace}(v_0)$

$\boxed{(e)^{\ell} \blacktriangleright_{\mathsf{A}} (e)^{\ell}}$

$(\mathit{unop}\{\mathcal{U}\}\ v_0)^{\ell_0} \quad \blacktriangleright_{\mathsf{A}} \quad (\mathsf{TagErr}\ \bullet)^{\ell_0}$

if $v_1 = \mathit{rem\text{-}trace}(v_0)$ and $v_1 \notin \mathsf{mon}\ (\ell \blacktriangleleft (\tau\times\tau) \blacktriangleleft \ell)\ v$ and $\delta(\mathit{unop}, v_1)$ is undefined

$(\mathit{unop}\{\mathcal{U}\}\ v_0)^{\ell_0} \quad \blacktriangleright_{\mathsf{A}} \quad (\mathit{add\text{-}trace}(\mathit{get\text{-}trace}(v_0), \delta(\mathit{unop}, v_1)))^{\ell_0}$

if $v_1 = \mathit{rem\text{-}trace}(v_0)$ and $\delta(\mathit{unop}, v_1)$ is defined

$(\mathsf{fst}\{\mathcal{U}\}\ \llparenthesis \mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b}_0\ \llparenthesis \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (v_1)^{\ell_2} \rrparenthesis^{\ell_3^*} \rrparenthesis^{\ell_4^*})^{\ell_5} \quad \blacktriangleright_{\mathsf{A}} \quad (\mathsf{trace}\ \overline{b}_0\ \llparenthesis \mathsf{stat}\ b_1\ (\mathsf{fst}\{\tau_1\}\ v_1)^{\ell_2} \rrparenthesis^{\ell_3^*})^{\ell_4^*\ell_5}$

where $\tau_1 = \mathit{fst}(\tau_0)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)$

$(\mathsf{snd}\{\mathcal{U}\}\ \llparenthesis \mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b}_0\ \llparenthesis \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (v_1)^{\ell_2} \rrparenthesis^{\ell_3^*} \rrparenthesis^{\ell_4^*})^{\ell_5} \quad \blacktriangleright_{\mathsf{A}} \quad (\mathsf{trace}\ \overline{b}_0\ \llparenthesis \mathsf{stat}\ b_1\ (\mathsf{snd}\{\tau_1\}\ v_1)^{\ell_2} \rrparenthesis^{\ell_3^*})^{\ell_4^*\ell_5}$

where $\tau_1 = \mathit{snd}(\tau_0)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)$

$(\mathit{binop}\{\mathcal{U}\}\ v_0\ v_1)^{\ell_0} \quad \blacktriangleright_{\mathsf{A}} \quad (\mathsf{TagErr}\ \bullet)^{\ell_0}$

if $v_2 = \mathit{rem\text{-}trace}(v_0)$ and $v_3 = \mathit{rem\text{-}trace}(v_1)$ and $\delta(\mathit{binop}, v_2, v_3)$ is undefined

$(\mathit{binop}\{\mathcal{U}\}\ v_0\ v_1)^{\ell_0} \quad \blacktriangleright_{\mathsf{A}} \quad (\delta(\mathit{binop}, v_2, v_3))^{\ell_0}$

if $v_2 = \mathit{rem\text{-}trace}(v_0)$ and $v_3 = \mathit{rem\text{-}trace}(v_1)$ and $\delta(\mathit{binop}, v_2, v_3)$ is defined

$(\mathsf{app}\{\mathcal{U}\}\ \llparenthesis \mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b}_0\ \llparenthesis v_0 \rrparenthesis^{\ell_0^*} \rrparenthesis^{\ell_1^*}\ v_1)^{\ell_2} \quad \blacktriangleright_{\mathsf{A}} \quad (\mathsf{TagErr}\ \bullet)^{\ell_0}$

if $v_0 \notin (\lambda x.\, e) \cup (\mathsf{mon}\ (\ell \blacktriangleleft (\tau \Rightarrow \tau)^{\ell} \blacktriangleleft \ell)\ v)$

$(\mathsf{app}\{\mathcal{U}\}\ \llparenthesis \mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b}_0\ \llparenthesis \lambda x_0.\, e_0 \rrparenthesis^{\ell_0^*} \rrparenthesis^{\ell_1^*}\ v_1)^{\ell_2} \quad \blacktriangleright_{\mathsf{A}} \quad (\mathsf{trace}\ \overline{b}_0\ \llparenthesis e_0[x_0 \leftarrow v_2] \rrparenthesis^{\ell_0^*})^{\ell_1^*\ell_2}$

where $v_2 = \mathit{add\text{-}trace}(\mathit{rev}(\overline{b}_0), \llparenthesis v_1 \rrparenthesis^{\ell_2 \mathit{rev}(\ell_1^*) \mathit{rev}(\ell_0^*)})$

$(\mathsf{app}\{\mathcal{U}\}\ \llparenthesis \mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b}_0\ \llparenthesis \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (v_0)^{\ell_2} \rrparenthesis^{\ell_3^*} \rrparenthesis^{\ell_4^*}\ v_1)^{\ell_5} \quad \blacktriangleright_{\mathsf{A}} \quad \llparenthesis \mathsf{trace}\ \overline{b}_0\ \llparenthesis \mathsf{stat}\ b_1\ (\mathsf{app}\{\tau_3\}\ v_0\ (\mathsf{dyn}\ b_2\ v_2))^{\ell_2} \rrparenthesis^{\ell_3^*} \rrparenthesis^{\ell_4^*\ell_5}$

where $\tau_2 = \mathit{cod}(\tau_0)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)$ and $b_2 = (\ell_1 \blacktriangleleft \mathit{dom}(\tau_0) \blacktriangleleft \ell_0)$ and $\tau_3 = \mathit{forget}(\tau_2)$

and $v_2 = (\mathit{add\text{-}trace}(\mathit{rev}(\overline{b}_0), \llparenthesis v_1 \rrparenthesis^{\ell_5 \mathit{rev}(\ell_3^* \ell_4^*)}))^{\ell_2}$

$(\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0)^{\ell_2} \quad \blacktriangleright_{\mathsf{A}} \quad (\mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0)^{\ell_2}$

if $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$ and $v_0 \in \llparenthesis \lambda(x\!:\!\tau).\, e \rrparenthesis^{\ell^*} \cup \llparenthesis \langle v, v \rangle \rrparenthesis^{\ell^*}$

$(\mathsf{stat}\ b_0\ \llparenthesis \mathsf{mon}\ b_1\ \llparenthesis \mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b}_2\ v_0 \rrparenthesis^{\ell_0^*} \rrparenthesis^{\ell_1^*})^{\ell_2} \quad \blacktriangleright_{\mathsf{A}} \quad (\mathsf{trace}\ (b_0 b_1 \overline{b}_2)\ \llparenthesis v_0 \rrparenthesis^{\ell_0^* \ell_1^* \ell_2})^{\ell_2}$

if $b_0 = (\ell_3 \blacktriangleleft \tau_0 \blacktriangleleft \ell_4)$ and $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

and $v_0 \in \llparenthesis \lambda x.\, e \rrparenthesis^{\ell^*} \cup \llparenthesis \langle v, v \rangle \rrparenthesis^{\ell^*} \cup \llparenthesis \mathsf{mon}\ b\ (\lambda(x\!:\!\tau).\, e) \rrparenthesis^{\ell^*} \cup \llparenthesis \mathsf{mon}\ b\ \langle v, v \rangle \rrparenthesis^{\ell^*}$

$(\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ \llparenthesis i_0 \rrparenthesis^{\ell_2^*})^{\ell_3} \quad \blacktriangleright_{\mathsf{A}} \quad (i_0)^{\ell_3}$

if $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

$(\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ \llparenthesis v_0 \rrparenthesis^{\ell_2^*})^{\ell_3} \quad \blacktriangleright_{\mathsf{A}} \quad (\mathsf{TagErr}\ \circ)^{\ell_3}$

if $\neg \mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

$(\mathsf{trace}\ \overline{b}_0\ v_0)^{\ell_0} \quad \blacktriangleright_{\mathsf{A}} \quad (v_1)^{\ell_0}$

where $v_1 = \mathit{add\text{-}trace}(\overline{b}_0, v_0)$

5 Natural THEOREMS, LEMMAS, AND PROOFS

5.1 Natural Theorems

Theorem 5.1 (type soundness). *If* $e_0 : \tau_0$ **wf** *then one of the following holds:*

- $e_0 \rightarrow^*_{\mathsf{N}} v_0$ *and* $\cdot \vdash_{\mathsf{N}} v_0 : \tau_0$
- e_0 *diverges*
- $e_0 \rightarrow^*_{\mathsf{N}} E_0[\mathsf{dyn}\ b_1\ E[e_1]]$ *and* $e_1 \blacktriangleright_{\mathsf{N}} \mathsf{TagErr}\bullet$
- $e_0 \rightarrow^*_{\mathsf{N}} \mathsf{DivErr}$
- $e_0 \rightarrow^*_{\mathsf{N}} \mathsf{BndryErr}\,(\overline{b}_1, v_1)$

Proof. By progress and preservation lemmas (lemma 5.6 & lemma 5.7).

□

Theorem 5.2 (dynamic soundness). *If* $e_0 : \mathcal{U}$ **wf** *then one of the following holds:*

- $e_0 \rightarrow^*_{\mathsf{N}} v_0$ *and* $\cdot \vdash_{\mathsf{N}} v_0 : \mathcal{U}$
- e_0 *diverges*
- $e_0 \rightarrow^*_{\mathsf{N}} E_0[e_1]$ *and* $e_1 \blacktriangleright_{\mathsf{N}} \mathsf{TagErr}\bullet$
- $e_0 \rightarrow^*_{\mathsf{N}} \mathsf{DivErr}$
- $e_0 \rightarrow^*_{\mathsf{N}} \mathsf{BndryErr}\,(\overline{b}_1, v_1)$

Proof. By progress and preservation lemmas (lemma 5.6 & lemma 5.7).

□

Theorem 5.3. *If* $e_0 : \tau?\ \overline{\mathbf{wf}}$ *then* $\mathit{forget}(e_0) : \tau?\ \mathbf{wf}$ *and* $e_0 \longrightarrow_{\overline{\mathsf{N}}} e_1$ *iff* $\mathit{forget}(e_0) \rightarrow_{\mathsf{N}} \mathit{forget}(e_1)$

Proof. By the definition of $\longrightarrow_{\overline{\mathsf{N}}}$.

□

Theorem 5.4 (complete monitoring). *If* $e_0 : \tau?\ \overline{\mathbf{wf}}$ *and* $e_0 \longrightarrow^*_{\overline{\mathsf{N}}} e_1$ *then* $\cdot; \ell \Vdash e_1$.

Proof. By lemma 5.20 and lemma 5.21.

□

Theorem 5.5 (correct blame). *If* $e_0 : \tau?\ \overline{\mathbf{wf}}$ *and* $e_0 \longrightarrow^*_{\overline{\mathsf{N}}} \mathsf{BndryErr}\,(\overline{b}_0, v_0)$ *then:*

- $\mathit{senders}(\overline{b}_0) = \mathit{owners}(v_0)$
- $\overline{b}_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
- *either* $\mathit{has\text{-}boundary}((\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1), e_0)$ *or* $\mathit{has\text{-}boundary}((\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0), e_0)$

Proof.

1. Either $e_0 \longrightarrow^*_{\overline{\mathsf{N}}} v_0$ where $v_0 : \tau?\ \overline{\mathbf{wf}}$ or e_0 diverges, or $e_0 \longrightarrow^*_{\overline{\mathsf{N}}} \mathsf{Err}$
   by lemma 5.20 and lemma 5.21
2. assume $e_0 \longrightarrow^*_{\overline{\mathsf{N}}} \mathsf{BndryErr}\,(\overline{b}_0, v_0)$
   2.1. $\exists\, \ell_0, \tau_0, \ell_1 . \overline{b}_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
      2.1.1. qed

by definition of $\longrightarrow^{*}_{\overline{\mathsf{N}}}$

2.2. $\mathit{owners}(v_0) = \ell_1$

2.2.1. QED

by lemma 5.21

2.3. $\exists\, v_1, E_0$ such that $e_0 \longrightarrow^{*}_{\overline{\mathsf{N}}} E_0[\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_1]$

by lemma 5.24

2.4. $(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) \in e_0$ or $(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0) \in e_0$

by lemma 5.35

3. QED

□

5.2 Natural **Lemmas**

Lemma 5.6 ($\vdash_{\mathsf{N}}$ progress). *If* $\cdot \vdash_{\mathsf{N}} e_0 : \tau?$ *then one of the following holds:*

- $e_0 \in v$
- $e_0 \in \mathsf{Err}$
- $\exists\, e_1$ *such that* $e_0 \rightarrow_{\mathsf{N}} e_1$

Proof. By case analysis of e_0.

By lemma 5.10 it suffices to consider the following cases.

1. CASE $e_0 \in v$

1.1. QED

2. CASE $e_0 = E_0[\mathsf{Err}]$

2.1. QED

3. CASE $e_0 = E_0[\mathsf{app}\{\tau_1\}\ v_0\ v_1]$

3.1. $v_0 \in (\lambda(x\!:\!\tau).\, e) \cup (\mathsf{mon}\ b\ v)$

by inversion $\vdash_{\mathsf{N}}$

3.2. SCASE $v_0 = \lambda(x_2\!:\!\tau_2).\, e_2$

3.2.1. QED

$e_0 \rhd_{\mathsf{N}} E_0[e_2[x_2 \leftarrow v_1]]$

3.3. SCASE $v_0 = \mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2) \blacktriangleleft \ell_1)\ v_2$

3.3.1. QED

$e_0 \rhd_{\mathsf{N}} E_0[\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\mathcal{U}\}\ v_2\ (\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ v_1))]$

4. CASE $e_0 = E_0[\mathsf{app}\{\mathcal{U}\}\ v_0\ v_1]$

4.1. SCASE $v_0 = \lambda x_2.\, e_2$

4.1.1. QED

$e_0 \blacktriangleright_{\mathsf{N}} E_0[e_2[x_2 \leftarrow v_1]]$

4.2. SCASE $v_0 = \mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2) \blacktriangleleft \ell_1)\ v_2$

4.2.1. QED

$e_0 \blacktriangleright_{\mathsf{N}} E_0[\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\tau_2\}\ v_2\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ v_1))]$

4.3. SCASE $v_0 \notin (\lambda(x\!:\!\tau).\, e) \cup (\mathsf{mon}\ b\ v)$

4.3.1. QED

$e_0 \blacktriangleright_{\mathsf{N}} E_0[\mathsf{TagErr}\,\bullet]$

5. CASE $e_0 = E_0[\mathit{unop}\{\tau?\}\, v_0]$
   5.1. QED
      by lemma 5.11 and lemma 5.13
6. CASE $e_0 = E_0[\mathit{binop}\{\tau?\}\, v_0\, v_1]$
   6.1. QED
      by lemma 5.11 and lemma 5.13
7. CASE $e_0 = E_0[\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, v_0]$
   7.1. QED
      by lemma 5.11 and lemma 5.15
8. CASE $e_0 = E_0[\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, v_0]$
   8.1. QED
      by lemma 5.11 and lemma 5.16

□

Lemma 5.7 ($\vdash_{\mathsf{N}}$ static preservation). *If* $\cdot \vdash_{\mathsf{N}} e_0 : \tau?$ *and* $e_0 \rightarrow_{\mathsf{N}} e_1$ *then* $\cdot \vdash_{\mathsf{N}} e_1 : \tau?$.

Proof. By lemma 5.8 and lemma 5.9.

□

Lemma 5.8 ($\rhd_{\mathsf{N}}$ preservation). *If* $\cdot \vdash_{\mathsf{N}} e_0 : \tau_0$ *and* $e_0 \rhd_{\mathsf{N}} e_1$ *then* $\cdot \vdash_{\mathsf{N}} e_1 : \tau_0$.

Proof. By case analysis of $\rhd_{\mathsf{N}}$.

1. CASE $\mathit{unop}\{\tau_0\}\, v_0 \rhd_{\mathsf{N}} \delta_N(\mathit{unop}, v_0)$
   1.1. QED
      by lemma 5.14
2. CASE $\mathit{binop}\{\tau_0\}\, v_0\, v_1 \rhd_{\mathsf{N}} \delta_N(\mathit{binop}, v_0, v_1)$
   2.1. QED
      by lemma 5.14
3. CASE $\mathsf{app}\{\tau_0\}\,(\lambda(x_1\!:\!\tau_1).\, e_1)\, v_2 \rhd_{\mathsf{N}} e_1[x_1 \leftarrow v_2]$
   3.1. QED
      by lemma 5.19
4. CASE $\mathsf{app}\{\tau_0\}\,(\mathsf{mon}\,(\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_0) \blacktriangleleft \ell_1)\, v_0)\, v_1$
      $\rhd_{\mathsf{N}}\ \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\,(\mathsf{app}\{\mathcal{U}\}\, v_0\,(\mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\, v_1))$
   4.1. QED

$$\dfrac{\dfrac{\dfrac{\text{by inversion} \vdash_{\mathsf{N}}}{\cdot \vdash_{\mathsf{N}} v_0 : \mathcal{U}} \quad \dfrac{\dfrac{\text{by inversion} \vdash_{\mathsf{N}}}{\cdot \vdash_{\mathsf{N}} v_1 : \tau_1}}{\cdot \vdash_{\mathsf{N}} \mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\, v_1 : \mathcal{U}}}{\cdot \vdash_{\mathsf{N}} \mathsf{app}\{\mathcal{U}\}\, v_0\,(\mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\, v_1) : \mathcal{U}}}{\cdot \vdash_{\mathsf{N}} \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\,(\mathsf{app}\{\mathcal{U}\}\, v_0\,(\mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\, v_1)) : \tau_0}$$

5. CASE $\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} v_1$
   5.1. QED
   by lemma 5.17

□

Lemma 5.9 ($\blacktriangleright_{\mathsf{N}}$ preservation). *If* $\cdot \vdash_{\mathsf{N}} e_0 : \mathcal{U}$ *and* $e_0 \blacktriangleright_{\mathsf{N}} e_1$ *then* $\cdot \vdash_{\mathsf{N}} e_1 : \mathcal{U}$.

Proof. By case analysis of $\blacktriangleright_{\mathsf{N}}$.

1. CASE $unop\{\mathcal{U}\}\ v_0 \blacktriangleright_{\mathsf{N}} \mathsf{TagErr}\,\bullet$
   1.1. QED
   $\cdot \vdash_{\mathsf{N}} \mathsf{TagErr}\,\bullet : \mathcal{U}$
2. CASE $\delta_N(unop, v_0)$ is defined and $unop\{\mathcal{U}\}\ v_0 \blacktriangleright_{\mathsf{N}} \delta_N(unop, v_0)$
   2.1. QED
   by lemma 5.14
3. CASE $binop\{\mathcal{U}\}\ v_0\ v_1 \blacktriangleright_{\mathsf{N}} \mathsf{TagErr}\,\bullet$
   3.1. QED
   $\cdot \vdash_{\mathsf{N}} \mathsf{TagErr}\,\bullet : \mathcal{U}$
4. CASE $\delta_N(binop, v_0, v_1)$ is defined and $binop\{\mathcal{U}\}\ v_0\ v_1 \blacktriangleright_{\mathsf{N}} \delta_N(binop, v_0, v_1)$
   4.1. QED
   by lemma 5.14
5. CASE $\mathsf{app}\{\mathcal{U}\}\ (\lambda x_1.\, e_1)\ v_2 \blacktriangleright_{\mathsf{N}} e_1[x_1 \leftarrow v_2]$
   5.1. QED
   by lemma 5.19
6. CASE $\mathsf{app}\{\mathcal{U}\}\ (\mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_0) \blacktriangleleft \ell_1)\ v_0)\ v_1$
   $\blacktriangleright_{\mathsf{N}} \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\tau_0\}\ v_0\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ v_1))$
   6.1. QED

$$\dfrac{\dfrac{\dfrac{\text{by inversion } \vdash_{\mathsf{N}}}{\cdot \vdash_{\mathsf{N}} v_0 : \tau_1 \Rightarrow \tau_0} \quad \dfrac{\dfrac{\text{by inversion } \vdash_{\mathsf{N}}}{\cdot \vdash_{\mathsf{N}} v_1 : \mathcal{U}}}{\cdot \vdash_{\mathsf{N}} \mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ v_1 : \tau_1}}{\cdot \vdash_{\mathsf{N}} \mathsf{app}\{\tau_0\}\ v_0\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ v_1) : \tau_0}}{\cdot \vdash_{\mathsf{N}} \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\tau_0\}\ v_0\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ v_1)) : \mathcal{U}}$$

7. CASE $\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{N}} v_1$
   7.1. QED
   by lemma 5.18

□

Lemma 5.10 (unique decomposition). *If* $\cdot \vdash_{\mathsf{N}} e_0 : \tau?$ *then either:*

- $e_0 \in v$
- $e_0 = E_0[\mathsf{app}\{\tau?\}\ v_0\ v_1]$
- $e_0 = E_0[unop\{\tau?\}\ v_0]$

- $e_0 = E_0[\mathit{binop}\{\tau?\}\, v_0\, v_1]$
- $e_0 = E_0[\mathsf{dyn}\ b_1\ v_1]$
- $e_0 = E_0[\mathsf{stat}\ b_1\ v_1]$
- $e_0 = E_0[\mathsf{Err}]$

Proof. By induction on the structure of e_0.

1. CASE $e_0 = x_0$
   1.1. CONTRADICTION:
   $\cdot \vdash_{\mathsf{N}} e_0 : \tau?$
2. CASE $e_0 = v_0$
   2.1. QED
3. CASE $e_0 = \langle e_1, e_2 \rangle$
   3.1. SCASE $e_1 \notin v$
      3.1.1. QED
         by the induction hypothesis
   3.2. SCASE $e_1 \in v$ and $e_2 \notin v$
      3.2.1. QED
         by the induction hypothesis
   3.3. SCASE $e_1 \in v$ and $e_2 \in v$
      3.3.1. QED
         $e_0 \in v$
4. CASE $e_0 = \mathsf{app}\{\tau?\}\, e_1\, e_2$
   4.1. QED
      by the induction hypothesis
5. CASE $e_0 = \mathit{unop}\{\tau?\}\, e_1$
   5.1. QED
      by the induction hypothesis
6. CASE $e_0 = \mathit{binop}\{\tau?\}\, e_1\, e_2$
   6.1. QED
      by the induction hypothesis
7. CASE $e_0 = \mathsf{dyn}\ b_1\ e_1$
   7.1. QED
      by the induction hypothesis
8. CASE $e_0 = \mathsf{stat}\ b_1\ e_1$
   8.1. QED
      by the induction hypothesis
9. CASE $e_0 \in \mathsf{Err}$
   9.1. QED

□

Lemma 5.11. *If* $\cdot \vdash_{\mathsf{N}} E_0[e_0] : \tau?$ *then one of the following holds:*

- $\cdot \vdash_N e_0 : \mathcal{U}$
- $\exists \tau_0 \,.\, \cdot \vdash_N e_0 : \tau_0$

PROOF. By induction on the structure of E_0 and case analysis of $\vdash_N$.

1. CASE $E_0 = [\,]$
   1.1. QED
2. CASE $E_0 = \langle E_1, e_2 \rangle$
   2.1. $\cdot \vdash_N E_1[e_0] : \tau?$
   by inversion $\vdash_N$
   2.2. QED
   by the induction hypothesis
3. CASE $E_0 = \langle v_1, E_2 \rangle$
   3.1. QED
   by the induction hypothesis
4. CASE $E_0 = \mathsf{app}\{\tau?\}\, E_1\, e_2$
   4.1. QED
   by the induction hypothesis
5. CASE $E_0 = \mathsf{app}\{\tau?\}\, v_1\, E_2$
   5.1. QED
   by the induction hypothesis
6. CASE $E_0 = \mathit{unop}\{\tau?\}\, E_1$
   6.1. QED
   by the induction hypothesis
7. CASE $E_0 = \mathit{binop}\{\tau?\}\, E_1\, e_2$
   7.1. QED
   by the induction hypothesis
8. CASE $E_0 = \mathit{binop}\{\tau?\}\, v_1\, E_2$
   8.1. QED
   by the induction hypothesis
9. CASE $E_0 = \mathsf{dyn}\, b_1\, E_1$
   9.1. QED
   by the induction hypothesis
10. CASE $E_0 = \mathsf{stat}\, b_1\, E_1$
   10.1. QED
   by the induction hypothesis

□

LEMMA 5.12 ($\vdash_N$ REPLACEMENT).

- *If* $\cdot \vdash_N E_0[e_0] : \tau?$ *and the derivation contains a proof of* $\cdot \vdash_N e_0 : \tau_0$ *and* $\cdot \vdash_N e_1 : \tau_0$ *then* $\cdot \vdash_N E_0[e_1] : \tau?$.
- *If* $\cdot \vdash_N E_0[e_0] : \tau?$ *and the derivation contains a proof of* $\cdot \vdash_N e_0 : \mathcal{U}$ *and* $\cdot \vdash_N e_1 : \mathcal{U}$ *then* $\cdot \vdash_N E_0[e_1] : \tau?$.

PROOF. By induction on E_0.

1. CASE $E_0 = [\,]$
   1.1. QED
2. CASE $E_0 = \langle E_1, e_2 \rangle$
   2.1. QED
   by the induction hypothesis
3. CASE $E_0 = \langle v_1, E_2 \rangle$
   3.1. QED
   by the induction hypothesis
4. CASE $E_0 = \mathsf{app}\{\tau?\}\, E_1\, e_2$
   4.1. QED
   by the induction hypothesis
5. CASE $E_0 = \mathsf{app}\{\tau?\}\, v_1\, E_2$
   5.1. QED
   by the induction hypothesis
6. CASE $E_0 = unop\{\tau?\}\, E_1$
   6.1. QED
   by the induction hypothesis
7. CASE $E_0 = binop\{\tau?\}\, E_1\, e_2$
   7.1. QED
   by the induction hypothesis
8. CASE $E_0 = binop\{\tau?\}\, v_1\, E_2$
   8.1. QED
   by the induction hypothesis
9. CASE $E_0 = \mathsf{dyn}\, b_1\, E_1$
   9.1. QED
   by the induction hypothesis
10. CASE $E_0 = \mathsf{stat}\, b_1\, E_1$
   10.1. QED
   by the induction hypothesis

□

LEMMA 5.13 (δ_N TYPE PROGRESS).

- *If* $\cdot \vdash_{\mathsf{N}} unop\{\tau_1\}\, v_0 : \tau_0$ *then* $\delta_N(unop, v_0)$ *is defined.*
- *if* $\cdot \vdash_{\mathsf{N}} binop\{\tau_1\}\, v_0\, v_1 : \tau_0$ *then* $\delta_N(binop, v_0, v_1)$ *is defined.*
- *If* $\cdot \vdash_{\mathsf{N}} unop\{\mathcal{U}\}\, v_0 : \mathcal{U}$ *then* $unop\{\mathcal{U}\}\, v_0 \blacktriangleright_{\mathsf{N}} e_1$.
- *if* $\cdot \vdash_{\mathsf{N}} binop\{\mathcal{U}\}\, v_0\, v_1 : \mathcal{U}$ *then* $binop\{\mathcal{U}\}\, v_0\, v_1 \blacktriangleright_{\mathsf{N}} e_1$.

PROOF. By case analysis of δ_N, $\vdash_{\mathsf{N}}$, and $\blacktriangleright_{\mathsf{N}}$.

1. CASE $\cdot \vdash_{\mathsf{N}} \mathsf{fst}\{\tau_1\}\, v_0 : \tau_0$
   1.1. $v_0 = \langle v_1, v_2 \rangle$
   by $\vdash_{\mathsf{N}}$ canonical forms

1.2. QED

$\delta_N(unop, v_0) = v_1$

2. CASE $\cdot \vdash_{\mathsf{N}} \mathsf{snd}\{\tau_1\}\, v_0 : \tau_0$

2.1. $v_0 = \langle v_1, v_2 \rangle$

by $\vdash_{\mathsf{N}}$ canonical forms

2.2. QED

$\delta_N(unop, v_0) = v_2$

3. CASE $\cdot \vdash_{\mathsf{N}} \mathsf{sum}\{\tau_1\}\, v_0\, v_1 : \tau_0$

3.1. $v_0 \in i$ and $v_1 \in i$

by $\vdash_{\mathsf{N}}$ canonical forms

3.2. QED

$\delta_N(unop, v_0, v_1) \in i$

4. CASE $\cdot \vdash_{\mathsf{N}} \mathsf{quotient}\{\tau_1\}\, v_0\, v_1 : \tau_0$

4.1. $v_0 \in i$ and $v_1 \in i$

by $\vdash_{\mathsf{N}}$ canonical forms

4.2. QED

$\delta_N(unop, v_0, v_1) \in i \cup \mathsf{DivErr}$

5. CASE $\cdot \vdash_{\mathsf{N}} unop\{\mathcal{U}\}\, v_0 : \mathcal{U}$

5.1. SCASE $\delta_N(unop, v_0)$ is defined

5.1.1. QED

$unop\{\mathcal{U}\}\, v_0 \blacktriangleright_{\mathsf{N}} \delta_N(unop, v_0)$

5.2. SCASE $\delta_N(unop, v_0)$ is not defined

5.2.1. QED

$unop\{\mathcal{U}\}\, v_0 \blacktriangleright_{\mathsf{N}} \mathsf{TagErr}\,\bullet$

6. CASE $\cdot \vdash_{\mathsf{N}} binop\{\mathcal{U}\}\, v_0\, v_1 : \mathcal{U}$

6.1. SCASE $\delta_N(binop, v_0, v_1)$ is defined

6.1.1. QED

$binop\{\mathcal{U}\}\, v_0\, v_1 \blacktriangleright_{\mathsf{N}} \delta_N(binop, v_0, v_1)$

6.2. SCASE $\delta_N(binop, v_0, v_1)$ is not defined

6.2.1. QED

$binop\{\mathcal{U}\}\, v_0\, v_1 \blacktriangleright_{\mathsf{N}} \mathsf{TagErr}\,\bullet$

□

LEMMA 5.14 (δ_N TYPE PRESERVATION).

- *If* $\cdot \vdash_{\mathsf{N}} unop\{\tau_1\}\, v_0 : \tau_0$ *then* $\cdot \vdash_{\mathsf{N}} \delta_N(unop, v_0) : \tau_0$.
- *If* $\cdot \vdash_{\mathsf{N}} binop\{\tau_1\}\, v_0\, v_1 : \tau_0$ *then* $\cdot \vdash_{\mathsf{N}} \delta_N(binop, v_0, v_1) : \tau_0$.
- *If* $\cdot \vdash_{\mathsf{N}} unop\{\mathcal{U}\}\, v_0 : \mathcal{U}$ *and* $\delta_N(unop, v_0)$ *is defined then* $\cdot \vdash_{\mathsf{N}} \delta_N(unop, v_0) : \mathcal{U}$.
- *If* $\cdot \vdash_{\mathsf{N}} binop\{\mathcal{U}\}\, v_0\, v_1 : \mathcal{U}$ *and* $\delta_N(binop, v_0, v_1)$ *is defined then* $\cdot \vdash_{\mathsf{N}} \delta_N(binop, v_0, v_1) : \mathcal{U}$.

PROOF. By case analysis of δ_N and $\vdash_{\mathsf{N}}$.

1. CASE $\cdot \vdash_{\mathsf{N}} \mathsf{fst}\{\tau_0\}\, v_0 : \tau_0$

1.1. $v_0 = \langle v_1, v_2 \rangle$ and $\cdot \vdash_N v_1 : \tau_0$
by inversion $\vdash_N$
1.2. QED
2. CASE $\cdot \vdash_N \mathsf{snd}\{\tau_0\}\, v_0 : \tau_0$
2.1. $v_0 = \langle v_1, v_2 \rangle$ and $\cdot \vdash_N v_2 : \tau_0$
by $\vdash_N$ canonical forms and $\vdash_N$ inversion
2.2. QED
3. CASE $\cdot \vdash_N \mathsf{sum}\{\tau_1\}\, v_0\, v_1 : \tau_0$
3.1. $\tau_0 \in \mathsf{Int} \cup \mathsf{Nat}$
by inversion $\vdash_N$
3.2. SCASE $\tau_0 = \mathsf{Int}$
3.2.1. $\cdot \vdash_N v_0 : \mathsf{Int}$ and $\cdot \vdash_N v_1 : \mathsf{Int}$
by inversion $\vdash_N$
3.2.2. $v_0 \in i$ and $v_1 \in i$
by $\vdash_N$ canonical forms
3.2.3. QED
$\delta_N(binop, v_0, v_1) \in i$
3.3. SCASE $\tau_0 = \mathsf{Nat}$
3.3.1. $\cdot \vdash_N v_0 : \mathsf{Nat}$ and $\cdot \vdash_N v_1 : \mathsf{Nat}$
by inversion $\vdash_N$
3.3.2. $v_0 \in n$ and $v_1 \in n$
by $\vdash_N$ canonical forms
3.3.3. QED
$\delta_N(binop, v_0, v_1) \in n$
4. CASE $\cdot \vdash_N \mathsf{quotient}\{\tau_1\}\, v_0\, v_1 : \tau_0$
4.1. $\tau_0 \in \mathsf{Int} \cup \mathsf{Nat}$
by inversion $\vdash_N$
4.2. SCASE $\tau_0 = \mathsf{Int}$
4.2.1. $\cdot \vdash_N v_0 : \mathsf{Int}$ and $\cdot \vdash_N v_1 : \mathsf{Int}$
by inversion $\vdash_N$
4.2.2. $v_0 \in i$ and $v_1 \in i$
by $\vdash_N$ canonical forms
4.2.3. QED
$\delta_N(binop, v_0, v_1) \in i \cup \mathsf{DivErr}$
4.3. SCASE $\tau_0 = \mathsf{Nat}$
4.3.1. $\cdot \vdash_N v_0 : \mathsf{Nat}$ and $\cdot \vdash_N v_1 : \mathsf{Nat}$
by inversion $\vdash_N$
4.3.2. $v_0 \in n$ and $v_1 \in n$
by $\vdash_N$ canonical forms
4.3.3. QED
$\delta_N(binop, v_0, v_1) \in n \cup \mathsf{DivErr}$

5. CASE $\cdot \vdash_{\mathrm{N}} \mathsf{fst}\{\mathcal{U}\}\, v_0 : \mathcal{U}$
   5.1. $v_0 = \langle v_1, v_2 \rangle$
   because $\delta_N(unop, v_0)$ is defined
   5.2. $\cdot \vdash_{\mathrm{N}} v_1 : \mathcal{U}$
   by inversion $\vdash_{\mathrm{N}}$
   5.3. QED
   $\delta_N(unop, v_0) = v_1$
6. CASE $\cdot \vdash_{\mathrm{N}} \mathsf{snd}\{\mathcal{U}\}\, v_0 : \mathcal{U}$
   6.1. $v_0 = \langle v_1, v_2 \rangle$
   because $\delta_N(unop, v_0)$ is defined
   6.2. $\cdot \vdash_{\mathrm{N}} v_2 : \mathcal{U}$
   by inversion $\vdash_{\mathrm{N}}$
   6.3. QED
   $\delta_N(unop, v_0) = v_2$
7. CASE $\cdot \vdash_{\mathrm{N}} \mathsf{sum}\{\mathcal{U}\}\, v_0\, v_1 : \mathcal{U}$
   7.1. $\delta_N(binop, v_0, v_1) \in i$
   by definition δ_N
   7.2. QED
   $\cdot \vdash_{\mathrm{N}} \delta_N(binop, v_0, v_1) : \mathcal{U}$
8. CASE $\cdot \vdash_{\mathrm{N}} \mathsf{quotient}\{\mathcal{U}\}\, v_0\, v_1 : \mathcal{U}$
   8.1. $\delta_N(binop, v_0, v_1) \in i \cup \mathsf{DivErr}$
   by definition δ_N
   8.2. QED
   $\cdot \vdash_{\mathrm{N}} \delta_N(binop, v_0, v_1) : \mathcal{U}$

□

LEMMA 5.15. *If* $\cdot \vdash_{\mathrm{N}} \mathsf{dyn}\, b_0\, v_0 : \tau_0$ *and* $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ *then* $\exists\, e_1$ *such that* $\mathsf{dyn}\, b_0\, v_0 \rhd_{\mathrm{N}} e_1$.

PROOF. By case analysis of $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$.

1. CASE $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, \lambda x_1.\, e_1)$
   1.1. QED
   $\mathsf{dyn}\, b_0\, v_0 \rhd_{\mathrm{N}} \mathsf{mon}\, b_0\, v_0$
2. CASE $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, \lambda(x_1 : \tau_1).\, e_1)$
   2.1. CONTRADICTION:
   $\cdot \vdash_{\mathrm{N}} \mathsf{dyn}\, b_0\, v_0 : \tau_0$
3. CASE $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, \mathsf{mon}\, b_1\, v_1)$
   3.1. QED
   $\mathsf{dyn}\, b_0\, v_0 \rhd_{\mathrm{N}} \mathsf{mon}\, b_0\, v_0$
4. CASE $\mathit{tag\text{-}match}(\lfloor (\tau_1 \times \tau_2) \rfloor, \langle v_1, v_2 \rangle)$
   4.1. QED
   $\mathsf{dyn}\, b_0\, v_0 \rhd_{\mathrm{N}} \langle \mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\, v_1, \mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\, v_2 \rangle$
5. CASE $\mathit{tag\text{-}match}(\lfloor \mathsf{Int} \rfloor, v_0)$

5.1. QED
$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} v_0$

6. CASE *tag-match*$(\lfloor \mathsf{Nat} \rfloor, v_0)$
6.1. QED
$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} v_0$

7. CASE $\neg$*tag-match*$(\lfloor \tau_0 \rfloor, v_0)$
7.1. QED
$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}\,(b_0, v_0)$

□

Lemma 5.16. *If* $\cdot \vdash_{\mathsf{N}} \mathsf{stat}\ b_0\ v_0 : \mathcal{U}$ *and* $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ *then* $\exists\, e_1$ *such that* $\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{N}} e_1$.

Proof. By case analysis on v_0.

1. CASE $v_0 \in \lambda x.\, e$
1.1. Contradiction:
$\cdot \vdash_{\mathsf{N}} \mathsf{stat}\ b_0\ v_0 : \mathcal{U}$

2. CASE $v_0 \in \lambda(x\!:\!\tau).\, e$
2.1. QED
$\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{N}} \mathsf{mon}\ b_0\ v_0$

3. CASE $v_0 \in \mathsf{mon}\ b\ e$
3.1. QED
$\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{N}} \mathsf{mon}\ b_0\ v_0$

4. CASE $v_0 = \langle v_1, v_2 \rangle$
4.1. $\tau_0 = \tau_1 \times \tau_2$
by inversion $\vdash_{\mathsf{N}}$
4.2. QED
$\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{N}} \langle \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ v_1, \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2 \rangle$

5. CASE $v_0 \in i$
5.1. QED
$\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{N}} v_0$

□

Lemma 5.17 (N-dyn preservation). *If* $\cdot \vdash_{\mathsf{N}} \mathsf{dyn}\ b_0\ v_0 : \tau_0$ *and* $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ *and* $\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} e_1$ *then* $\cdot \vdash_{\mathsf{N}} e_1 : \tau_0$.

Proof. By case analysis of $\rhd_{\mathsf{N}}$.

1. CASE $\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{mon}\ b_0\ v_0$
and $v_0 \in (\lambda x.\, e) \cup (\mathsf{mon}\ b\ v)$
1.1. QED

$$\cfrac{\cfrac{\text{by inversion } \vdash_{\mathsf{N}}}{\cdot \vdash_{\mathsf{N}} v_0 : \mathcal{U}}}{\cdot \vdash_{\mathsf{N}} \mathsf{mon}\ b_0\ v_0 : \tau_0}$$

2. CASE $\tau_0 = \tau_1 \times \tau_2$

and $\mathsf{dyn}\,(\ell_0 \blacktriangleleft (\tau_1 \times \tau_2) \blacktriangleleft \ell_1)\ \langle v_1, v_2 \rangle \rhd_{\mathsf{N}} \langle \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ v_1, \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2 \rangle$

2.1. QED

$$\frac{\dfrac{\dfrac{\text{by inversion } \vdash_{\mathsf{N}}}{\cdot \vdash_{\mathsf{N}} v_1 : \mathcal{U}}}{\cdot \vdash_{\mathsf{N}} \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ v_1 : \tau_1} \qquad \dfrac{\dfrac{\text{by inversion } \vdash_{\mathsf{N}}}{\cdot \vdash_{\mathsf{N}} v_2 : \mathcal{U}}}{\cdot \vdash_{\mathsf{N}} \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2 : \tau_2}}{\cdot \vdash_{\mathsf{N}} \langle \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ v_1, \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2 \rangle : \tau_1 \times \tau_2}$$

3. CASE $\mathsf{dyn}\ b_0\ i_0 \rhd_{\mathsf{N}} i_0$

and $\textit{tag-match}(\lfloor \tau_0 \rfloor, i_0)$

3.1. QED

by case analysis of $\textit{tag-match}(\lfloor \tau_0 \rfloor, i_0)$

4. CASE $\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}\,(b_0, v_0)$

4.1. QED

$\cdot \vdash_{\mathsf{N}} \mathsf{BndryErr}\,(b_0, v_0) : \tau_0$

□

LEMMA 5.18 (N-stat PRESERVATION). *If* $\cdot \vdash_{\mathsf{N}} \mathsf{stat}\ b_0\ v_0 : \mathcal{U}$ *and* $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ *and* $\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{N}} e_1$ *then* $\cdot \vdash_{\mathsf{N}} e_1$.

PROOF. By case analysis of $\blacktriangleright_{\mathsf{N}}$.

1. CASE $v_0 \in (\lambda(x\!:\!\tau).\,e) \cup (\mathsf{mon}\ b\ v)$

and $\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{N}} \mathsf{mon}\ b_0\ v_0$

1.1. QED

$$\frac{\dfrac{\text{by inversion } \vdash_{\mathsf{N}}}{\cdot \vdash_{\mathsf{N}} v_0 : \tau_0}}{\cdot \vdash_{\mathsf{N}} \mathsf{mon}\ b_0\ v_0 : \mathcal{U}}$$

2. CASE $\tau_0 = \tau_1 \times \tau_2$

and $\mathsf{stat}\,(\ell_0 \blacktriangleleft (\tau_1 \times \tau_2) \blacktriangleleft \ell_1)\ \langle v_1, v_2 \rangle \blacktriangleright_{\mathsf{N}} \langle \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ v_1, \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2 \rangle$

2.1. QED

$$\frac{\dfrac{\dfrac{\text{by inversion } \vdash_{\mathsf{N}}}{\cdot \vdash_{\mathsf{N}} v_1 : \tau_1}}{\cdot \vdash_{\mathsf{N}} \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ v_1 : \mathcal{U}} \qquad \dfrac{\dfrac{\text{by inversion } \vdash_{\mathsf{N}}}{\cdot \vdash_{\mathsf{N}} v_2 : \tau_2}}{\cdot \vdash_{\mathsf{N}} \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2 : \mathcal{U}}}{\cdot \vdash_{\mathsf{N}} \langle \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ v_1, \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2 \rangle : \mathcal{U}}$$

3. CASE $\mathsf{stat}\ b_0\ i_0 \blacktriangleright_{\mathsf{N}} i_0$

3.1. QED

$\cdot \vdash_{\mathsf{N}} i_0 : \mathcal{U}$

4. CASE $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0 \blacktriangleright_{\mathsf{N}} \mathsf{TagErr} \circ$

4.1. CONTRADICTION:

$\cdot \vdash_{\mathsf{N}} v_0 : \tau_0$

□

Lemma 5.19.

- *If* $(x_0 : \tau_0), \Gamma_0 \vdash_{\mathsf{N}} e_1 : \tau?$ *and* $\cdot \vdash_{\mathsf{N}} v_0 : \tau_0$ *then* $\Gamma_0 \vdash_{\mathsf{N}} e_1[x_0 \leftarrow v_0] : \tau?$
- *If* $(x_0 : \mathcal{U}), \Gamma_0 \vdash_{\mathsf{N}} e_1 : \tau?$ *and* $\cdot \vdash_{\mathsf{N}} v_0 : \mathcal{U}$ *then* $\Gamma_0 \vdash_{\mathsf{N}} e_1[x_0 \leftarrow v_0] : \tau?$

Proof. By induction on e_1.

1. case $e_1 = x_2$
   1.1. scase $x_0 = x_2$
      1.1.1. qed
         $e_1[x_0 \leftarrow v_0] = v_0$
   1.2. scase $x_0 \neq x_2$
      1.2.1. qed
         $e_1[x_0 \leftarrow v_0] = e_1$
2. case $e_1 = i_0$
   2.1. qed
      $e_1[x_0 \leftarrow v_0] = e_1$
3. case $e_1 = \lambda x_2.\, e_2$
   3.1. scase $x_0 = x_2$
      3.1.1. qed
         by the induction hypothesis
   3.2. scase $x_0 \neq x_2$
      3.2.1. qed
         $e_1[x_0 \leftarrow v_0] = e_1$
4. case $e_1 = \lambda(x_2 : \tau_2).\, e_2$
   4.1. scase $x_0 = x_2$
      4.1.1. qed
         by the induction hypothesis
   4.2. scase $x_0 \neq x_2$
      4.2.1. qed
         $e_1[x_0 \leftarrow v_0] = e_1$
5. case $e_1 = \langle e_2, e_3 \rangle$
   5.1. qed
      by the induction hypothesis
6. case $e_1 = \mathsf{app}\{\tau?\}\ e_2\ e_3$
   6.1. qed
      by the induction hypothesis
7. case $e_1 = unop\{\tau?\}\ e_2$
   7.1. qed
      by the induction hypothesis

8. CASE $e_1 = \mathit{binop}\{\tau?\}\ e_2\ e_3$
   8.1. QED
   by the induction hypothesis
9. CASE $e_1 = \mathsf{dyn}\ b_2\ e_2$
   9.1. QED
   by the induction hypothesis
10. CASE $e_1 = \mathsf{stat}\ b_2\ e_2$
   10.1. QED
   by the induction hypothesis

□

LEMMA 5.20 ($\Vdash$-PROGRESS). *If $(e_0)^{\ell_0} : \tau?\ \overline{\mathbf{wf}}$ and $\cdot ; \ell_0 \Vdash e_0$ then one of the following holds:*

- $e_0 \in (\!(v)\!)^{\ell}$
- $e_0 \in E[\mathsf{Err}]^{\ell}$
- $\exists\, e_1$ *such that* $e_0 \longrightarrow_{\overline{\mathsf{N}}} e_1$

PROOF. By case analysis of e_0.

By lemma 5.25, it suffices to consider the following cases.

1. CASE $e_0 \in (\!(v)\!)^{\ell}$
   1.1. QED
2. CASE $e_0 \in E[\mathsf{Err}]^{\ell}$
   2.1. QED
3. CASE $e_0 = E[\mathsf{app}\{\tau_1\}\ (\!(v_0)\!)^{\ell_0}\ (\!(v_1)\!)^{\ell_0}]^{\ell_0}$
   3.1. $v_0 \in (\lambda(x:\tau).\, e) \cup (\mathsf{mon}\ b\ v)$
   by $\vdash_{\mathsf{N}}$ canonical forms and $\vdash_{\mathsf{N}}$ inversion
   3.2. SCASE $v_0 = \lambda(x_2 : \tau_2).\, e_2$
      3.2.1. QED
      $e_0 \rhd_{\overline{\mathsf{N}}} E[e_2[x_2 \leftarrow (v_1)^{\ell_0 \ell_0 \ell_0}]]^{\ell_0}$
   3.3. SCASE $v_0 = \mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_2 \Rightarrow \tau_3) \blacktriangleleft \ell_1)\ v_2$
      3.3.1. LET $b_3 = (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)$
      and $b_4 = (\ell_1 \blacktriangleleft \tau_2 \blacktriangleleft \ell_0)$
      3.3.2. QED
      $e_0 \rhd_{\overline{\mathsf{N}}} E[\mathsf{dyn}\ b_3\ (\mathsf{app}\{\mathcal{U}\}\ v_2\ (\mathsf{stat}\ b_4\ (\!(v_1)\!)^{\ell_0 \ell_0 \ell_0}))^{\ell_2}]^{\ell_0}$
4. CASE $e_0 = E[\mathsf{app}\{\mathcal{U}\}\ (\!(v_0)\!)^{\ell_0}\ (\!(v_1)\!)^{\ell_0}]^{\ell_0}$
   4.1. SCASE $v_0 = \lambda x_2.\, e_2$
      4.1.1. QED
      $e_0 \blacktriangleright_{\overline{\mathsf{N}}} E[e_2[x_2 \leftarrow (v_0)^{\ell_0 \ell_0 \ell_0}]]^{\ell_0}$
   4.2. SCASE $v_0 = \mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_2 \Rightarrow \tau_3) \blacktriangleleft \ell_1)\ v_2$
      4.2.1. LET $b_3 = (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)$
      and $b_4 = (\ell_1 \blacktriangleleft \tau_2 \blacktriangleleft \ell_0)$
      and $\tau_4 = \mathit{forget}(\tau_3)$

4.2.2. QED

$e_0 \blacktriangleright_{\overline{\mathsf{N}}} E[\mathsf{stat}\ b_3\ (\mathsf{app}\{\tau_4\}\ v_2\ (\mathsf{dyn}\ b_4\ (\!(v_1)\!)^{\ell_0 \ell_0 \ell_0}))^{\ell_2}]^{\ell_0}$

4.3. SCASE $v_0 \notin (\lambda x.\, e) \cup (\mathsf{mon}\ b\ v)$

4.3.1. QED

$e_0 \blacktriangleright_{\overline{\mathsf{N}}} E[\mathsf{TagErr}\,\bullet]^{\ell_0}$

5. CASE $e_0 = E[\mathit{unop}\{\tau?\}\ (\!(v_0)\!)^{\ell_0}]^{\ell_0}$

5.1. QED

by lemma 5.11 and lemma 5.28

6. CASE $e_0 = E[\mathit{binop}\{\tau?\}\ (\!(v_0)\!)^{\ell_0}\ (\!(v_1)\!)^{\ell_0}]^{\ell_0}$

6.1. QED

by lemma 5.11 and lemma 5.28

7. CASE $e_0 = E[\mathsf{dyn}\ b_1\ (\!(v_1)\!)^{\ell_0}]^{\ell_0}$

7.1. QED

by lemma 5.11 and lemma 5.30

8. CASE $e_0 = E[\mathsf{stat}\ b_1\ (\!(v_1)\!)^{\ell_0}]^{\ell_0}$

8.1. QED

by lemma 5.11 and lemma 5.31

□

Lemma 5.21 ($\Vdash$-preservation). *If* $(e_0)^{\ell_0} : \tau?\ \overline{\mathbf{wf}}$ *and* $e_0 \longrightarrow_{\overline{\mathsf{N}}} e_1$ *then* $\cdot; \ell_0 \Vdash e_1$

Proof. By lemma 5.22 and lemma 5.23.

□

Lemma 5.22. *If* $(e_0)^{\ell_0} : \tau_0\ \overline{\mathbf{wf}}$ *and* $e_0 \rhd_{\overline{\mathsf{N}}} e_1$ *then* $\cdot; \ell_0 \Vdash e_1$

Proof. By case analysis of $\rhd_{\overline{\mathsf{N}}}$.

1. CASE $\delta_N(\mathit{unop}, (\!(v_0)\!)^{\ell_0})$ is defined
and $(\mathit{unop}\{\tau?\}\ (\!(v_0)\!)^{\ell_0})^{\ell_0} \rhd_{\overline{\mathsf{N}}} (\delta_N(\mathit{unop}, (\!(v_0)\!)^{\ell_0}))^{\ell_0}$

1.1. QED

by lemma 5.29

2. CASE $\delta_N(\mathit{binop}, (\!(v_0)\!)^{\ell_0}, (\!(v_1)\!)^{\ell_0})$ is defined
and $(\mathit{binop}\{\tau?\}\ (\!(v_0)\!)^{\ell_0}\ (\!(v_1)\!)^{\ell_0})^{\ell_0} \rhd_{\overline{\mathsf{N}}} (\delta_N(\mathit{binop}, (\!(v_0)\!)^{\ell_0}, (\!(v_1)\!)^{\ell_0}))^{\ell_0}$

2.1. QED

by lemma 5.29

3. CASE $(\mathsf{app}\{\tau_0\}\ (\!(\lambda(x_0 : \tau_1).\, e_0)\!)^{\ell_0}\ (\!(v_1)\!)^{\ell_0})^{\ell_0} \rhd_{\overline{\mathsf{N}}} (e_0[x_0 \leftarrow (\!(v_1)\!)^{\ell_0 \ell_0 \ell_0}])^{\ell_0}$

3.1. QED

by lemma 5.34

4. CASE $(\mathsf{app}\{\tau_0\}\ (\!(\mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2)^{\ell_1} \blacktriangleleft \ell_1)\ (v_0)^{\ell_2})\!)^{\ell_0}\ (\!(v_1)\!)^{\ell_0})^{\ell_0}$
$\rhd_{\overline{\mathsf{N}}} (\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\mathcal{U}\}\ v_0\ (\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ (\!(v_1)\!)^{\ell_0 \ell_0 \ell_0}))^{\ell_2})^{\ell_0}$

4.0.1. $\ell_1 = \ell_2$

by inversion $\Vdash$

4.0.2. QED

$$\dfrac{\dfrac{\text{by inversion } \Vdash}{\cdot; \ell_1 \Vdash v_0} \quad \dfrac{\dfrac{\dfrac{\text{by inversion } \Vdash}{\cdot; \ell_0 \Vdash v_1}}{\cdot; \ell_0 \Vdash (\!(v_1)\!)^{\ell_0 \ell_0 \ell_0}}}{\cdot; \ell_1 \Vdash \mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ (\!(v_1)\!)^{\ell_0 \ell_0 \ell_0}}}{\dfrac{\cdot; \ell_1 \Vdash \mathsf{app}\{\mathcal{U}\}\ v_0\ (\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ (\!(v_1)\!)^{\ell_0 \ell_0 \ell_0})}{\dfrac{\cdot; \ell_1 \Vdash (\mathsf{app}\{\mathcal{U}\}\ v_0\ (\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ (\!(v_1)\!)^{\ell_0 \ell_0 \ell_0}))^{\ell_2}}{\dfrac{\cdot; \ell_0 \Vdash \mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\mathcal{U}\}\ v_0\ (\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ (\!(v_1)\!)^{\ell_0 \ell_0 \ell_0}))^{\ell_2}}{\cdot; \ell_0 \Vdash (\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\mathcal{U}\}\ v_0\ (\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ (\!(v_1)\!)^{\ell_0 \ell_0 \ell_0}))^{\ell_2})^{\ell_0}}}}}$$

5. CASE $(\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (\!(v_0)\!)^{\ell_2})^{\ell_0} \triangleright_{\overline{\mathsf{N}}} e_2$
   by lemma 5.32

□

Lemma 5.23. *If* $(e_0)^{\ell_0} : \mathcal{U}\ \overline{\mathbf{wf}}$ *and* $e_0 \blacktriangleright_{\overline{\mathsf{N}}} e_1$ *then* $\cdot; \ell_0 \Vdash e_1$

Proof. By case analysis of $\blacktriangleright_{\overline{\mathsf{N}}}$.

1. CASE $\delta_N(unop, (\!(v_0)\!)^{\ell_0})$ is not defined
   and $(unop\{\tau?\}\ (\!(v_0)\!)^{\ell_0})^{\ell_0} \blacktriangleright_{\overline{\mathsf{N}}} (\mathsf{TagErr}\ \bullet)^{\ell_0}$
   1.1. QED
2. CASE $\delta_N(unop, (\!(v_0)\!)^{\ell_0})$ is defined
   and $(unop\{\tau?\}\ (\!(v_0)\!)^{\ell_0})^{\ell_0} \blacktriangleright_{\overline{\mathsf{N}}} (\delta_N(unop, (\!(v_0)\!)^{\ell_0}))^{\ell_0}$
   2.1. QED
   by lemma 5.29
3. CASE $\delta_N(binop, (\!(v_0)\!)^{\ell_0}, (\!(v_1)\!)^{\ell_0})$ is not defined
   and $(binop\{\tau?\}\ (\!(v_0)\!)^{\ell_0}\ (\!(v_1)\!)^{\ell_0})^{\ell_0} \blacktriangleright_{\overline{\mathsf{N}}} (\mathsf{TagErr}\ \bullet)^{\ell_0}$
   3.1. QED
4. CASE $\delta_N(binop, (\!(v_0)\!)^{\ell_0}, (\!(v_1)\!)^{\ell_0})$ is defined
   and $(binop\{\tau?\}\ (\!(v_0)\!)^{\ell_0}\ (\!(v_1)\!)^{\ell_0})^{\ell_0} \blacktriangleright_{\overline{\mathsf{N}}} (\delta_N(binop, (\!(v_0)\!)^{\ell_0}, (\!(v_1)\!)^{\ell_0}))^{\ell_0}$
   4.1. QED
   by lemma 5.29
5. CASE $(\mathsf{app}\{\mathcal{U}\}\ (\!(\lambda x_0.\, e_0)\!)^{\ell_0}\ (\!(v_1)\!)^{\ell_0})^{\ell_0} \blacktriangleright_{\overline{\mathsf{N}}} (e_0[x_0 \leftarrow (\!(v_1)\!)^{\ell_0 \ell_0 \ell_0}])^{\ell_0}$
   5.1. QED
   by lemma 5.34
6. CASE $(\mathsf{app}\{\mathcal{U}\}\ (\!(\mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1)^{\ell_1} \blacktriangleleft \ell_1)\ (v_0)^{\ell_2})\!)^{\ell_0}\ (\!(v_1)\!)^{\ell_0})^{\ell_0} \blacktriangleright_{\overline{\mathsf{N}}}$
   $(\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (\mathsf{app}\{forget(\tau_1)\}\ v_0\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0)\ (\!(v_1)\!)^{\ell_0 \ell_0 \ell_0}))^{\ell_2})^{\ell_0}$
   6.1. $\ell_1 = \ell_2$
   by inversion $\Vdash$
   6.2. QED

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{by inversion } \Vdash}{\cdot; \ell_0 \Vdash v_0} \qquad \dfrac{\dfrac{\dfrac{\text{by inversion } \Vdash}{\cdot; \ell_0 \Vdash v_1}}{\cdot; \ell_0 \Vdash (\!(v_1)\!)^{\ell_0 \ell_0 \ell_0}}}{\cdot; \ell_0 \Vdash \mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0)\ (\!(v_1)\!)^{\ell_0 \ell_0 \ell_0}}}{\cdot; \ell_0 \Vdash \mathsf{app}\{\mathit{forget}(\tau_1)\}\ v_0\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0)\ (\!(v_1)\!)^{\ell_0 \ell_0 \ell_0})}}{\cdot; \ell_0 \Vdash (\mathsf{app}\{\mathit{forget}(\tau_1)\}\ v_0\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0)\ (\!(v_1)\!)^{\ell_0 \ell_0 \ell_0}))^{\ell_2}}}{\cdot; \ell_0 \Vdash \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\mathit{forget}(\tau_1)\}\ v_0\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0)\ (\!(v_1)\!)^{\ell_0 \ell_0 \ell_0}))^{\ell_2}}}{\cdot; \ell_0 \Vdash (\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\mathit{forget}(\tau_1)\}\ v_0\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0)\ (\!(v_1)\!)^{\ell_0 \ell_0 \ell_0}))^{\ell_2})^{\ell_0}}$$

7. CASE $(\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (\!(v_0)\!)^{\ell_2})^{\ell_0} \blacktriangleright_{\overline{\mathsf{N}}} e_2$
   7.1. QED
   by lemma 5.33

□

LEMMA 5.24 (UNWIND BOUNDARY ERROR). *If* $\cdot \vdash_{\mathsf{N}} e_0 : \tau?$ *and* $e_0 \rightarrow^*_{\mathsf{N}} \mathsf{BndryErr}\,(b_0, v_1)$ *then* $\exists v_1, E$ *such that* $e_0 \rightarrow^*_{\mathsf{N}} E_0[\mathsf{dyn}\ b_0\ v_1] \rightarrow_{\mathsf{N}} E_0[\mathsf{BndryErr}\,(b_0, v_1)]$

PROOF. By the definitions of $\rhd_{\mathsf{N}}$ and $\blacktriangleright_{\mathsf{N}}$, only dyn expression can step to a boundary error.

□

LEMMA 5.25. *If* $\cdot \vdash_{\mathsf{N}} e_0 : \tau?$ *and* $\cdot; \ell_0 \Vdash e_0$ *then either:*

- $e_0 \in (\!(v)\!)^{\ell}$
- $e_0 = E_0[\mathsf{app}\{\tau?\}\ (\!(v_0)\!)^{\ell_0}\ (\!(v_1)\!)^{\ell_0}]^{\ell_0}$
- $e_0 = E_0[\mathit{unop}\{\tau?\}\ (\!(v_0)\!)^{\ell_0}]^{\ell_0}$
- $e_0 = E_0[\mathit{binop}\{\tau?\}\ (\!(v_0)\!)^{\ell_0}\ (\!(v_1)\!)^{\ell_0}]^{\ell_0}$
- $e_0 = E_0[\mathsf{dyn}\ b_1\ (\!(v_1)\!)^{\ell_0}]^{\ell_0}$
- $e_0 = E_0[\mathsf{stat}\ b_1\ (\!(v_1)\!)^{\ell_0}]^{\ell_0}$
- $e_0 = E_0[\mathsf{Err}]^{\ell_0}$

PROOF. By induction on the structure of e_0.

1. CASE $e_0 = (\!(x_0)\!)^{\ell_0}$
   1.1. CONTRADICTION:
   $\cdot; \ell_0 \Vdash e_0$
2. CASE $e_0 \in (\!(v)\!)^{\ell_0}$
   2.1. QED
3. CASE $e_0 = (\!(\langle e_1, e_2 \rangle)\!)^{\ell_0}$
   3.1. $\cdot \vdash_{\mathsf{N}} e_1 : \tau?$ and $\cdot \vdash_{\mathsf{N}} e_2 : \tau?$
   by inversion $\vdash_{\mathsf{N}}$
   3.2. $\cdot; \ell_0 \Vdash e_1$ and $\cdot; \ell_0 \Vdash e_2$
   by inversion $\Vdash$

3.3. SCASE $e_1 \notin (\!(v)\!)^{\overline{\ell}}$

3.3.1. QED

by the induction hypothesis

3.4. SCASE $e_1 \in (\!(v)\!)^{\overline{\ell}}$ and $e_2 \notin (\!(v)\!)^{\overline{\ell}}$

3.4.1. QED

by the induction hypothesis

3.5. SCASE $e_1 \in (\!(v)\!)^{\overline{\ell}}$ and $e_2 \in (\!(v)\!)^{\overline{\ell}}$

3.5.1. QED

$e_0 \in (\!(v)\!)^{\overline{\ell}}$

4. CASE $e_0 = \mathsf{app}\{\tau?\}\, e_1\, e_2$

4.1. QED

by the induction hypothesis

5. CASE $e_0 = \mathit{unop}\{\tau?\}\, e_1$

5.1. QED

by the induction hypothesis

6. CASE $e_0 = \mathit{binop}\{\tau?\}\, e_1\, e_2$

6.1. QED

by the induction hypothesis

7. CASE $e_0 = \mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, (e_1)^{\ell_1}$

7.1. $\cdot; \ell_1 \Vdash e_1$

by inversion $\Vdash$

7.2. QED

by the induction hypothesis

8. CASE $e_0 = \mathsf{stat}\, b_1\, e_1$

8.1. QED

by the induction hypothesis

□

Lemma 5.26. *If* $\cdot; \ell_0 \Vdash E_0[e_0]$ *then* $\exists\, \ell_1$ *such that* $\cdot; \ell_1 \Vdash e_0$

Proof. By induction on the structure of E_0.

1. $E_0 = [\,]$

1.1. QED

2. $E_0 = \langle E_1, e_2 \rangle$

2.1. QED

by the induction hypothesis

3. $E_0 = \langle e_1, E_2 \rangle$

3.1. QED

by the induction hypothesis

4. $E_0 = \mathit{unop}\{\tau?\}\, E_1$

4.1. QED

by the induction hypothesis

5. $E_0 = \mathit{binop}\{\tau?\}\, E_1\, e_2$
   5.1. QED
   by the induction hypothesis
6. $E_0 = \mathit{binop}\{\tau?\}\, e_1\, E_2$
   6.1. QED
   by the induction hypothesis
7. $E_0 = \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\,(E_1)^{\ell_3}$
   7.1. $\cdot; \ell_2 \Vdash E_1[e_0]$
   7.2. QED
   by the induction hypothesis
8. $E_0 = \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\,(E_1)^{\ell_3}$
   8.1. QED
   by the induction hypothesis
9. $E_0 = (E_1)^{\ell_0}$
   9.1. QED
   by the induction hypothesis

□

Lemma 5.27 ($\Vdash$ replacement). *If $\cdot; \ell_0 \Vdash E_0[e_0]$ and the derivation contains a proof of $\cdot; \ell_1 \Vdash e_0$ and $\cdot; \ell_1 \Vdash e_1$ then $L_0; \ell_0 \Vdash E_0[e_1]$*

Proof. By induction on the structure of E_0.

1. $E_0 = [\,]$
   1.1. QED
2. $E_0 = \langle E_1, e_2 \rangle$
   2.1. QED
   by the induction hypothesis
3. $E_0 = \langle e_1, E_2 \rangle$
   3.1. QED
   by the induction hypothesis
4. $E_0 = \mathit{unop}\{\tau?\}\, E_1$
   4.1. QED
   by the induction hypothesis
5. $E_0 = \mathit{binop}\{\tau?\}\, E_1\, e_2$
   5.1. QED
   by the induction hypothesis
6. $E_0 = \mathit{binop}\{\tau?\}\, e_1\, E_2$
   6.1. QED
   by the induction hypothesis
7. $E_0 = \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\,(E_1)^{\ell_3}$
   7.1. $\cdot; \ell_2 \Vdash E_1[e_0]$
   7.2. QED

by the induction hypothesis

8. $E_0 = \text{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\,(E_1)^{\ell_3}$
   8.1. QED
   by the induction hypothesis

9. $E_0 = (E_1)^{\ell_0}$
   9.1. QED
   by the induction hypothesis

□

Lemma 5.28 (δ_N label progress).

- *If* $\cdot \vdash_{\mathsf{N}} unop\{\tau_1\}\, v_0 : \tau_0$ *and* $\cdot; \ell_0 \Vdash unop\{\tau_1\}\, v_0$ *then* $\delta_N(unop, v_0)$ *is defined and* $unop\{\tau_1\}\, v_0 \rhd_{\overline{\mathsf{N}}} \delta_N(unop, v_0)$.
- *if* $\cdot \vdash_{\mathsf{N}} binop\{\tau_1\}\, v_0\, v_1 : \tau_0$ *and* $\cdot; \ell_0 \Vdash binop\{\tau_1\}\, v_0\, v_1$ *then* $\delta_N(binop, v_0, v_1)$ *is defined and* $binop\{\tau_1\}\, v_0\, v_1 \rhd_{\overline{\mathsf{N}}} \delta_N(binop, v_0, v_1)$.
- *If* $\cdot \vdash_{\mathsf{N}} unop\{\mathcal{U}\}\, v_0 : \mathcal{U}$ *and* $\cdot; \ell_0 \Vdash unop\{\mathcal{U}\}\, v_0$ *then* $unop\{\mathcal{U}\}\, v_0 \blacktriangleright_{\overline{\mathsf{N}}} e_1$.
- *if* $\cdot \vdash_{\mathsf{N}} binop\{\mathcal{U}\}\, v_0\, v_1 : \mathcal{U}$ *and* $\cdot; \ell_0 \Vdash binop\{\mathcal{U}\}\, v_0\, v_1$ *then* $binop\{\mathcal{U}\}\, v_0\, v_1 \blacktriangleright_{\overline{\mathsf{N}}} e_1$.

Proof. By case analysis of δ_N, $\vdash_{\mathsf{N}}$, $\Vdash$, and $\blacktriangleright_{\overline{\mathsf{N}}}$.

1. case $\cdot \vdash_{\mathsf{N}} unop\{\tau_1\}\, v_0 : \tau_0$
   1.1. $v_0 = (\!(v_1)\!)^{\ell_0}$
   by inversion $\Vdash$
   1.2. $v_1 = \langle v_2, v_3 \rangle$
   by inversion $\vdash_{\mathsf{N}}$
   1.3. QED
   by definition δ_N
2. case $\cdot \vdash_{\mathsf{N}} binop\{\tau_1\}\, v_0\, v_1 : \tau_0$
   2.1. $v_0 = (\!(v_2)\!)^{\ell_0}$ and $v_1 = (\!(v_3)\!)^{\ell_0}$
   by inversion $\Vdash$
   2.2. $v_0 \in i$ and $v_1 \in i$
   by inversion $\vdash_{\mathsf{N}}$
   2.3. QED
   by definition δ_N
3. case $\cdot \vdash_{\mathsf{N}} unop\{\mathcal{U}\}\, v_0 : \mathcal{U}$
   3.1. QED
   by inversion $\Vdash$
4. case $\cdot \vdash_{\mathsf{N}} binop\{\mathcal{U}\}\, v_0\, v_1 : \mathcal{U}$
   4.1. QED
   by inversion $\Vdash$

□

Lemma 5.29 (δ_N label preservation).

- *If* $\cdot; \ell_0 \Vdash unop\{\tau?\}\, v_0$ *and* $(unop\{\tau?\}\, v_0)^{\ell_0}\, (\rhd_{\overline{\mathsf{N}}} \cup \blacktriangleright_{\overline{\mathsf{N}}})\, (e_1)^{\ell_0}$ *then* $\cdot; \ell_0 \Vdash e_1$.

- *If* $\cdot; \ell_0 \Vdash \mathit{binop}\{\tau?\}\, v_0\, v_1$ *and* $(\mathit{binop}\{\tau?\}\, v_0\, v_1)^{\ell_0}\ (\rhd_{\overline{\mathsf{N}}} \cup \blacktriangleright_{\overline{\mathsf{N}}})\ (e_1)^{\ell_0}$ *then* $\cdot; \ell_0 \Vdash e_1$.

Proof. By case analysis of $(\rhd_{\overline{\mathsf{N}}} \cup \blacktriangleright_{\overline{\mathsf{N}}})$.

1. $(\mathsf{fst}\{\tau?\}\, (\!|\langle v_1, v_2\rangle|\!)^{\ell_0})^{\ell_0}\ (\rhd_{\overline{\mathsf{N}}} \cup \blacktriangleright_{\overline{\mathsf{N}}})\ (\!|v_1|\!)^{\ell_0}$
   1.1. $\cdot; \ell_0 \Vdash v_1$
   by inversion $\Vdash$
   1.2. QED
2. $(\mathsf{snd}\{\tau?\}\, (\!|\langle v_1, v_2\rangle|\!)^{\ell_0})^{\ell_0}\ (\rhd_{\overline{\mathsf{N}}} \cup \blacktriangleright_{\overline{\mathsf{N}}})\ (\!|v_2|\!)^{\ell_0}$
   2.1. $\cdot; \ell_0 \Vdash v_2$
   by inversion $\Vdash$
   2.2. QED
3. $(\mathsf{sum}\{\tau?\}\, (\!|i_1|\!)^{\ell_1}\, (\!|i_2|\!)^{\ell_2})^{\ell_0}\ (\rhd_{\overline{\mathsf{N}}} \cup \blacktriangleright_{\overline{\mathsf{N}}})\ (i_1 + i_2)^{\ell_0}$
   3.1. QED
4. $(\mathsf{quotient}\{\tau?\}\, (\!|i_1|\!)^{\ell_1}\, (\!|i_2|\!)^{\ell_2})^{\ell_0}\ (\rhd_{\overline{\mathsf{N}}} \cup \blacktriangleright_{\overline{\mathsf{N}}})\ (\mathsf{DivErr})^{\ell_0}$
   4.1. QED
5. $(\mathsf{quotient}\{\tau?\}\, (\!|i_1|\!)^{\ell_1}\, (\!|i_2|\!)^{\ell_2})^{\ell_0}\ (\rhd_{\overline{\mathsf{N}}} \cup \blacktriangleright_{\overline{\mathsf{N}}})\ (\lfloor i_1/i_2 \rfloor)^{\ell_0}$
   5.1. QED

□

Lemma 5.30. *If* $\cdot \vdash_{\mathsf{N}} \mathsf{dyn}\ b_0\ v_0 : \tau_0$ *and* $\cdot; \ell_0 \Vdash \mathsf{dyn}\ b_0\ v_0$ *then* $\exists\, e_1$ *such that* $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_{\overline{\mathsf{N}}} (e_1)^{\ell_0}$.

Proof. By inversion of $\Vdash$ and case analysis of $\mathit{tag\text{-}match}(\lfloor\tau_0\rfloor, v_0)$.

1. $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
   and $\cdot; \ell_1 \Vdash v_0$
   by inversion $\Vdash$
2. $v_0 = (\!|v_1|\!)^{\ell_1}$
   by inversion $\Vdash$
3. CASE $v_1 = \lambda x_1.\, e_1$ and $\mathit{tag\text{-}match}(\lfloor\tau_0\rfloor, v_1)$
   3.1. QED
   $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_{\overline{\mathsf{N}}} (\mathsf{mon}\ b_0\ v_0)^{\ell_0}$
4. CASE $v_1 = \lambda(x_1\!:\!\tau_1).\, e_1$ and $\mathit{tag\text{-}match}(\lfloor\tau_0\rfloor, v_1)$
   4.1. Contradiction:
   $\cdot \vdash_{\mathsf{N}} \mathsf{dyn}\ b_0\ v_0 : \tau_0$
5. CASE $v_1 = \mathsf{mon}\ b_1\ v_2$ and $\mathit{tag\text{-}match}(\lfloor\tau_0\rfloor, v_1)$
   5.1. QED
   $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_{\overline{\mathsf{N}}} (\mathsf{mon}\ b_0\ v_0)^{\ell_0}$
6. CASE $v_1 = \langle v_2, v_3\rangle$ and $\mathit{tag\text{-}match}(\lfloor(\tau_1\times\tau_2)^{\ell_1}\rfloor, v_1)$
   6.1. QED
   $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_{\overline{\mathsf{N}}} (\langle \mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ v_2, \mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_3\rangle)^{\ell_0}$
7. CASE $v_1 \in i$ and $\mathit{tag\text{-}match}(\lfloor\mathsf{Int}\rfloor, v_1)$
   7.1. QED
   $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_{\overline{\mathsf{N}}} (v_1)^{\ell_0}$

8. CASE $v_1 \in n$ and *tag-match*$(\lfloor \mathsf{Nat} \rfloor, v_1)$
   8.1. QED
   $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_{\overline{\mathsf{N}}} (v_1)^{\ell_0}$
9. CASE $\neg$*tag-match*$(\lfloor \tau_0 \rfloor, v_1)$
   9.1. QED
   $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_{\overline{\mathsf{N}}} (\mathsf{BndryErr}\,(b_0, v_0))^{\ell_0}$

□

LEMMA 5.31. *If* $\cdot \vdash_{\mathsf{N}} \mathsf{stat}\ b_0\ v_0 : \mathcal{U}$ *and* $\cdot; \ell_0 \Vdash \mathsf{stat}\ b_0\ v_0$ *then* $\exists\, e_1$ *such that* $(\mathsf{stat}\ b_0\ v_0)^{\ell_0} \blacktriangleright_{\overline{\mathsf{N}}} (e_1)^{\ell_0}$.

PROOF. By case analysis on v_0.

1. $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
   and $\cdot; \ell_1 \Vdash v_0$
   by inversion $\Vdash$
2. $v_0 = (\!(v_1)\!)^{\ell_1}$
   by inversion $\Vdash$
3. CASE $v_1 \in \lambda x.\, e$
   3.1. CONTRADICTION:
   $\cdot \vdash_{\mathsf{N}} \mathsf{stat}\ b_0\ v_0 : \mathcal{U}$
4. CASE $v_1 \in \lambda(x\!:\!\tau).\, e$
   4.1. QED
   $(\mathsf{stat}\ b_0\ v_0)^{\ell_0} \blacktriangleright_{\overline{\mathsf{N}}} (\mathsf{mon}\ b_0\ v_0)^{\ell_0}$
5. CASE $v_1 \in \mathsf{mon}\ b\ e$
   5.1. QED
   $(\mathsf{stat}\ b_0\ v_0)^{\ell_0} \blacktriangleright_{\overline{\mathsf{N}}} (\mathsf{mon}\ b_0\ v_0)^{\ell_0}$
6. CASE $v_1 = \langle v_2, v_3 \rangle$
   6.1. $\tau_0 = (\tau_1 \times \tau_2)^{\ell_1}$
   by inversion $\vdash_{\mathsf{N}}$
   6.2. QED
   $(\mathsf{stat}\ b_0\ v_0)^{\ell_0} \blacktriangleright_{\overline{\mathsf{N}}} (\langle \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ v_2, \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_3 \rangle)^{\ell_0}$
7. CASE $v_1 \in i$
   7.1. QED
   $(\mathsf{stat}\ b_0\ v_0)^{\ell_0} \blacktriangleright_{\overline{\mathsf{N}}} (v_1)^{\ell_0}$

□

LEMMA 5.32. *If* $\cdot \vdash_{\mathsf{N}} \mathsf{dyn}\ b_0\ v_0 : \tau_0$ *and* $\cdot; \ell_0 \Vdash \mathsf{dyn}\ b_0\ v_0$ *and* $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_{\overline{\mathsf{N}}} (e_1)^{\ell_0}$ *then* $\cdot; \ell_0 \Vdash e_1$.

PROOF. By case analysis of $\rhd_{\overline{\mathsf{N}}}$.

1. $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
   and $\cdot; \ell_1 \Vdash v_0$
   by inversion $\Vdash$
2. $v_0 = (\!(v_1)\!)^{\ell_1}$

by inversion $\Vdash$

3. CASE $v_1 \in (\lambda x.\, e) \cup (\mathsf{mon}\ b\ v)$
   and $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_{\overline{\mathsf{N}}} (\mathsf{mon}\ b_0\ v_0)^{\ell_0}$
   3.1. QED

$$\dfrac{\dfrac{\text{by inversion } \Vdash}{\cdot; \ell_1 \Vdash v_0}}{\cdot; \ell_0 \Vdash \mathsf{mon}\ b_0\ v_0}$$

4. CASE $v_1 = \langle v_2, v_3 \rangle$
   and $b_0 = (\ell_0 \blacktriangleleft (\tau_2 \times \tau_3)^{\ell_1} \blacktriangleleft \ell_1)$
   and $(\mathsf{dyn}\ b_0\ (\!(\langle v_0, v_1 \rangle)\!)^{\ell_2})^{\ell_0} \rhd_{\overline{\mathsf{N}}} (\langle \mathsf{dyn}\ b_3\ (\!(v_2)\!)^{\ell_1}, \mathsf{dyn}\ b_4\ (\!(v_3)\!)^{\ell_1} \rangle)^{\ell_0}$
   4.1. $b_3 = (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)$
   and $b_4 = (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)$
   by definition $\rhd_{\overline{\mathsf{N}}}$
   4.2. QED

$$\dfrac{\dfrac{\dfrac{\text{by inversion } \Vdash}{\cdot; \ell_1 \Vdash (\!(v_2)\!)^{\ell_1}}}{\cdot; \ell_0 \Vdash \mathsf{dyn}\ b_3\ (\!(v_2)\!)^{\ell_1}} \qquad \dfrac{\dfrac{\text{by inversion } \Vdash}{\cdot; \ell_1 \Vdash (\!(v_3)\!)^{\ell_1}}}{\cdot; \ell_0 \Vdash \mathsf{dyn}\ b_4\ (\!(v_3)\!)^{\ell_1}}}{\cdot; \ell_0 \Vdash \langle \mathsf{dyn}\ b_3\ (\!(v_2)\!)^{\ell_1}, \mathsf{dyn}\ b_4\ (\!(v_3)\!)^{\ell_1} \rangle}$$

5. CASE $v_1 \in i$
   and $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_{\overline{\mathsf{N}}} (v_1)^{\ell_0}$
   5.1. QED
6. CASE $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_{\overline{\mathsf{N}}} (\mathsf{BndryErr}\,(b_0, v_0))^{\ell_0}$
   6.1. QED

□

LEMMA 5.33 (N-stat PRESERVATION). *If* $\cdot \vdash_{\mathsf{N}} \mathsf{stat}\ b_0\ v_0 : \mathcal{U}$ *and* $\cdot; \ell_0 \Vdash \mathsf{stat}\ b_0\ v_0$ *and* $(\mathsf{stat}\ b_0\ v_0)^{\ell_0} \blacktriangleright_{\overline{\mathsf{N}}} (e_1)^{\ell_0}$ *then* $\cdot; \ell_0 \Vdash e_1$.

PROOF. By case analysis of $\blacktriangleright_{\overline{\mathsf{N}}}$.

1. $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
   and $\cdot; \ell_1 \Vdash v_0$
   and $v_0 = (\!(v_1)\!)^{\ell_1}$
   by inversion $\Vdash$
2. CASE $v_1 \in (\lambda(x\!:\!\tau).\, e) \cup (\mathsf{mon}\ b\ v)$
   and $(\mathsf{stat}\ b_0\ v_0)^{\ell_0} \blacktriangleright_{\overline{\mathsf{N}}} (\mathsf{mon}\ b_0\ v_0)^{\ell_0}$
   2.1. QED

$$\dfrac{\dfrac{\text{by inversion } \Vdash}{\cdot;\ell_1 \Vdash v_0}}{\cdot;\ell_0 \Vdash \mathsf{mon}\ b_0\ v_0}$$

3. CASE $v_1 = \langle v_2, v_3 \rangle$
   and $\tau_0 = (\tau_2 \times \tau_3)^{\ell_1}$
   and $(\mathsf{stat}\ b_0\ v_0)^{\ell_0} \blacktriangleright_{\overline{\mathsf{N}}} (\langle \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2, \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3 \rangle)^{\ell_0}$
   3.1. QED

$$\dfrac{\dfrac{\dfrac{\text{by inversion } \Vdash}{\cdot;\ell_1 \Vdash v_2}}{\cdot;\ell_0 \Vdash \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2} \qquad \dfrac{\dfrac{\text{by inversion } \Vdash}{\cdot;\ell_1 \Vdash v_3}}{\cdot;\ell_0 \Vdash \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3}}{\cdot;\ell_0 \Vdash \langle \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2, \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3 \rangle}$$

4. CASE $v_1 \in i$
   and $(\mathsf{stat}\ b_0\ v_0)^{\ell_0} \blacktriangleright_{\overline{\mathsf{N}}} (v_1)^{\ell_0}$
   qedstep

□

LEMMA 5.34. *If* $(x_0 : \tau?_0), \Gamma_0 \vdash_{\mathsf{N}} e_1 : \tau?$ *and* $(x_0 : \ell_0), L_0; \ell_1 \Vdash e_1$ *and* $\cdot \vdash_{\mathsf{N}} v_0 : \tau?_0$ *and* $\cdot; \ell_0 \Vdash v_0$ *then* $\Gamma_0 \vdash_{\mathsf{N}} e_1[x_0 \leftarrow v_0] : \tau?$ *and* $L_0 \Vdash e_1[x_0 \leftarrow v_0]$.

PROOF. By induction on the structure of e_0.

1. $e_0 = x_2$
   1.1. SCASE $x_0 = x_2$
      1.1.1. QED
   1.2. SCASE $x_0 \neq x_2$
      1.2.1. QED
         $e_1[x_0 \leftarrow v_0] = e_1$
2. CASE $e_0 \in i$
   2.1. QED
      $e_1[x_0 \leftarrow v_0] = e_1$
3. CASE $e_0 = \lambda x_2.\, e_2$
   or$e_0 = \lambda(x_2 : \tau_2).\, e_2$
   3.1. SCASE $x_0 = x_2$
      3.1.1. QED
         $e_1[x_0 \leftarrow v_0] = e_1$
   3.2. SCASE $x_0 \neq x_2$
      3.2.1. QED
         by the induction hypothesis
4. CASE $e_0 = \langle e_1, e_2 \rangle$
   4.1. QED

by the induction hypothesis

5. CASE $e_0 = \mathsf{app}\{\tau?\}\, e_1\, e_2$

5.1. QED

by the induction hypothesis

6. CASE $e_0 = \mathit{unop}\{\tau?\}\, e_1$

6.1. QED

by the induction hypothesis

7. CASE $e_0 = \mathit{binop}\{\tau?\}\, e_1\, e_2$

7.1. QED

by the induction hypothesis

8. CASE $e_0 = \mathsf{dyn}\, b_1\, e_1$

8.1. QED

by the induction hypothesis

9. CASE $e_0 = \mathsf{stat}\, b_1\, e_1$

9.1. QED

by the induction hypothesis

10. CASE $e_0 = (e_1)^{\ell_1}$

10.1. $\ell_0 = \ell_1$

by inversion $\Vdash$

10.2. QED

by the induction hypothesis

□

LEMMA 5.35 (BOUNDARY PRESERVATION). *If* $e_0 : \tau?$ **wf** *and* $e_0 \rightarrow^*_{\mathsf{N}} E_0[\mathsf{dyn}\, b_0\, v_1]$ *then either has-boundary*(e_0, b_0) *or has-boundary*$(e_0, \mathit{flip}(b_0))$.

PROOF. By case analysis of $\rhd_{\mathsf{N}}$ and $\blacktriangleright_{\mathsf{N}}$, evaluation does not create new labels and only creates a new boundary by flipping an existing boundary.

□

LEMMA 5.36. *If* $e_0 : \tau?$ **wf** *and* $e_0 \rightarrow^*_{\mathsf{N}} E[\mathsf{mon}\, (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1)^{\ell_1} \blacktriangleleft \ell_2)\, v_0]$ *then tag-match*$(\lfloor \tau_0 \Rightarrow \tau_1 \rfloor, v_0)$

PROOF. Surface expressions do not contain monitors, and the only ways to create one via $\blacktriangleright_{\mathsf{N}}$ and $\rhd_{\mathsf{N}}$ require *tag-match* as a precondition.

□

6 Transient THEOREMS, LEMMAS, AND PROOFS

6.1 Transient Theorems

Theorem 6.1 (incomplete monitoring). *There exists* $e_0 : \tau ?\ \overline{\mathbf{wf}}$ *such that* $e_0; \emptyset; \emptyset \longrightarrow^*_{\mathsf{T}} e_1; \mathcal{H}_1; \mathcal{B}_1$ *and for all* O_1 *such that* $O_1 \Vdash_{\mathsf{T}} \mathcal{H}_1$*, we have* $O_1; \cdot; \ell \nVdash_{\mathsf{T}} e_1$.

Proof. Let

$$\begin{array}{lcl} e_f & = & \mathsf{stat}\ (\ell_0 \blacktriangleleft (\mathsf{Int} \Rightarrow \mathsf{Int}) \blacktriangleleft \ell_1)\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft (\mathsf{Int} \Rightarrow \mathsf{Int}) \blacktriangleleft \ell_2)\ (\lambda x_0.\ (\mathsf{sum}\{x_0\}\ 1\ ))^{\ell_2})^{\ell_1} \\ e_0 & = & (\mathsf{app}\{\mathcal{U}\}\ e_0\ (\lambda x_0.\ (\mathsf{sum}\{x_0\}\ 1\ )))^{\ell_0} \\ v_f & = & (\!(\mathsf{p}_0)\!)^{\ell_2 \ell_1 \ell_0} \\ e_1 & = & (\mathsf{app}\{\mathcal{U}\}\ v_f\ (\lambda x_1.\ 0))^{\ell_0} \end{array}$$

By the reduction rules:

$e_0; \emptyset; \emptyset \longrightarrow^*_{\mathsf{T}} e_1; \mathcal{H}_0; \mathcal{B}_0$

where $\mathcal{H}_0 = \{\mathsf{p}_0 \mapsto \lambda x_0.\ (\mathsf{sum}\{x_0\}\ 1\ )\}$

and $\mathcal{B}_0 = \{\mathsf{p}_0 \mapsto \{(\ell_0 \blacktriangleleft (\mathsf{Int} \Rightarrow \mathsf{Int}) \blacktriangleleft \ell_1), (\ell_1 \blacktriangleleft (\mathsf{Int} \Rightarrow \mathsf{Int}) \blacktriangleleft \ell_2)\}\}$

In the result, $\Vdash_{\mathsf{T}}$ fails because v_f has multiple owners.

□

Theorem 6.2 (unsound blame). *There exists* $e_0 : \tau ?\ \overline{\mathbf{wf}}$ *such that* $e_0; \emptyset; \emptyset \longrightarrow^*_{\mathsf{T}} \mathsf{BndryErr}\ ((\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1), v_1); \mathcal{H}_1; \mathcal{B}_1$ *and* $\mathit{senders}(\overline{b}_1) \nsubseteq \mathit{owners}(v_1)$.

Proof. Consider the following example e_0 where the let-expressions are sugar for untyped function applications:

$$\begin{array}{lcl} e_0 & = & (\mathsf{let}\ f_0 = (\lambda x_0.\ \langle x_0, x_0 \rangle)\ \mathsf{in} \\ & & \quad \mathsf{let}\ f_1 = (\mathsf{stat}\ (\ell_0 \blacktriangleleft (\mathsf{Int} \Rightarrow \mathsf{Int}) \blacktriangleleft \ell_1)\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft (\mathsf{Int} \Rightarrow \mathsf{Int}) \blacktriangleleft \ell_0)\ (f_0)^{\ell_0})^{\ell_1})\ \mathsf{in} \\ & & \quad \mathsf{stat}\ (\ell_0 \blacktriangleleft \mathsf{Int} \blacktriangleleft \ell_2)\ (\mathsf{app}\{\mathsf{Int}\}\ (\mathsf{dyn}\ (\ell_2 \blacktriangleleft (\mathsf{Int} \Rightarrow \mathsf{Int}) \blacktriangleleft \ell_0)\ (f_0)^{\ell_0})\ 5)^{\ell_2})^{\ell_0} \end{array}$$

By the reduction rules:

$e_0; \emptyset; \emptyset$

$\longrightarrow^*_{\mathsf{T}} (\mathsf{stat}\ (\ell_0 \blacktriangleleft \mathsf{Int} \blacktriangleleft \ell_2)\ (\mathsf{app}\{\mathsf{Int}\}\ (\!(\mathsf{p}_0)\!)^{\ell_0 \ell_2}\ 5)^{\ell_2})^{\ell_0}; \mathcal{H}_0; \mathcal{B}_0$

$\longrightarrow^*_{\mathsf{T}} (\mathsf{stat}\ (\ell_0 \blacktriangleleft \mathsf{Int} \blacktriangleleft \ell_2)\ (\mathsf{check}\ \mathsf{Int}\ (\!(\mathsf{p}_1)\!)^{\ell_0 \ell_2}\ \mathsf{p}_0)^{\ell_2})^{\ell_0}; \mathcal{H}_1; \mathcal{B}_1$

$\longrightarrow^*_{\mathsf{T}} \mathsf{BndryErr}\ (\mathcal{B}_0(\mathsf{p}_1) \cup \mathcal{B}_0(\mathsf{p}_0), (\!(\mathsf{p}_1)\!)^{\ell_0 \ell_2}); \mathcal{H}_1; \mathcal{B}_0$

where $\mathcal{H}_1 = \{\mathsf{p}_0 \mapsto \lambda x_0.\ \langle x_0, x_0 \rangle, \mathsf{p}_1 \mapsto \langle 5, 5 \rangle\}$

and $\mathcal{B}_1 = \{\mathsf{p}_0 \mapsto (\ell_1 \blacktriangleleft (\mathsf{Int} \Rightarrow \mathsf{Int}) \blacktriangleleft \ell_0), (\ell_0 \blacktriangleleft (\mathsf{Int} \Rightarrow \mathsf{Int}) \blacktriangleleft \ell_1), (\ell_2 \blacktriangleleft (\mathsf{Int} \Rightarrow \mathsf{Int}) \blacktriangleleft \ell_0), \mathsf{p}_1 \mapsto \emptyset\}$

Thus $\mathit{senders}(\mathcal{B}_0(\mathsf{p}_1) \cup \mathcal{B}_0(\mathsf{p}_0)) = \{\ell_0, \ell_1\} \nsubseteq \{\ell_0, \ell_2\} = \mathit{owners}((\!(\mathsf{p}_1)\!)^{\ell_0 \ell_2})$.

□

Theorem 6.3 (incomplete blame). *There exists* $e_0 : \tau ?\ \overline{\mathbf{wf}}$ *such that* $e_0; \emptyset; \emptyset \longrightarrow^*_{\mathsf{T}} \mathsf{BndryErr}\ ((\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1), v_1); \mathcal{H}_1; \mathcal{B}_1$ *and* $\mathit{senders}(\overline{b}_1) \nsupseteq \mathit{owners}(v_1)$.

Proof. Consider the following example e_0 where the let-expressions are sugar for untyped function applications:

$$
\begin{array}{l}
(\mathsf{let}\ f_0 = \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\ (\lambda x_0.\, x_0))\ \mathsf{in} \\
\ \mathsf{let}\ f_1 = \mathsf{stat}\ (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_3)\ (\mathsf{dyn}\ (\ell_3 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_4)\ (\lambda x_1.\, x_1))\ \mathsf{in} \\
\ \mathsf{stat}\ (\ell_0 \blacktriangleleft (\mathsf{Int}{\times}\mathsf{Int}) \blacktriangleleft \ell_5) \\
\qquad (\mathsf{app}\{\mathsf{Int}{\times}\mathsf{Int}\}\ (\mathsf{dyn}\ (\ell_5 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ (\mathsf{app}\{\mathcal{U}\}\ f_1\ f_0)^{\ell_0})\ 42)^{\ell_5})^{\ell_0}; \emptyset; \emptyset \\
\longrightarrow^*_{\overline{\mathsf{T}}}\ (\mathsf{stat}\ (\ell_0 \blacktriangleleft (\mathsf{Int}{\times}\mathsf{Int}) \blacktriangleleft \ell_5) \\
\qquad\qquad (\mathsf{app}\{\mathsf{Int}{\times}\mathsf{Int}\}\ (\mathsf{dyn}\ (\ell_5 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ (\mathsf{app}\{\mathcal{U}\}\ (\!(\mathsf{p}_1)\!)^{\ell_4\ell_3\ell_0}\ (\!(\mathsf{p}_0)\!)^{\ell_2\ell_1\ell_0})^{\ell_0})\ 42)^{\ell_5})^{\ell_0}; \mathcal{H}_0; \mathcal{B}_0 \\
\longrightarrow^*_{\overline{\mathsf{T}}}\ (\mathsf{stat}\ (\ell_0 \blacktriangleleft (\mathsf{Int}{\times}\mathsf{Int}) \blacktriangleleft \ell_5)\ (\mathsf{app}\{\mathsf{Int}{\times}\mathsf{Int}\}\ (\!(\mathsf{p}_0)\!)^{\overline{\ell}_0}\ 42)^{\ell_5})^{\ell_0}; \mathcal{H}_1; \mathcal{B}_1 \\
\longrightarrow^*_{\overline{\mathsf{T}}}\ (\mathsf{stat}\ (\ell_0 \blacktriangleleft (\mathsf{Int}{\times}\mathsf{Int}) \blacktriangleleft \ell_5)\ (\mathsf{check}\ (\mathsf{Int}{\times}\mathsf{Int})\ (\!(42)\!)^{\overline{\ell}_1}\ \mathsf{p}_0)^{\ell_5})^{\ell_0}; \mathcal{H}_2; \mathcal{B}_2 \\
\longrightarrow^*_{\overline{\mathsf{T}}}\ \mathsf{BndryErr}\ (\mathcal{B}_2(\mathsf{p}_0), (\!(42)\!)^{\overline{\ell}_1}); \mathcal{H}_2; \mathcal{B}_2 \\
\quad \text{where } \tau_0 = (\mathsf{Int} \Rightarrow \mathsf{Int}) \\
\quad \text{and } \tau_1 = (\mathsf{Int} \Rightarrow \mathsf{Int}{\times}\mathsf{Int}) \\
\quad \text{and } \overline{\ell}_0 = \ell_2\ell_1\ell_0\ell_3\ell_4\ell_3\ell_0\ell_5 \\
\quad \text{and } \overline{\ell}_1 = \ell_5\overline{\ell}_0(\mathit{rev}(\overline{\ell}_0)) \\
\quad \text{and } \mathcal{H}_2 = \{(\mathsf{p}_0 \mapsto \lambda x_0.\, x_0), (\mathsf{p}_1 \mapsto \lambda x_1.\, x_1)\} \\
\quad \text{and } \mathcal{B}_2 = \{(\mathsf{p}_0 \mapsto \{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1), (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2), (\ell_5 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\}), \\
\qquad\qquad\qquad (\mathsf{p}_1 \mapsto \{(\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_3), (\ell_3 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_4)\})\}
\end{array}
$$

Hence, $\mathit{senders}(\mathcal{B}_2(\mathsf{p}_0)) = \{\ell_0, \ell_1, \ell_2\} \not\supseteq \{\ell_0, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5\} = \mathit{owners}((\!(42)\!)^{\overline{\ell}_1})$. The crucial labels ℓ_3 and ℓ_4 are nowhere to be found.

□

Theorem 6.4 (all-path sound blame). *If* $e_0 : \tau?\ \overline{\mathbf{wf}}$*:*

- $e_0 \longrightarrow^*_{\overline{\mathsf{T}}} E_0[\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)\ v_1]; \mathcal{H}_0; \mathcal{B}_0; O_0 \rhd_{\overline{\mathsf{T}}} \mathsf{BndryErr}\ ((\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2), v_1); \mathcal{H}_1; \mathcal{B}_1; O_1$ *then*
  (1) $\mathit{has\text{-}boundary}(e_0, (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2))$
  (2) $v_1 = (\!(v)\!)^{\ell_n \cdots \ell_2}$
- $e_0 \longrightarrow^*_{\overline{\mathsf{T}}} E_0[\mathsf{app}\{\tau?\}\ v_0\ v_1]; \mathcal{H}_0; \mathcal{B}_0; O_0 \rhd_{\overline{\mathsf{T}}} \mathsf{BndryErr}\ (\overline{b}_1, v_1); \mathcal{H}_1; \mathcal{B}_1; O_1$ *then*
  (1) $\forall\, b_1 \in \overline{b}_1$*, either* $\mathit{has\text{-}boundary}(e_0, b_1)$ *or* $\mathit{has\text{-}boundary}(e_0, \mathit{flip}(b_1))$
  (2) $v_1 = (\!(\mathsf{p}_1)\!)^{\overline{\ell}_0}, \exists\, \overline{\ell}_1 .\ \overline{\ell}_1 \simeq \overline{b}_1$ *and* $\overline{\ell}_0 \cup O_1(\mathsf{p}_1) \subseteq \overline{\ell}_1$*.*
- $e_0 \longrightarrow^*_{\overline{\mathsf{T}}} E_0[\mathsf{check}(\tau_0, v_0, \mathsf{p}_1)]; \mathcal{H}_0; \mathcal{B}_0; O_0 \rhd_{\overline{\mathsf{T}}} \mathsf{BndryErr}\ (\overline{b}_1, v_0); \mathcal{H}_0; \mathcal{B}_0; O_0$ *then*
  (1) $\forall\, b_1 \in \overline{b}_1$*, either* $\mathit{has\text{-}boundary}(e_0, b_1)$ *or* $\mathit{has\text{-}boundary}(e_0, \mathit{flip}(b_1))$
  (2) $v_0 = (\!(v_1)\!)^{\overline{\ell}_0}, \exists\, \overline{\ell}_1 .\ \overline{\ell}_1 \simeq \overline{b}_1$*, and* $\overline{\ell}_0 \cup O_1(v_1) \cup O_1(\mathsf{p}_1) \subseteq \mathit{senders}(\overline{b}_1)$*.*

Proof Sketch. The proof relies on a straight-forward subject reduction argument showing that the evaluation trace of any well-formed term e_0 satisfies two invariants:

- for all $\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)\ v_1$ that occur in all configurations $e; \mathcal{H}; \mathcal{B}; O$ in the trace, $\mathit{has\text{-}boundary}(e_0, (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2))$ and $v_1 = (\!(v)\!)^{\ell_n \cdots \ell_2}$;
- for all configurations $e; \mathcal{H}; \mathcal{B}; O$ in the trace, (i) if $e \neq \mathsf{BndryErr}\ (\overline{b}, v)$ then for all p in the domain of O, $O(\mathsf{p}) \subseteq \ell^*$ where $\ell^* \simeq \mathcal{B}(\mathsf{p}_0)$ and (ii) $\forall\, b_1 \in \mathcal{B}(\mathsf{p}_0)$, either $\mathit{has\text{-}boundary}(e_0, b_1)$ or $\mathit{has\text{-}boundary}(e_0, \mathit{flip}(b_1))$.

We employ the first invariant to establish the first case of the theorem and the second for the others. □

Theorem 6.5 (all-path incomplete blame). *There exists* e_0 *such that* $\cdot; \ell_\bullet \Vdash e_0$,

- $e_0 \longrightarrow^{*}_{\overline{\mathsf{T}}} E_0[\mathsf{check}(\tau_0, v_0, \mathsf{p}_1)]; \mathcal{H}_0; \mathcal{B}_0; O_0 \rhd_{\overline{\mathsf{T}}} \mathsf{BndryErr}\,(\overline{b}_1, v_0); \mathcal{H}_0; \mathcal{B}_0; O_0$ *and*

  (1) $\forall\, b_1 \in \overline{b}_1$, *either* $\mathit{has\text{-}boundary}(e_0, b_1)$ *or* $\mathit{has\text{-}boundary}(e_0, \mathit{flip}(b_1))$

  (2) $v_0 = (\!(v_1)\!)^{\ell^*}$, $v \neq \mathsf{p}$, $\ell_1 \ldots \ell_n \simeq \overline{b}_1$ *and* $\ell_1 \ldots \ell_n \not\supseteq O_1(\mathsf{p}_1) \cup \ell^*$.

Proof. Consider the following example e_0 where the let-expressions are sugar for untyped function applications:

$$
\begin{aligned}
e_0 \quad = \quad & (\mathsf{let}\ f_0 = \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\ (\lambda x_0 .\, x_0))\ \mathsf{in} \\
& \ \mathsf{let}\ f_1 = \mathsf{stat}\ (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_3)\ (\mathsf{dyn}\ (\ell_3 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_4)\ (\lambda x_1 .\, x_1))\ \mathsf{in} \\
& \ \mathsf{stat}\ (\ell_0 \blacktriangleleft (\mathsf{Int} \times \mathsf{Int}) \blacktriangleleft \ell_5) \\
& (\mathsf{app}\{\mathsf{Int} \times \mathsf{Int}\}\ (\mathsf{dyn}\ (\ell_5 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ (\mathsf{app}\{\mathcal{U}\}\ f_1\ f_0)^{\ell_0})\ 42)^{\ell_5})^{\ell_0}
\end{aligned}
$$

With a straight-forward application of the reduction rules we obtain:

$$
\begin{aligned}
& e_0; \emptyset; \emptyset; \emptyset \\
\rightarrow^{*}_{\mathsf{T}}\ & \mathsf{stat}\ (\ell_0 \blacktriangleleft (\mathsf{Int} \times \mathsf{Int}) \blacktriangleleft \ell_5) \\
& \quad (\mathsf{app}\{\mathsf{Int} \times \mathsf{Int}\}\ (\mathsf{dyn}\ (\ell_5 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ (\mathsf{app}\{\mathcal{U}\}\ (\!(\mathsf{p}_1)\!)^{\ell_4 \ell_3 \ell_0}\ (\!(\mathsf{p}_0)\!)^{\ell_2 \ell_1 \ell_0})^{\ell_0})\ 42)^{\ell_5})^{\ell_0}; \mathcal{H}_0; \mathcal{B}_0; O_0 \\
\rightarrow^{*}_{\mathsf{T}}\ & (\mathsf{stat}\ (\ell_0 \blacktriangleleft \mathsf{Int} \times \mathsf{Int} \blacktriangleleft \ell_5)\ (\mathsf{app}\{\mathsf{Int} \times \mathsf{Int}\}\ (\!(\mathsf{p}_0)\!)^{\overline{\ell}_0}\ 42)^{\ell_5})^{\ell_0}; \mathcal{H}_1; \mathcal{B}_1; O_1 \\
\rightarrow^{*}_{\mathsf{T}}\ & (\mathsf{stat}\ (\ell_0 \blacktriangleleft \mathsf{Int} \times \mathsf{Int} \blacktriangleleft \ell_5)\ (\mathsf{check}\ \mathsf{Int} \times \mathsf{Int}\ (\!(42)\!)^{\overline{\ell}_1}\ \mathsf{p}_0)^{\ell_5})^{\ell_0}; \mathcal{H}_2; \mathcal{B}_2; O_2 \\
\rhd_{\mathsf{T}}\ & \mathsf{BndryErr}\,(\mathcal{B}_2(\mathsf{p}_0), (\!(42)\!)^{\overline{\ell}_1}); \mathcal{H}_2; \mathcal{B}_2; O_2 \\
& \textit{where } \tau_0 = (\mathsf{Int} \Rightarrow \mathsf{Int}) \textit{ and } \tau_1 = (\mathsf{Int} \Rightarrow \mathsf{Int} \times \mathsf{Int}) \\
& \textit{and } \overline{\ell}_0 = \ell_2 \ell_1 \ell_0 \ell_3 \ell_4 \ell_3 \ell_0 \ell_5 \textit{ and } \overline{\ell}_1 = \ell_5 \overline{\ell}_0 (\mathit{rev}(\overline{\ell}_0)) \\
& \textit{and } \mathcal{H}_2 = \{(\mathsf{p}_0 \mapsto \lambda x_0 .\, x_0), (\mathsf{p}_1 \mapsto \lambda x_1 .\, x_1)\} \\
& \textit{and } \mathcal{B}_2 = \{(\mathsf{p}_0 \mapsto \{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1), (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2), (\ell_5 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\}), \\
& \qquad\qquad (\mathsf{p}_1 \mapsto \{(\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_3), (\ell_3 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_4)\})\}
\end{aligned}
$$

Thus $\{\ell_0, \ell_1, \ell_2, \ell_5\} \simeq \mathcal{B}_2(\mathsf{p}_0)$ and $\{\ell_0, \ell_1, \ell_2, \ell_5\} \not\supseteq O_2(\mathsf{p}_0) \cup \{\ell_0, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5\}$ independently of the details of O_2.

□

Theorem 6.6 (type soundness). *If* $e_0 : \tau?\ \mathbf{wf}$ *then one of the following holds:*

- $e_0; \emptyset; \emptyset \rightarrow^{*}_{\mathsf{T}} v_0; \mathcal{H}_0; \mathcal{B}_0$ *and* $\exists\, \mathcal{T}_0 .\ \mathcal{T}_0; \cdot \vdash_{\mathsf{T}} v_0 : \lfloor \tau? \rfloor$
- $e_0; \emptyset; \emptyset$ *diverges*
- $e_0; \emptyset; \emptyset \rightarrow^{*}_{\mathsf{T}} E_0[\mathsf{dyn}\ b_1\ E[e_1]]; \mathcal{H}_0; \mathcal{B}_0$ *and* $e_1; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{TagErr}\,\bullet; \mathcal{H}_1; \mathcal{B}_1$
- $e_0; \emptyset; \emptyset \rightarrow^{*}_{\mathsf{T}} \mathsf{DivErr}; \mathcal{H}_0; \mathcal{B}_0$
- $e_0; \emptyset; \emptyset \rightarrow^{*}_{\mathsf{T}} \mathsf{BndryErr}\,(\overline{b}_1, v_1); \mathcal{H}_0; \mathcal{B}_0$

Proof Sketch. By progress and preservation lemmas; preservation constructs a heap typing ($\mathcal{T}$) at each step. □

7 Amnesic THEOREMS, LEMMAS, AND PROOFS

7.1 Amnesic Theorems

THEOREM 7.1. *If* $e_0 : \tau? \; \overline{\mathbf{wf}}$ *then* $\mathit{forget}(e_0) : \tau? \; \mathbf{wf}$ *and* $e_0 \longrightarrow_{\overline{\mathsf{A}}} e_1$ *iff* $\mathit{forget}(e_0) \rightarrow_{\mathsf{A}} \mathit{forget}(e_1)$

PROOF. By the definition of $\longrightarrow_{\overline{\mathsf{A}}}$.

□

THEOREM 7.2 (TYPE SOUNDNESS). *If* $e_0 : \tau_0 \; \mathbf{wf}$ *then one of the following holds:*

- $e_0 \rightarrow^*_{\mathsf{A}} v_0$ *and* $\cdot \vdash_{\mathsf{A}} v_0 : \tau_0$
- e_0 *diverges*
- $e_0 \rightarrow^*_{\mathsf{A}} E_0[\mathsf{dyn}\; b_1\; E[e_1]]$ *and* $e_1 \blacktriangleright_{\mathsf{A}} \mathsf{TagErr}\bullet$
- $e_0 \rightarrow^*_{\mathsf{A}} \mathsf{DivErr}$
- $e_0 \rightarrow^*_{\mathsf{A}} \mathsf{BndryErr}\,(\overline{b}_1, v_1)$

PROOF. By progress and preservation lemmas (lemma 7.11 and lemma 7.12).

□

THEOREM 7.3 (DYNAMIC SOUNDNESS). *If* $e_0 : \mathcal{U} \; \mathbf{wf}$ *then one of the following holds:*

- $e_0 \rightarrow^*_{\mathsf{A}} v_0$ *and* $\cdot \vdash_{\mathsf{A}} v_0 : \mathcal{U}$
- e_0 *diverges*
- $e_0 \rightarrow^*_{\mathsf{A}} E_0[e_1]$ *and* $e_1 \blacktriangleright_{\mathsf{A}} \mathsf{TagErr}\bullet$
- $e_0 \rightarrow^*_{\mathsf{A}} \mathsf{DivErr}$
- $e_0 \rightarrow^*_{\mathsf{A}} \mathsf{BndryErr}\,(\overline{b}_1, v_1)$

PROOF. By progress and preservation lemmas (lemma 7.11 & lemma 7.12).

□

THEOREM 7.4 (INCOMPLETE MONITORING). *There exist* $e_0, e_1, \ell_0, \tau?$ *such that* $(e_0)^{\ell_0} : \tau? \; \overline{\mathbf{wf}}$ *and* $e_0 \longrightarrow^*_{\overline{\mathsf{A}}} e_1$ *and* $\cdot; \ell \not\Vdash e_1$.

PROOF. Let

$$\begin{aligned}
e_f &= \mathsf{stat}\;(\ell_0 \blacktriangleleft (\mathsf{Int} \Rightarrow \mathsf{Int}) \blacktriangleleft \ell_1)\;(\mathsf{dyn}\;(\ell_1 \blacktriangleleft (\mathsf{Int} \Rightarrow \mathsf{Int}) \blacktriangleleft \ell_2)\;(\lambda x_0.\,(\mathsf{sum}\{\mathsf{Int}\}\;x_0\;1))^{\ell_2})^{\ell_1} \\
e_0 &= (\mathsf{app}\{\mathcal{U}\}\;e_0\;(\lambda x_0.\,(\mathsf{sum}\{\mathsf{Int}\}\;x_0\;1)))^{\ell_0} \\
v_f &= (\mathsf{trace}_{\mathsf{v}}\;\{(\ell_0 \blacktriangleleft (\mathsf{Int} \Rightarrow \mathsf{Int}) \blacktriangleleft \ell_1), (\ell_1 \blacktriangleleft (\mathsf{Int} \Rightarrow \mathsf{Int}) \blacktriangleleft \ell_2)\}\;((\lambda x_0.\,(\mathsf{sum}\{\mathsf{Int}\}\;x_0\;1)))^{\ell_2 \ell_1})^{\ell_0} \\
e_1 &= (\mathsf{app}\{\mathcal{U}\}\;v_0\;(\lambda x_1.\,0))^{\ell_0}
\end{aligned}$$

With a straight-forward application of the reduction rules we obtain:

$$\begin{aligned}
(e_f)^{\ell_0} &\longrightarrow^*_{\overline{\mathsf{A}}} (\mathsf{stat}\;(\ell_0 \blacktriangleleft (\mathsf{Int} \Rightarrow \mathsf{Int}) \blacktriangleleft \ell_1)\;(\mathsf{mon}\;(\ell_1 \blacktriangleleft (\mathsf{Int} \Rightarrow \mathsf{Int}) \blacktriangleleft \ell_2)\;(\lambda x_0.\,(\mathsf{sum}\{\mathsf{Int}\}\;x_0\;1))^{\ell_2})^{\ell_1})^{\ell_0} \\
&\longrightarrow^*_{\overline{\mathsf{A}}} (\mathsf{trace}_{\mathsf{v}}\;\{(\ell_0 \blacktriangleleft (\mathsf{Int} \Rightarrow \mathsf{Int}) \blacktriangleleft \ell_1), (\ell_1 \blacktriangleleft (\mathsf{Int} \Rightarrow \mathsf{Int}) \blacktriangleleft \ell_2)\}\;((\lambda x_0.\,(\mathsf{sum}\{\mathsf{Int}\}\;x_0\;1)))^{\ell_2 \ell_1})^{\ell_0} \\
&= v_f
\end{aligned}$$

therefore

$$e_0 \quad \longrightarrow^*_{\overline{\mathsf{A}}} e_1$$

□

THEOREM 7.5 (SOUND AND COMPLETE BLAME). *If* $\cdot; \ell_\bullet \Vdash e_0$ *and* $e_0 \longrightarrow^*_{\overline{\mathsf{A}}} \mathsf{BndryErr}\,((\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1), v_1)$ *then*

- *either has-boundary*$((\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1), e_0)$ *or has-boundary*$((\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0), e_0)$*, and*
- *senders*$(\overline{b}_1) =$ *owners*(v_1)

Theorem 7.6 (blame correctness). *If* $(e_0)^{\ell_0} : \tau?\ \overline{\mathbf{wf}}$ *then one of the following holds:*

- $e_0 \longrightarrow^*_{\overline{\mathsf{A}}} v_0$ *and* $\cdot; \ell_0 \Vdash_{\mathsf{A}} v_0$
- e_0 *diverges*
- $e_0 \longrightarrow^*_{\overline{\mathsf{A}}} \mathsf{TagErr}\,\bullet$
- $e_0 \longrightarrow^*_{\overline{\mathsf{A}}} \mathsf{DivErr}$
- $e_0 \longrightarrow^*_{\overline{\mathsf{A}}} E_0[\mathsf{dyn}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)\, v_1] \longrightarrow_{\overline{\mathsf{A}}} \mathsf{BndryErr}\,(\overline{b}_1, v_1)$ *and furthermore:*
  (1) $\forall\, b_1 \in \overline{b}_1$*, either has-boundary*$(e_0, b_1)$ *or has-boundary*$(e_0, \mathit{flip}(b_1))$
  (2) *one of the following holds:*
    (a) $v_1 \notin (\mathsf{trace_v}\, \overline{b}\, (\!(v)\!)^{\overline{\ell}})$ *and* $\cdot; \ell_2 \Vdash_{\mathsf{A}} v_1$
    (b) $v_1 = (\mathsf{trace_v}\, \overline{b}_2\, (\!(v_2)\!)^{\overline{\ell}_2 \ell_2})$ *and* $\overline{b}_2 \simeq \ell_2 \overline{\ell}_2 \ell_2$ *and* $\cdot;$ *last*$(\ell_3) \Vdash_{\mathsf{A}} v_2$

Proof.
1. suffices assume $e_0 \longrightarrow^*_{\overline{\mathsf{A}}} E_0[\mathsf{dyn}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)\, v_1] \longrightarrow_{\overline{\mathsf{A}}} \mathsf{BndryErr}\,(\overline{b}_1, v_1)$
   prove (1) $\forall\, b_1 \in \overline{b}_1$, either *has-boundary*$(e_0, b_1)$ or *has-boundary*$(e_0, \mathit{flip}(b_1))$
   (2) one of the following holds:
   (a) $v_1 \notin (\mathsf{trace_v}\, \overline{b}\, (\!(v)\!)^{\overline{\ell}})$ and $\cdot; \ell_2 \Vdash_{\mathsf{A}} v_1$
   (b) $v_1 = (\mathsf{trace_v}\, \overline{b}_2\, (\!(v_2)\!)^{\overline{\ell}_2 \ell_2})$ and $\overline{b}_2 \simeq \ell_2 \overline{\ell}_2 \ell_2$ and $\cdot;$ *last*$(\ell_3) \Vdash_{\mathsf{A}} v_2$
   by lemma 7.26 and lemma 7.27
2. $(\ell_1 \blacktriangleleft \ell_2 \blacktriangleleft\ \Vdash_{\mathsf{A}})\tau_1$ and $\ell_2 \Vdash_{\mathsf{A}} v_1$
   by lemma 7.27
3. $\forall b_1 \in \overline{b}_1$ either $b_1 \in e_0$ or $\mathit{flip}(b_1) \in e_0$
   by lemma 7.41
4. either $\overline{b}_1 = (\ell_1 \blacktriangleleft \ell_2 \blacktriangleleft)$
   or $v_1 = (\mathsf{trace_v}\, \overline{b}_0\, v_0)$ and $\overline{b}_1 = (\ell_1 \blacktriangleleft \ell_2 \blacktriangleleft)\overline{b}_0$
   by the definition of $\longrightarrow_{\overline{\mathsf{A}}}$

□

Corollary 7.7 (minimal blame info). *If* $e_0 : \tau?\ \mathbf{wf}$ *and* $e_0 \rightarrow^*_{\mathsf{A}} \mathsf{BndryErr}\,(\overline{b}_1, v_1)$ *then* $\overline{b}_1 \neq \cdot$

Proof. by Theorem 7.6

□

Corollary 7.8 (blame/ownership match). *If* $L_0; \ell_0 \Vdash_{\mathsf{A}} e_0$ *then for all subterms* $(\mathsf{trace_v}\, \overline{b}_1\, e_1)$ *there exists* $\overline{\ell}_1$ *such that* $e_1 = (\!(e_2)\!)^{\overline{\ell}_1}$ *and* $\overline{b}_1 \simeq \overline{\ell}_1$

Proof. by definition of $\Vdash_{\mathsf{A}}$

□

Theorem 7.9. *If* $e_0 : \tau_0\ \mathbf{wf}$ *and* $e_0 \rightarrow^*_{\mathsf{A}} v_1$ *then mon-depth*$(v_1) \leq 2$

Proof. 1. suffices if $\mathsf{dyn}\, b_0\, v_2 \rhd_{\mathsf{A}} v_3$ then *mon-depth*$(v_3) \leq 2$

because the only way to increase the *mon-depth* of a value is by crossing a boundary

2. QED

by lemma 7.10 and the definition of $\blacktriangleright_{\mathsf{A}}$

□

THEOREM 7.10. *If* $e_0 : \mathcal{U}$ **wf** *and* $e_0 \rightarrow^*_{\mathsf{A}} v_1$ *then* $\mathit{mon\text{-}depth}(v_1) \leq 1$

PROOF. 1. SUFFICES if stat $b_0\ v_2 \blacktriangleright_{\mathsf{A}} v_3$ then $\mathit{mon\text{-}depth}(v_3) \leq 1$

because the only way to increase the *mon-depth* of a value is by crossing a boundary

2. QED

by definition of $\blacktriangleright_{\mathsf{A}}$

□

7.2 Amnesic **Lemmas**

LEMMA 7.11 ($\vdash_{\mathsf{A}}$ PROGRESS). *If* $\cdot \vdash_{\mathsf{A}} e_0 : \tau?$ *then one of the following holds:*

- $e_0 \in v$
- $e_0 \in \mathsf{Err}$
- $\exists\, e_1$ *such that* $e_0 \rightarrow_{\mathsf{A}} e_1$

PROOF. By case analysis of e_0.

By lemma 7.15 it suffices to consider the following cases.

1. CASE $e_0 \in v$
   1.1. QED
2. CASE $e_0 = E_0[\mathsf{Err}]$
   2.1. QED
3. CASE $e_0 = E_0[\mathsf{app}\{\tau_1\}\ v_0\ v_1]$
   3.1. $v_0 \in (\lambda(x : \tau).\ e) \cup (\mathsf{mon}\ b\ v)$
      by lemma 7.16 and inversion $\vdash_{\mathsf{A}}$
   3.2. SCASE $v_0 = \lambda(x_2 : \tau_2).\ e_2$
      3.2.1. QED
         $e_0 \rhd_{\mathsf{A}} E_0[e_2[x_2 \leftarrow v_1]]$
   3.3. SCASE $v_0 = \mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2) \blacktriangleleft \ell_1)\ v_2$
      3.3.1. QED
         $e_0 \rhd_{\mathsf{A}} E_0[\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\mathcal{U}\}\ v_2\ (\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ v_1))]$
4. CASE $e_0 = E_0[\mathsf{app}\{\tau_1\}\ v_0\ v_1]$
   4.1. SCASE $v_0 = \mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_0\ (\lambda(x_2 : \tau_2).\ e_2)$
      4.1.1. QED
         $e_0 \blacktriangleright_{\mathsf{A}} E_0[\mathsf{trace}\ \overline{b}_0\ (e_2[x_2 \leftarrow v_1])]$
   4.2. SCASE $v_0 = \mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_0\ (\mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2) \blacktriangleleft \ell_1)\ v_2)$
      4.2.1. QED
         $e_0 \blacktriangleright_{\mathsf{A}} E_0[\mathsf{trace}\ \overline{b}_0\ (\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\mathit{forget}(\tau_2)\}\ v_2\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ v_1)))]$
   4.3. SCASE $v_0 \notin (\mathsf{trace}^?_{\mathsf{v}}\ \overline{b}\ (\lambda(x : \tau).\ e)) \cup (\mathsf{trace}^?_{\mathsf{v}}\ \overline{b}\ (\mathsf{mon}\ b\ v))$

4.3.1. QED

$e_0 \blacktriangleright_{\mathsf{A}} E_0[\mathsf{TagErr}\,\bullet]$

5. CASE $e_0 = E_0[\mathit{unop}\{\tau?\}\, v_0]$

   5.1. QED

   by lemma 7.16 and lemma 7.18

6. CASE $e_0 = E_0[\mathit{binop}\{\tau?\}\, v_0\, v_1]$

   6.1. QED

   by lemma 7.16 and lemma 7.18

7. CASE $e_0 = E_0[\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, v_0]$

   7.1. QED

   by lemma 7.16 and lemma 7.20

8. CASE $e_0 = E_0[\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, v_0]$

   8.1. QED

   by lemma 7.16 and lemma 7.21

9. CASE $e_0 = E_0[\mathsf{trace}\,\overline{b}_0\, v_0]$

   9.1. QED

   $e_0 \blacktriangleright_{\mathsf{A}} E_0[\mathit{add\text{-}trace}(\overline{b}_0, v_0)]$

□

LEMMA 7.12 ($\vdash_{\mathsf{A}}$ TYPE PRESERVATION). *If* $\cdot \vdash_{\mathsf{A}} e_0 : \tau?$ *and* $e_0 \rightarrow_{\mathsf{A}} e_1$ *then* $\cdot \vdash_{\mathsf{A}} e_1 : \tau?$.

PROOF. By lemma 7.13 and lemma 7.14.

□

LEMMA 7.13 ($\triangleright_{\mathsf{A}}$ PRESERVATION). *If* $\cdot \vdash_{\mathsf{A}} e_0 : \tau_0$ *and* $e_0 \triangleright_{\mathsf{A}} e_1$ *then* $\cdot \vdash_{\mathsf{A}} e_1 : \tau_0$.

PROOF. By case analysis of $\triangleright_{\mathsf{A}}$.

1. CASE $\delta_A(\mathit{unop}, v_0)$ is defined
   and $\mathit{unop}\{\tau?\}\, v_0 \triangleright_{\mathsf{A}} \delta_A(\mathit{unop}, v_0)$

   1.1. QED

   by lemma 7.19

2. CASE $\mathsf{fst}\{\tau_0\}\,(\mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\, v_0) \triangleright_{\mathsf{A}} \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\,(\mathsf{fst}\{\mathcal{U}\}\, v_0)$

   2.1. QED

$$\dfrac{\dfrac{\dfrac{\text{by inversion } \vdash_{\mathsf{A}}}{\cdot v_0 : \mathcal{U}}}{\cdot \vdash_{\mathsf{A}} \mathsf{fst}\{\mathcal{U}\}\, v_0 : \mathcal{U}}}{\cdot \vdash_{\mathsf{A}} \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\,(\mathsf{fst}\{\mathcal{U}\}\, v_0) : \tau_0}$$

3. CASE $\mathsf{snd}\{\tau_0\}\,(\mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\, v_0) \triangleright_{\mathsf{A}} \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\,(\mathsf{snd}\{\mathcal{U}\}\, v_0)$

   3.1. QED

$$\dfrac{\dfrac{\dfrac{\text{by inversion} \vdash_{\mathsf{A}}}{\cdot v_0 : \mathcal{U}}}{\cdot \vdash_{\mathsf{A}} \mathsf{snd}\{\mathcal{U}\}\, v_0 : \mathcal{U}}}{\cdot \vdash_{\mathsf{A}} \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\,(\mathsf{snd}\{\mathcal{U}\}\, v_0) : \tau_0}$$

4. CASE $binop\{\tau?\}\, v_0\, v_1 \rhd_{\mathsf{A}} \delta_A(binop, v_0, v_1)$
   4.1. QED
      by lemma 7.19
5. CASE $\mathsf{app}\{\tau_0\}\,(\lambda(x_1 : \tau_1).\, e_1)\, v_2 \rhd_{\mathsf{A}} e_1[x_1 \leftarrow v_2]$
   5.1. QED
      by lemma 7.25
6. CASE $\mathsf{app}\{\tau_0\}\,(\mathsf{mon}\,(\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2) \blacktriangleleft \ell_1)\, v_0)\, v_1$
   $\rhd_{\mathsf{A}} \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\,(\mathsf{app}\{\mathcal{U}\}\, v_0\,(\mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\, v_1))$
   6.1. QED

$$\dfrac{\dfrac{\dfrac{\text{by inversion} \vdash_{\mathsf{A}}}{\cdot \vdash_{\mathsf{A}} v_0 : \mathcal{U}} \qquad \dfrac{\dfrac{\text{by inversion} \vdash_{\mathsf{A}}}{\cdot \vdash_{\mathsf{A}} v_1 : \tau_1}}{\cdot \vdash_{\mathsf{A}} \mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\, v_1 : \mathcal{U}}}{\cdot \vdash_{\mathsf{A}} \mathsf{app}\{\mathcal{U}\}\, v_0\,(\mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\, v_1) : \mathcal{U}}}{\cdot \vdash_{\mathsf{A}} \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\,(\mathsf{app}\{\mathcal{U}\}\, v_0\,(\mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\, v_1)) : \tau_0}$$

7. CASE $\mathsf{dyn}\, b_0\, v_0 \rhd_{\mathsf{A}} v_1$
   7.1. QED
      by lemma 7.22

□

LEMMA 7.14 ($\blacktriangleright_{\mathsf{A}}$ PRESERVATION). *If* $\cdot \vdash_{\mathsf{A}} e_0 : \mathcal{U}$ *and* $e_0 \blacktriangleright_{\mathsf{A}} e_1$ *then* $\cdot \vdash_{\mathsf{A}} e_1 : \mathcal{U}$.

PROOF. By case analysis of $\blacktriangleright_{\mathsf{A}}$.

1. CASE $unop\{\tau?\}\, v_0 \blacktriangleright_{\mathsf{A}} \mathsf{TagErr}\,\bullet$
   1.1. QED
      $\cdot \vdash_{\mathsf{A}} \mathsf{TagErr}\,\bullet : \mathcal{U}$
2. CASE $unop\{\tau?\}\, v_0 \blacktriangleright_{\mathsf{A}} \delta_A(unop, v_0)$
   2.1. QED
      by lemma 7.19
3. CASE $\mathsf{fst}\{\mathcal{U}\}\,(\mathsf{trace}^?_{\mathsf{v}}\, \overline{b}_0\,(\mathsf{mon}\,(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\, v_1)) \blacktriangleright_{\mathsf{A}} \mathsf{trace}\, \overline{b}_0\,(\mathsf{stat}\, b_7\,(\mathsf{fst}\{\mathit{fst}(\tau_0)\}\, v_1))$
   where $b_7 = (\ell_1 \blacktriangleleft \mathit{fst}(\tau_0) \blacktriangleleft \ell_2)$
   3.1. QED

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{by inversion } \vdash_{\mathsf{A}}}{\cdot \vdash_{\mathsf{A}} v_1 : \tau_0}}{\cdot \vdash_{\mathsf{A}} \mathsf{fst}\{\mathit{fst}(\tau_0)\}\ v_1 : \mathit{fst}(\tau_0)}}{\cdot \vdash_{\mathsf{A}} \mathsf{stat}\ b_7\ (\mathsf{fst}\{\mathit{fst}(\tau_0)\}\ v_1) : \mathcal{U}}}{\cdot \vdash_{\mathsf{A}} \mathsf{trace}\ \overline{b}_0\ (\mathsf{stat}\ b_7\ (\mathsf{fst}\{\mathit{fst}(\tau_0)\}\ v_1)) : \mathcal{U}}$$

4. CASE $\mathsf{snd}\{\mathcal{U}\}\ (\mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_0\ (\mathsf{mon}\ (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\ v_1)) \blacktriangleright_{\mathsf{A}} \mathsf{trace}\ \overline{b}_0\ (\mathsf{stat}\ b_7\ (\mathsf{snd}\{\mathit{snd}(\tau_0)\}\ v_1))$
   where $b_7 = (\ell_1 \blacktriangleleft \mathit{snd}(\tau_0) \blacktriangleleft \ell_2)$

   4.1. QED

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{by inversion } \vdash_{\mathsf{A}}}{\cdot \vdash_{\mathsf{A}} v_1 : \tau_0}}{\cdot \vdash_{\mathsf{A}} \mathsf{snd}\{\mathit{snd}(\tau_0)\}\ v_1 : \mathit{snd}(\tau_0)}}{\cdot \vdash_{\mathsf{A}} \mathsf{stat}\ b_7\ (\mathsf{snd}\{\mathit{snd}(\tau_0)\}\ v_1) : \mathcal{U}}}{\cdot \vdash_{\mathsf{A}} \mathsf{trace}\ \overline{b}_0\ (\mathsf{stat}\ b_7\ (\mathsf{snd}\{\mathit{snd}(\tau_0)\}\ v_1)) : \mathcal{U}}$$

5. CASE $\mathit{binop}\{\tau?\}\ v_0\ v_1 \blacktriangleright_{\mathsf{A}} \mathsf{TagErr}\bullet$

   5.1. QED

   $\cdot \vdash_{\mathsf{A}} \mathsf{TagErr}\bullet : \mathcal{U}$

6. CASE $\mathit{binop}\{\tau?\}\ v_0\ v_1 \blacktriangleright_{\mathsf{A}} \delta_A(\mathit{binop}, v_0, v_1)$

   6.1. QED

   by lemma 7.19

7. CASE $\mathsf{app}\{\mathcal{U}\}\ (\mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_0\ (\lambda x_1.\, e_1))\ v_2$
   $\blacktriangleright_{\mathsf{A}} \mathsf{trace}\ \overline{b}_0\ (e_1[x_1 \leftarrow v_2])$

   7.1. QED

   by lemma 7.25

8. CASE $\mathsf{app}\{\mathcal{U}\}\ (\mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_0\ (\mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_0) \blacktriangleleft \ell_1)\ v_0))\ v_1$
   $\blacktriangleright_{\mathsf{A}} \mathsf{trace}\ \overline{b}_0\ (\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\tau_0\}\ v_0\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ v_1)))$

   8.1. QED

$$\dfrac{\dfrac{\dfrac{\text{by inversion } \vdash_{\mathsf{A}}}{\cdot \vdash_{\mathsf{A}} v_0 : \tau_1 \Rightarrow \tau_0} \quad \dfrac{\dfrac{\text{by inversion } \vdash_{\mathsf{A}}}{\cdot \vdash_{\mathsf{A}} v_1 : \mathcal{U}}}{\cdot \vdash_{\mathsf{A}} \mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ v_1 : \tau_1}}{\cdot \vdash_{\mathsf{A}} \mathsf{app}\{\tau_0\}\ v_0\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ v_1) : \tau_0}}{\cdot \vdash_{\mathsf{A}} \mathsf{trace}\ \overline{b}_0\ (\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\tau_0\}\ v_0\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ v_1))) : \mathcal{U}}$$

9. CASE $\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{A}} v_1$

   9.1. QED

   by lemma 7.23

10. CASE $\mathsf{trace}\ \overline{b}_0\ v_0 \blacktriangleright_{\mathsf{A}} \mathit{add\text{-}trace}(\overline{b}_0, v_0)$

10.1. QED

by lemma 7.24

□

LEMMA 7.15 (UNIQUE DECOMPOSITION). *If* $\cdot \vdash_{\mathsf{A}} e_0 : \tau?$ *then either:*

- $e_0 \in v$
- $e_0 = E_0[\mathsf{app}\{\tau?\}\ v_0\ v_1]$
- $e_0 = E_0[\mathit{unop}\{\tau?\}\ v_0]$
- $e_0 = E_0[\mathit{binop}\{\tau?\}\ v_0\ v_1]$
- $e_0 = E_0[\mathsf{dyn}\ b_1\ v_1]$
- $e_0 = E_0[\mathsf{stat}\ b_1\ v_1]$
- $e_0 = E_0[\mathsf{trace}\ \overline{b}_1\ v_1]$
- $e_0 = E_0[\mathsf{Err}]$

PROOF. By induction on the structure of e_0.

1. CASE $e_0 = x_0$
   1.1. CONTRADICTION:
   $\cdot \vdash_{\mathsf{A}} e_0 : \tau?$
2. CASE $e_0 = v_0$
   2.1. QED
3. CASE $e_0 = \langle e_1, e_2 \rangle$
   3.1. SCASE $e_1 \notin v$
      3.1.1. QED
         by the induction hypothesis
   3.2. SCASE $e_1 \in v$ and $e_2 \notin v$
      3.2.1. QED
         by the induction hypothesis
   3.3. SCASE $e_1 \in v$ and $e_2 \in v$
      3.3.1. QED
         $e_0 \in v$
4. CASE $e_0 = \mathsf{app}\{\tau?\}\ e_1\ e_2$
   4.1. QED
      by the induction hypothesis
5. CASE $e_0 = \mathit{unop}\{\tau?\}\ e_1$
   5.1. QED
      by the induction hypothesis
6. CASE $e_0 = \mathit{binop}\{\tau?\}\ e_1\ e_2$
   6.1. QED
      by the induction hypothesis
7. CASE $e_0 = \mathsf{dyn}\ b_1\ e_1$
   7.1. QED

by the induction hypothesis

8. CASE $e_0 = \mathsf{stat}\ b_1\ e_1$

 8.1. QED

 by the induction hypothesis

9. CASE $e_0 = \mathsf{trace}\ \overline{b_1}\ e_1$

 9.1. QED

 by the induction hypothesis

10. CASE $e_0 \in \mathsf{Err}$

 10.1. QED

□

Lemma 7.16. *If* $\cdot \vdash_{\mathsf{A}} E_0[e_0] : \tau?$ *then one of the following holds:*

- $\cdot \vdash_{\mathsf{A}} e_0 : \mathcal{U}$
- $\exists\, \tau_0 \,.\, \cdot \vdash_{\mathsf{A}} e_0 : \tau_0$

Proof. By induction on the structure of E_0 and case analysis of $\vdash_{\mathsf{A}}$.

1. CASE $E_0 = [\,]$

 1.1. QED

2. CASE $E_0 = \langle E_1, e_2 \rangle$

 2.1. $\cdot \vdash_{\mathsf{A}} E_1[e_0] : \tau?$

 by inversion $\vdash_{\mathsf{A}}$

 2.2. QED

 by the induction hypothesis

3. CASE $E_0 = \langle v_1, E_2 \rangle$

 3.1. QED

 by the induction hypothesis

4. CASE $E_0 = \mathsf{app}\{\tau?\}\ E_1\ e_2$

 4.1. QED

 by the induction hypothesis

5. CASE $E_0 = \mathsf{app}\{\tau?\}\ v_1\ E_2$

 5.1. QED

 by the induction hypothesis

6. CASE $E_0 = unop\{\tau?\}\ E_1$

 6.1. QED

 by the induction hypothesis

7. CASE $E_0 = binop\{\tau?\}\ E_1\ e_2$

 7.1. QED

 by the induction hypothesis

8. CASE $E_0 = binop\{\tau?\}\ v_1\ E_2$

 8.1. QED

 by the induction hypothesis

9. CASE $E_0 = \mathsf{dyn}\; b_1\; E_1$
   9.1. QED
      by the induction hypothesis
10. CASE $E_0 = \mathsf{stat}\; b_1\; E_1$
   10.1. QED
      by the induction hypothesis
11. CASE $E_0 = \mathsf{trace}\; \overline{b}_1\; E_1$
   11.1. QED
      by the induction hypothesis

□

LEMMA 7.17 ($\vdash_A$ REPLACEMENT).

- *If* $\cdot \vdash_A E_0[e_0] : \tau?$ *and the derivation contains a proof of* $\cdot \vdash_A e_0 : \tau_0$ *and* $\cdot \vdash_A e_1 : \tau_0$ *then* $\cdot \vdash_A E_0[e_1] : \tau?$.
- *If* $\cdot \vdash_A E_0[e_0] : \tau?$ *and the derivation contains a proof of* $\cdot \vdash_A e_0 : \mathcal{U}$ *and* $\cdot \vdash_A e_1 : \mathcal{U}$ *then* $\cdot \vdash_A E_0[e_1] : \tau?$.

PROOF. By induction on E_0.

1. CASE $E_0 = [\,]$
   1.1. QED
2. CASE $E_0 = \langle E_1, e_2 \rangle$
   2.1. QED
      by the induction hypothesis
3. CASE $E_0 = \langle v_1, E_2 \rangle$
   3.1. QED
      by the induction hypothesis
4. CASE $E_0 = \mathsf{app}\{\tau?\}\; E_1\; e_2$
   4.1. QED
      by the induction hypothesis
5. CASE $E_0 = \mathsf{app}\{\tau?\}\; v_1\; E_2$
   5.1. QED
      by the induction hypothesis
6. CASE $E_0 = \mathit{unop}\{\tau?\}\; E_1$
   6.1. QED
      by the induction hypothesis
7. CASE $E_0 = \mathit{binop}\{\tau?\}\; E_1\; e_2$
   7.1. QED
      by the induction hypothesis
8. CASE $E_0 = \mathit{binop}\{\tau?\}\; v_1\; E_2$
   8.1. QED
      by the induction hypothesis
9. CASE $E_0 = \mathsf{dyn}\; b_1\; E_1$
   9.1. QED

by the induction hypothesis

10. CASE $E_0 = \mathsf{stat}\, b_1\, E_1$

10.1. QED

by the induction hypothesis

11. CASE $E_0 = \mathsf{trace}\, \overline{b}_1\, E_1$

11.1. QED

by the induction hypothesis

□

LEMMA 7.18.

- *If* $\cdot \vdash_{\mathsf{A}} unop\{\tau_1\}\, v_0 : \tau_0$ *then* $unop\{\tau_1\}\, v_0 \rhd_{\mathsf{A}} e_1$.
- *if* $\cdot \vdash_{\mathsf{A}} binop\{\tau_1\}\, v_0\, v_1 : \tau_0$ *then* $binop\{\tau_1\}\, v_0\, v_1 \rhd_{\mathsf{A}} e_1$.
- *If* $\cdot \vdash_{\mathsf{A}} unop\{\mathcal{U}\}\, v_0 : \mathcal{U}$ *then* $unop\{\mathcal{U}\}\, v_0 \blacktriangleright_{\mathsf{A}} e_1$.
- *if* $\cdot \vdash_{\mathsf{A}} binop\{\mathcal{U}\}\, v_0\, v_1 : \mathcal{U}$ *then* $binop\{\mathcal{U}\}\, v_0\, v_1 \blacktriangleright_{\mathsf{A}} e_1$.

PROOF. By case analysis of δ_A, $\vdash_{\mathsf{A}}$, and $\blacktriangleright_{\mathsf{A}}$.

1. CASE $\cdot \vdash_{\mathsf{A}} \mathsf{fst}\{\tau_0\}\, v_0$

1.1. $v_0 \in \langle v, v\rangle \cup \mathsf{mon}\, (\ell \blacktriangleleft \tau \times \tau \blacktriangleleft \ell)\, v$

by $\vdash_{\mathsf{A}}$ canonical forms

1.2. SCASE $v_0 = \langle v_1, v_2\rangle$

1.2.1. QED

$\mathsf{fst}\{\tau_0\}\, v_0 \rhd_{\mathsf{A}} v_1$

1.3. SCASE $v_0 = \mathsf{mon}\, (\ell_0 \blacktriangleleft \tau_1 \times \tau_2 \blacktriangleleft \ell_1)\, v_1$

1.3.1. QED

$\mathsf{fst}\{\tau_0\}\, v_0 \rhd_{\mathsf{A}} \mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, (\mathsf{fst}\{\mathcal{U}\}\, v_0)$

2. CASE $\cdot \vdash_{\mathsf{A}} \mathsf{snd}\{\tau_0\}\, v_0$

2.1. QED

similar to the fst case

3. CASE $\cdot \vdash_{\mathsf{A}} \mathsf{fst}\{\mathcal{U}\}\, v_0$

3.1. SCASE $v_0 = \langle v_1, v_2\rangle$

3.1.1. QED

$\mathsf{fst}\{\mathcal{U}\}\, v_0 \blacktriangleright_{\mathsf{A}} v_1$

3.2. SCASE $v_0 = \mathsf{trace}^{?}_{\mathsf{v}}\, \overline{b}_0\, (\mathsf{mon}\, (\ell_1 \blacktriangleleft \tau_1 \times \tau_2 \blacktriangleleft \ell_2)\, v_1)$

3.2.1. QED

$\mathsf{fst}\{\mathcal{U}\}\, v_0 \blacktriangleright_{\mathsf{A}} \mathsf{trace}\, \overline{b}_0\, (\mathsf{stat}\, (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)\, (\mathsf{fst}\{\tau_1\}\, v_1))$

3.3. SCASE $v_0 \notin \langle v, v\rangle \cup (\mathsf{mon}\, (\ell \blacktriangleleft \tau \times \tau \blacktriangleleft \ell)\, v)$

3.3.1. QED

$\mathsf{fst}\{\mathcal{U}\}\, v_0 \blacktriangleright_{\mathsf{A}} \mathsf{TagErr}\, \bullet$

4. CASE $\cdot \vdash_{\mathsf{A}} \mathsf{snd}\{\mathcal{U}\}\, v_0$

4.1. QED

similar to the fst case

5. CASE $\cdot \vdash_{\mathsf{A}} binop\{\tau_1\}\, v_0\, v_1 : \tau_0$
   5.1. $v_0 \in i$ and $v_1 \in i$
   by $\vdash_{\mathsf{A}}$ canonical forms
   5.2. QED
   $binop\{\tau_1\}\, v_0\, v_1 \rhd_{\mathsf{A}} \delta_A(binop, v_0, v_1)$
6. CASE $\cdot \vdash_{\mathsf{A}} binop\{\mathcal{U}\}\, v_0\, v_1 : \mathcal{U}$
   6.1. SCASE $v_0 \in i$ and $v_1 \in i$
      6.1.1. QED
      $binop\{\mathcal{U}\}\, v_0\, v_1 \blacktriangleright_{\mathsf{A}} \delta_A(binop, v_0, v_1)$
   6.2. SCASE $v_0 \notin i$ or $v_1 \notin i$
      6.2.1. QED
      $binop\{\mathcal{U}\}\, v_0\, v_1 \blacktriangleright_{\mathsf{A}} \mathsf{TagErr}\bullet$

□

Lemma 7.19.

- *If* $\cdot \vdash_{\mathsf{A}} unop\{\tau_1\}\, v_0 : \tau_0$ *and* $unop\{\tau_1\}\, v_0 \rhd_{\mathsf{A}} e_1$ *then* $\cdot \vdash_{\mathsf{A}} e_1 : \tau_0$*.*
- *If* $\cdot \vdash_{\mathsf{A}} binop\{\tau_1\}\, v_0\, v_1 : \tau_0$ *and* $binop\{\tau_1\}\, v_0\, v_1 \rhd_{\mathsf{A}} e_2$ *then* $\cdot \vdash_{\mathsf{A}} e_2 : \tau_0$*.*
- *If* $\cdot \vdash_{\mathsf{A}} unop\{\mathcal{U}\}\, v_0 : \mathcal{U}$ *and* $unop\{\mathcal{U}\}\, v_0 \blacktriangleright_{\mathsf{A}} e_1$ *then* $\cdot \vdash_{\mathsf{A}} e_1 : \mathcal{U}$*.*
- *If* $\cdot \vdash_{\mathsf{A}} binop\{\mathcal{U}\}\, v_0\, v_1 : \mathcal{U}$ *and* $binop\{\mathcal{U}\}\, v_0\, v_1 \blacktriangleright_{\mathsf{A}} e_2$ *then* $\cdot \vdash_{\mathsf{A}} e_2 : \mathcal{U}$*.*

Proof. By case analysis of δ_A and $\vdash_{\mathsf{A}}$.

1. CASE $\cdot \vdash_{\mathsf{A}} \mathsf{fst}\{\tau_0\}\, v_0 : \tau_0$
   1.1. SCASE $\mathsf{fst}\{\tau_0\}\, \langle v_1, v_2\rangle \rhd_{\mathsf{A}} v_1$
      1.1.1. QED
      by inversion $\vdash_{\mathsf{A}}$
   1.2. SCASE $\mathsf{fst}\{\tau_0\}\, (\mathsf{mon}\, (\ell_0 \blacktriangleleft \tau_1 \times \tau_2 \blacktriangleleft \ell_1)\, v_1)$
   $\rhd_{\mathsf{A}} \mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\, (\mathsf{fst}\{\mathcal{U}\}\, v_1)$
      1.2.1. QED

$$\frac{\dfrac{\dfrac{\dfrac{\text{by inversion } \vdash_{\mathsf{A}}}{\cdot \vdash_{\mathsf{A}} v_1 : \mathcal{U}}}{\cdot \vdash_{\mathsf{A}} \mathsf{fst}\{\mathcal{U}\}\, v_1 : \mathcal{U}}}{\cdot \vdash_{\mathsf{A}} \mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\, (\mathsf{fst}\{\mathcal{U}\}\, v_1) : \tau_1} \qquad \dfrac{\text{by inversion } \vdash_{\mathsf{A}}}{\tau_1 \leqslant: \tau_0}}{\cdot \vdash_{\mathsf{A}} \mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\, (\mathsf{fst}\{\mathcal{U}\}\, v_1) : \tau_0}$$

2. CASE $\cdot \vdash_{\mathsf{A}} \mathsf{snd}\{\tau_0\}\, v_0 : \tau_0$
   2.1. QED
   similar to fst
3. CASE $\cdot \vdash_{\mathsf{A}} \mathsf{sum}\{\tau_1\}\, v_0\, v_1 : \tau_0$
   3.1. $\mathsf{sum}\{\tau_1\}\, v_0\, v_1 \rhd_{\mathsf{A}} \delta_A(\mathsf{sum}, v_0, v_1)$
   3.2. $\tau_0 \in \mathsf{Int} \cup \mathsf{Nat}$
   by inversion $\vdash_{\mathsf{A}}$

3.3. SCASE $\tau_0 = \mathsf{Int}$

3.3.1. $\cdot \vdash_{\mathsf{A}} v_0 : \mathsf{Int}$ and $\cdot \vdash_{\mathsf{A}} v_1 : \mathsf{Int}$

by inversion $\vdash_{\mathsf{A}}$

3.3.2. $v_0 \in i$ and $v_1 \in i$

by $\vdash_{\mathsf{A}}$ canonical forms

3.3.3. QED

$\cdot \vdash_{\mathsf{A}} \delta_A(binop, v_0, v_1) : \mathsf{Int}$

3.4. SCASE $\tau_0 = \mathsf{Nat}$

3.4.1. $\cdot \vdash_{\mathsf{A}} v_0 : \mathsf{Nat}$ and $\cdot \vdash_{\mathsf{A}} v_1 : \mathsf{Nat}$

by inversion $\vdash_{\mathsf{A}}$

3.4.2. $v_0 \in n$ and $v_1 \in n$

by $\vdash_{\mathsf{A}}$ canonical forms

3.4.3. QED

$\cdot \vdash_{\mathsf{A}} \delta_A(binop, v_0, v_1) : \mathsf{Nat}$

4. CASE $\cdot \vdash_{\mathsf{A}} \mathsf{quotient}\{\tau_1\}\, v_0\, v_1 : \tau_0$

4.1. $\mathsf{quotient}\{\tau_1\}\, v_0\, v_1 \rhd_{\mathsf{A}} \delta_A(\mathsf{quotient}, v_0, v_1)$

4.2. $\tau_0 \in \mathsf{Int} \cup \mathsf{Nat}$

by inversion $\vdash_{\mathsf{A}}$

4.3. SCASE $\tau_0 = \mathsf{Int}$

4.3.1. $\cdot \vdash_{\mathsf{A}} v_0 : \mathsf{Int}$ and $\cdot \vdash_{\mathsf{A}} v_1 : \mathsf{Int}$

by inversion $\vdash_{\mathsf{A}}$

4.3.2. $v_0 \in i$ and $v_1 \in i$

by $\vdash_{\mathsf{A}}$ canonical forms

4.3.3. QED

$\delta_A(binop, v_0, v_1) \in i \cup \mathsf{DivErr}$

4.4. SCASE $\tau_0 = \mathsf{Nat}$

4.4.1. $\cdot \vdash_{\mathsf{A}} v_0 : \mathsf{Nat}$ and $\cdot \vdash_{\mathsf{A}} v_1 : \mathsf{Nat}$

by inversion $\vdash_{\mathsf{A}}$

4.4.2. $v_0 \in n$ and $v_1 \in n$

by $\vdash_{\mathsf{A}}$ canonical forms

4.4.3. QED

$\delta_A(binop, v_0, v_1) \in n \cup \mathsf{DivErr}$

5. CASE $\cdot \vdash_{\mathsf{A}} \mathsf{fst}\{\mathcal{U}\}\, v_0 : \mathcal{U}$

5.1. SCASE $v_0 = \mathsf{trace}^{?}_{\mathsf{v}}\, \overline{b}_0\, \langle v_1, v_2 \rangle$

and $\mathsf{fst}\{\mathcal{U}\}\, v_0 \blacktriangleright_{\mathsf{A}} \textit{add-trace}\,(\overline{b}_0, v_1)$

5.1.1. $\cdot \vdash_{\mathsf{A}} v_1 : \mathcal{U}$

by inversion $\vdash_{\mathsf{A}}$

5.1.2. QED

by lemma 7.24

5.2. SCASE $v_0 = \mathsf{trace}^{?}_{\mathsf{v}}\, \overline{b}_0\, (\mathsf{mon}\, (\ell_1 \blacktriangleleft \tau_1 \times \tau_2 \blacktriangleleft \ell_2)\, v_1)$

and $\mathsf{fst}\{\mathcal{U}\}\, v_0 \blacktriangleright_{\mathsf{A}} \mathsf{trace}\, \overline{b}_0\, (\mathsf{stat}\, (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)\, (\mathsf{fst}\{\tau_1\}\, v_1))$

5.2.1. QED

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{by inversion} \vdash_{\mathsf{A}}}{\cdot \vdash_{\mathsf{A}} v_1 : \mathcal{U}}}{\cdot \vdash_{\mathsf{A}} \mathsf{fst}\{\tau_1\}\, v_1 : \mathcal{U}}}{\cdot \vdash_{\mathsf{A}} \mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)\,(\mathsf{fst}\{\tau_1\}\, v_1) : \mathcal{U}}}{\cdot \vdash_{\mathsf{A}} \mathsf{trace}\,\overline{b}_0\,(\mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)\,(\mathsf{fst}\{\tau_1\}\, v_1)) : \mathcal{U}}$$

5.3. SCASE $v_0 \notin (\mathsf{trace}^?_{\mathsf{v}}\,\overline{b}\,\langle v, v\rangle) \cup (\mathsf{trace}^?_{\mathsf{v}}\,\overline{b}\,(\mathsf{mon}\; b\; v))$ and $\mathsf{fst}\{\mathcal{U}\}\, v_0 \blacktriangleright_{\mathsf{A}} \mathsf{TagErr}\,\bullet$

5.3.1. QED

$\cdot \vdash_{\mathsf{A}} \mathsf{TagErr}\,\bullet : \mathcal{U}$

6. CASE $\cdot \vdash_{\mathsf{A}} \mathsf{snd}\{\mathcal{U}\}\, v_0 : \mathcal{U}$

6.1. QED

similar to fst

7. CASE $\cdot \vdash_{\mathsf{A}} \mathsf{sum}\{\mathcal{U}\}\, v_0\, v_1 : \mathcal{U}$

7.1. $\mathsf{sum}\{\mathcal{U}\}\, v_0\, v_1 \blacktriangleright_{\mathsf{A}} \delta_A(binop, v_0, v_1)$

7.2. $\delta_A(binop, v_0, v_1) \in i$

by definition δ_A

7.3. QED

$\cdot \vdash_{\mathsf{A}} \delta_A(binop, v_0, v_1) : \mathcal{U}$

8. CASE $\cdot \vdash_{\mathsf{A}} \mathsf{quotient}\{\mathcal{U}\}\, v_0\, v_1 : \mathcal{U}$

8.1. $\mathsf{quotient}\{\mathcal{U}\}\, v_0\, v_1 \blacktriangleright_{\mathsf{A}} \delta_A(binop, v_0, v_1)$

8.2. $\delta_A(binop, v_0, v_1) \in i \cup \mathsf{DivErr}$

by definition δ_A

8.3. QED

$\cdot \vdash_{\mathsf{A}} \delta_A(binop, v_0, v_1) : \mathcal{U}$

□

LEMMA 7.20. *If* $\cdot \vdash_{\mathsf{A}} \mathsf{dyn}\; b_0\; v_0 : \tau_0$ *and* $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ *then* $\exists\, e_1$ *such that* $\mathsf{dyn}\; b_0\; v_0 \rhd_{\mathsf{A}} e_1$.

PROOF. By case analysis of $\textit{tag-match}\,(\lfloor \tau_0 \rfloor, v_0)$.

1. CASE $\textit{tag-match}\,(\lfloor \tau_0 \rfloor, \mathsf{trace}^?_{\mathsf{v}}\,\overline{b}_2\,(\lambda x_1.\, e_1))$

1.1. QED

$\mathsf{dyn}\; b_0\; v_0 \rhd_{\mathsf{A}} \mathsf{mon}\; b_0\; v_0$

2. CASE $\textit{tag-match}\,(\lfloor \tau_0 \rfloor, \lambda(x_1 : \tau_1).\, e_1)$

2.1. CONTRADICTION:

$\cdot \vdash_{\mathsf{A}} \mathsf{dyn}\; b_0\; v_0 : \tau_0$

3. CASE $\textit{tag-match}\,(\lfloor \tau_0 \rfloor, \mathsf{trace}^?_{\mathsf{v}}\,\overline{b}_2\,(\mathsf{mon}\; b_1\; v_1))$

3.1. QED

$\mathsf{dyn}\; b_0\; v_0 \rhd_{\mathsf{A}} \mathsf{mon}\; b_0\; v_0$

4. CASE $\textit{tag-match}\,(\lfloor (\tau_1 \times \tau_2) \rfloor, \mathsf{trace}^?_{\mathsf{v}}\,\overline{b}_2\,\langle v_1, v_2\rangle)$

4.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{A}} \mathsf{mon}\ b_0\ v_0$

5. CASE *tag-match*$(\lfloor \mathsf{Int} \rfloor, \mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_1\ v_1)$

5.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{A}} v_1$

6. CASE *tag-match*$(\lfloor \mathsf{Nat} \rfloor, \mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_1\ v_1)$

6.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{A}} v_1$

7. CASE $\neg$*tag-match*$(\lfloor \tau_0 \rfloor, v_0)$

7.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{A}} \mathsf{BndryErr}\,(b_0, v_0)$

□

LEMMA 7.21. *If* $\cdot \vdash_{\mathsf{A}} \mathsf{stat}\ b_0\ v_0 : \mathcal{U}$ *and* $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ *then* $\exists\, e_1$ *such that* $\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{A}} e_1$.

PROOF. By case analysis on v_0.

1. CASE $v_0 \in \mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_2\ (\lambda x.\, e)$

1.1. CONTRADICTION:

$\cdot \vdash_{\mathsf{A}} \mathsf{stat}\ b_0\ v_0 : \mathcal{U}$

2. CASE $v_0 \in \lambda(x\!:\!\tau).\, e$

2.1. QED

$\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{A}} \mathsf{mon}\ b_0\ v_0$

3. CASE $v_0 = \mathsf{mon}\ b_1\ v_1$

3.1. SCASE $v_1 = \mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_2\ v_2$
and $v_2 \in (\lambda x.\, e) \cup \langle v, v\rangle$

3.1.1. QED

$\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{A}} \mathsf{trace}\ b_0 b_1 \overline{b}_2\ v_0$

3.2. SCASE $v_1 = \mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_2\ (\mathsf{mon}\ b_3\ v_2)$
and $v_2 \in (\lambda(x\!:\!\tau).\, v) \cup \langle v, v\rangle$

$\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{A}} \mathsf{trace}\ b_0 b_1 \overline{b}_2\ (\mathsf{mon}\ b_3\ v_2)$

4. CASE $v_0 = \langle v_1, v_2\rangle$

4.1. $\tau_0 = \tau_1 \times \tau_2$

by inversion $\vdash_{\mathsf{A}}$

4.2. QED

$\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{A}} \mathsf{mon}\ b_0\ v_0$

5. CASE $v_0 \in i$

5.1. QED

$\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{A}} v_0$

□

LEMMA 7.22. *If* $\cdot \vdash_{\mathsf{A}} \mathsf{dyn}\ b_0\ v_0 : \tau_0$ *and* $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ *and* $\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{A}} e_1$ *then* $\cdot \vdash_{\mathsf{A}} e_1 : \tau_0$.

PROOF. By case analysis of $\rhd_{\mathsf{A}}$.

1. CASE $\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{A}} \mathsf{mon}\ b_0\ v_0$
   1.1. QED

$$\dfrac{\dfrac{\text{by inversion } \vdash_{\mathsf{A}}}{\cdot \vdash_{\mathsf{A}} v_0 : \mathcal{U}}}{\cdot \vdash_{\mathsf{A}} \mathsf{mon}\ b_0\ v_0 : \tau_0}$$

2. CASE $\mathsf{dyn}\ b_0\ \mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_1\ i_0 \rhd_{\mathsf{A}} i_0$
   2.1. QED
   by case analysis of *tag-match*$(\lfloor \tau_0 \rfloor, i_0)$
3. CASE $\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{A}} \mathsf{BndryErr}\,(b_0, v_0)$
   3.1. QED
   $\cdot \vdash_{\mathsf{A}} \mathsf{BndryErr}\,(b_0, v_0) : \tau_0$

□

LEMMA 7.23. *If* $\cdot \vdash_{\mathsf{A}} \mathsf{stat}\ b_0\ v_0 : \mathcal{U}$ *and* $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ *and* $\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{A}} e_1$ *then* $\cdot \vdash_{\mathsf{A}} e_1$.

PROOF. By case analysis of $\blacktriangleright_{\mathsf{A}}$.

1. CASE $v_0 \in (\lambda(x:\tau).\, e) \cup (\langle v, v \rangle)$
   and $\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{A}} \mathsf{mon}\ b_0\ v_0$
   1.1. QED

$$\dfrac{\dfrac{\text{by inversion } \vdash_{\mathsf{A}}}{\cdot \vdash_{\mathsf{A}} v_0 : \tau_0}}{\cdot \vdash_{\mathsf{A}} \mathsf{mon}\ b_0\ v_0 : \mathcal{U}}$$

2. CASE $v_0 \in (\lambda x.\, e) \cup (\langle v, v \rangle)$
   and $\mathsf{stat}\ b_0\ (\mathsf{mon}\ b_1\ (\mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_2\ v_0)) \blacktriangleright_{\mathsf{A}} \mathsf{trace}\ b_0 b_1 \overline{b}_2\ v_0$
   2.1. QED

$$\dfrac{\dfrac{\text{by inversion } \vdash_{\mathsf{A}}}{\cdot \vdash_{\mathsf{A}} v_0 : \mathcal{U}}}{\cdot \vdash_{\mathsf{A}} \mathsf{trace}\ b_0 b_1 \overline{b}_2\ v_0 : \mathcal{U}}$$

3. CASE $v_0 \in (\lambda(x:\tau).\, e) \cup (\langle v, v \rangle)$
   and $\mathsf{stat}\ b_0\ (\mathsf{mon}\ b_1\ (\mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_2\,(\mathsf{mon}\ b_3\ v_0))) \blacktriangleright_{\mathsf{A}} \mathsf{trace}\ b_0 b_1 \overline{b}_2\,(\mathsf{mon}\ b_3\ v_0)$
   and $b_3 = (\ell_4 \blacktriangleleft \tau_3 \blacktriangleleft \ell_5)$
   3.1. QED

$$\dfrac{\dfrac{\dfrac{\text{by inversion } \vdash_{\mathsf{A}}}{\cdot \vdash_{\mathsf{A}} v_0 : \tau_3}}{\cdot \vdash_{\mathsf{A}} \mathsf{mon}\ b_3\ v_0 : \mathcal{U}}}{\cdot \vdash_{\mathsf{A}} \mathsf{trace}\ b_0 b_1 \overline{b}_2\,(\mathsf{mon}\ b_3\ v_0) : \mathcal{U}}$$

4. CASE $\mathsf{stat}\ b_0\ i_0 \blacktriangleright_{\mathsf{A}} i_0$

   4.1. QED

   $\cdot \vdash_{\mathsf{A}} i_0 : \mathcal{U}$

□

LEMMA 7.24. *If* $\cdot \vdash_{\mathsf{A}} \mathsf{trace}\,\overline{b_0}\, v_0 : \mathcal{U}$ *then* $\cdot \vdash_{\mathsf{A}} \mathit{add\text{-}trace}(\overline{b_0}, v_0) : \mathcal{U}$.

PROOF. By case analysis of *add-trace*.

1. CASE $\mathit{add\text{-}trace}(\cdot, v_0) = v_0$

   1.1. QED

2. CASE $\mathit{add\text{-}trace}(\overline{b_0}, (\!(\mathsf{trace}_{\mathsf{v}}\,\overline{b_1}\, v_1)\!)^{\overline{\ell_2}}) = \mathsf{trace}_{\mathsf{v}}\,\overline{b_0}\overline{b_1}\,(\!(v_1)\!)^{\overline{\ell_2}}$

   2.1. QED

   by $\cdot \vdash_{\mathsf{A}} v_1 : \mathcal{U}$

3. CASE $\mathit{add\text{-}trace}(\overline{b_0}, v_1) = \mathsf{trace}_{\mathsf{v}}\,\overline{b_0}\, v_1$

   3.1. QED

   by $\cdot \vdash_{\mathsf{A}} v_1 : \mathcal{U}$

□

LEMMA 7.25.

- *If* $(x_0 : \tau_0), \Gamma_0 \vdash_{\mathsf{A}} e_1 : \tau?$ *and* $\cdot \vdash_{\mathsf{A}} v_0 : \tau_0$ *then* $\Gamma_0 \vdash_{\mathsf{A}} e_1[x_0 \leftarrow v_0] : \tau?$
- *If* $(x_0 : \mathcal{U}), \Gamma_0 \vdash_{\mathsf{A}} e_1 : \tau?$ *and* $\cdot \vdash_{\mathsf{A}} v_0 : \mathcal{U}$ *then* $\Gamma_0 \vdash_{\mathsf{A}} e_1[x_0 \leftarrow v_0] : \tau?$

PROOF. By induction on e_1.

1. CASE $e_1 = x_2$

   1.1. SCASE $x_0 = x_2$

      1.1.1. QED

      $e_1[x_0 \leftarrow v_0] = v_0$

   1.2. SCASE $x_0 \neq x_2$

      1.2.1. QED

      $e_1[x_0 \leftarrow v_0] = e_1$

2. CASE $e_1 = i_0$

   2.1. QED

   $e_1[x_0 \leftarrow v_0] = e_1$

3. CASE $e_1 = \lambda x_2.\, e_2$

   3.1. SCASE $x_0 = x_2$

      3.1.1. QED

      by the induction hypothesis

   3.2. SCASE $x_0 \neq x_2$

      3.2.1. QED

      $e_1[x_0 \leftarrow v_0] = e_1$

4. CASE $e_1 = \lambda(x_2 : \tau_2).\, e_2$

   4.1. SCASE $x_0 = x_2$

4.1.1. QED

by the induction hypothesis

4.2. SCASE $x_0 \neq x_2$

4.2.1. QED

$e_1[x_0 \leftarrow v_0] = e_1$

5. CASE $e_1 = \langle e_2, e_3 \rangle$

5.1. QED

by the induction hypothesis

6. CASE $e_1 = \mathsf{app}\{\tau?\}\, e_2\, e_3$

6.1. QED

by the induction hypothesis

7. CASE $e_1 = \mathit{unop}\{\tau?\}\, e_2$

7.1. QED

by the induction hypothesis

8. CASE $e_1 = \mathit{binop}\{\tau?\}\, e_2\, e_3$

8.1. QED

by the induction hypothesis

9. CASE $e_1 = \mathsf{dyn}\, b_2\, e_2$

9.1. QED

by the induction hypothesis

10. CASE $e_1 = \mathsf{stat}\, b_2\, e_2$

10.1. QED

by the induction hypothesis

11. CASE $e_1 = \mathsf{trace}_{\mathsf{v}}\, \overline{b}_2\, v_2$

11.1. QED

by the induction hypothesis

12. CASE $e_1 = \mathsf{trace}\, \overline{b}_2\, e_2$

12.1. QED

by the induction hypothesis

□

LEMMA 7.26 ($\Vdash_{\mathsf{A}}$-PROGRESS). *If $\cdot \vdash_{\mathsf{A}} e_0 : \tau?$ and $\cdot; \ell_\bullet \Vdash_{\mathsf{A}} e_0$ then one of the following holds:*

- $e_0 \in (\!(v)\!)^{\overline{\ell}}$
- $e_0 \in E[\mathsf{Err}]^{\ell}$
- $\exists\, e_1$ *such that* $e_0 \rightarrow_{\mathsf{A}} e_1$

PROOF. By case analysis of e_0.

By lemma 7.30, it suffices to consider the following cases.

1. CASE $e_0 \in (\!(v)\!)^{\overline{\ell}}$

1.1. QED

2. CASE $e_0 \in E[\mathsf{Err}]^{\ell}$

2.1. QED

3. CASE $e_0 = E[\mathsf{app}\{\tau_0\}\ (\!(v_0)\!)^{\overline{\ell_0}}\ v_1]^{\ell_1}$

 3.1. $v_0 \in (\lambda(x\!:\!\tau).\,e) \cup (\mathsf{mon}\ b\ v)$

  by $\vdash_{\mathsf{A}}$ inversion and canonical forms

 3.2. SCASE $v_0 = \lambda(x_2\!:\!\tau_2).\,e_2$

  3.2.1. QED

   $e_0 \rhd_{\mathsf{A}} E[(\!(e_2[x_2 \leftarrow (v_1)^{\ell_1 rev(\overline{\ell_0})}])\!)^{\overline{\ell_0}}]^{\ell_1}$

 3.3. SCASE $v_0 = \mathsf{mon}\ (\ell_2 \blacktriangleleft (\tau_2 \Rightarrow \tau_3) \blacktriangleleft \ell_3)\ (v_2)^{\ell_4}$

  3.3.1. LET $b_3 = (\ell_2 \blacktriangleleft \tau_3 \blacktriangleleft \ell_3)$

   and $b_4 = (\ell_3 \blacktriangleleft \tau_2 \blacktriangleleft \ell_2)$

  3.3.2. QED

   $e_0 \rhd_{\mathsf{A}} E[(\!(\mathsf{dyn}\ b_3\ (\mathsf{app}\{\mathcal{U}\}\ v_2\ (\mathsf{stat}\ b_4\ (\!(v_1)\!)^{\ell_1 \ell_1 rev(\overline{\ell_0})}))^{\ell_4})\!)^{\overline{\ell_0}}]^{\ell_1}$

4. CASE $e_0 = E[\mathsf{app}\{\mathcal{U}\}\ (\!(v_0)\!)^{\overline{\ell_0}}\ v_1]^{\ell_1}$

 4.1. SCASE $v_0 = \mathsf{trace}^?_{\mathsf{v}}\ \overline{b_2}\ (\!(\lambda x_2.\,e_2)\!)^{\overline{\ell_3}}$

  4.1.1. LET $v_2 = \mathit{add\text{-}trace}(rev(\overline{b_2}), (\!(v_1)\!)^{\ell_1 rev(\overline{\ell_0}) rev(\overline{\ell_3})})$

  4.1.2. QED

   $e_0 \blacktriangleright_{\mathsf{A}} E[(\mathsf{trace}\ \overline{b_2}\ (\!(e_2[x_2 \leftarrow v_1])\!)^{\overline{\ell_3}})^{\overline{\ell_0}}]^{\ell_1}$

 4.2. SCASE $v_0 = \mathsf{trace}^?_{\mathsf{v}}\ \overline{b_2}\ (\!(\mathsf{mon}\ (\ell_3 \blacktriangleleft (\tau_2 \Rightarrow \tau_3)^{\ell_4} \blacktriangleleft \ell_4)\ (v_2)^{\ell_5})\!)^{\overline{\ell_6}}$

  4.2.1. LET $b_7 = (\ell_3 \blacktriangleleft \tau_3 \blacktriangleleft \ell_4)$

   and $b_8 = (\ell_4 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3)$

   and $\tau_4 = \mathit{forget}(\tau_3)$

  4.2.2. QED

   $e_0 \blacktriangleright_{\mathsf{A}} E[(\!(\mathsf{trace}\ \overline{b_2}\ (\!(\mathsf{stat}\ b_7\ (\mathsf{app}\{\tau_4\}\ v_2\ (\mathsf{dyn}\ b_8\ (\!(v_2)\!)^{last(\overline{\ell_6})}))^{\ell_3})\!)^{\overline{\ell_6}})\!)^{\overline{\ell_0}}]^{\ell_1}$

 4.3. SCASE $v_0 \notin (\lambda x.\,e) \cup (\mathsf{mon}\ b\ v)$

  4.3.1. QED

   $e_0 \blacktriangleright_{\mathsf{A}} E[\mathsf{TagErr}\ \bullet]^{\ell_0}$

5. CASE $e_0 = E[\mathit{unop}\{\tau?\}\ (\!(v_0)\!)^{\overline{\ell_0}}]^{\ell_1}$

 5.1. QED

  by lemma 7.16 and lemma 7.33

6. CASE $e_0 = E[\mathit{binop}\{\tau?\}\ (\!(v_0)\!)^{\overline{\ell_0}}\ (\!(v_1)\!)^{\overline{\ell_1}}]^{\ell_2}$

 6.1. QED

  by lemma 7.16 and lemma 7.33

7. CASE $e_0 = E[\mathsf{dyn}\ b_0\ (\!(v_1)\!)^{\overline{\ell_1}}]^{\ell_2}$

 7.1. QED

  by lemma 7.16 and lemma 7.35

8. CASE $e_0 = E[\mathsf{stat}\ b_0\ (\!(v_1)\!)^{\ell_1}]^{\ell_2}$

 8.1. QED

  by lemma 7.16 and lemma 7.36

9. CASE $e_0 = E[\mathsf{trace}\ \overline{b_0}\ v_0]^{\ell_2}$

9.1. QED

$e_0 \blacktriangleright_{\mathsf{A}} E[\mathit{add\text{-}trace}(\overline{b}_0, v_0)]^{\ell_2}$

□

Lemma 7.27 ($\Vdash_{\mathsf{A}}$-preservation). *If* $\cdot \vdash_{\mathsf{A}} e_0 : \tau?$ *and* $\cdot; \ell_\bullet \Vdash_{\mathsf{A}} e_0$ *and* $e_0 \rightarrow_{\mathsf{A}} e_1$ *then* $\cdot; \ell_\bullet \Vdash_{\mathsf{A}} e_1$

Proof. By lemma 7.28 and lemma 7.29.

□

Lemma 7.28. *If* $\cdot \vdash_{\mathsf{A}} e_0 : \tau_0$ *and* $\cdot; \ell_0 \Vdash_{\mathsf{A}} e_0$ *and* $e_0 \rhd_{\mathsf{A}} e_1$ *then* $\cdot; \ell_0 \Vdash_{\mathsf{A}} e_1$

Proof. By case analysis of $\rhd_{\mathsf{A}}$.

1. case $\delta_A(\mathit{unop}, v_0)$ is defined
   and $(\mathit{unop}\{\tau_1\}\ v_0)^{\ell_0} \rhd_{\mathsf{A}} (\delta_A(\mathit{unop}, v_0))^{\ell_0}$
   1.1. QED
   by lemma 7.34
2. case $\delta_A(\mathit{binop}, v_0, v_1)$ is defined
   and $(\mathit{binop}\{\tau_1\}\ v_0\ v_1)^{\ell_0} \rhd_{\mathsf{A}} (\delta_A(\mathit{binop}, v_0, v_1))^{\ell_0}$
   2.1. QED
   by lemma 7.34
3. case $(\mathsf{app}\{\tau_0\}\ (\!(\lambda(x_0\!:\!\tau_1).\, e_0)\!)^{\overline{\ell}_0}\ v_1)^{\ell_1} \rhd_{\mathsf{A}} (e_0[x_0 \leftarrow (\!(v_1)\!)^{\ell_1 rev(\overline{\ell}_0)}])^{\overline{\ell}_0 \ell_1}$
   3.1. $\overline{\ell}_0 = \ell_1 \cdots \ell_1$
   by inversion $\Vdash_{\mathsf{A}}$
   3.2. $\cdot; \ell_0 \Vdash_{\mathsf{A}} (\!(v_1)\!)^{\ell_1 rev(\overline{\ell}_0)}$
   by 3.1 and inversion $\Vdash_{\mathsf{A}}$
   3.3. QED
   by lemma 7.40

$$\dfrac{\dfrac{\text{by lemma 7.40}}{\cdot; \ell_1 \Vdash_{\mathsf{A}} e_0[x_0 \leftarrow (\!(v_1)\!)^{\ell_1 rev(\overline{\ell}_0)}]}}{\cdot; \ell_1 \Vdash_{\mathsf{A}} (e_0[x_0 \leftarrow (\!(v_1)\!)^{\ell_1 rev(\overline{\ell}_0)}])^{\overline{\ell}_0 \ell_1}}$$

4. case $(\mathsf{app}\{\tau_0\}\ (\!(\mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2)^{\ell_1} \blacktriangleleft \ell_1)\ (v_0)^{\ell_2})\!)^{\overline{\ell}_3}\ v_1)^{\ell_4}$
   $\rhd_{\mathsf{A}} (\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\mathcal{U}\}\ v_0\ (\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ (\!(v_1)\!)^{\ell_4 rev(\overline{\ell}_3)}))^{\ell_2})^{\overline{\ell}_3 \ell_4}$
   4.1. $\ell_1 = \ell_2$
   and $\overline{\ell}_3 = \ell_4 \cdots \ell_4$
   by inversion $\Vdash_{\mathsf{A}}$
   4.2. QED

$$\dfrac{\dfrac{\dfrac{\text{by inversion } \Vdash_{\mathsf{A}}}{\cdot; \ell_1 \Vdash_{\mathsf{A}} v_0} \qquad \dfrac{\dfrac{\dfrac{\text{by inversion } \Vdash_{\mathsf{A}}}{\cdot; \ell_0 \Vdash_{\mathsf{A}} v_1}}{\cdot; \ell_0 \Vdash_{\mathsf{A}} (\!(v_1)\!)^{\ell_4 rev(\overline{\ell}_3)}}}{\cdot; \ell_1 \Vdash_{\mathsf{A}} \mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\; (\!(v_1)\!)^{\ell_4 rev(\overline{\ell}_3)}}}{\dfrac{\cdot; \ell_1 \Vdash_{\mathsf{A}} \mathsf{app}\{\mathcal{U}\}\; v_0\; (\mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\; (\!(v_1)\!)^{\ell_4 rev(\overline{\ell}_3)})}{\dfrac{\cdot; \ell_1 \Vdash_{\mathsf{A}} (\mathsf{app}\{\mathcal{U}\}\; v_0\; (\mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\; (\!(v_1)\!)^{\ell_4 rev(\overline{\ell}_3)}))^{\ell_2}}{\cdot; \ell_0 \Vdash_{\mathsf{A}} \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; (\mathsf{app}\{\mathcal{U}\}\; v_0\; (\mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\; (\!(v_1)\!)^{\ell_4 rev(\overline{\ell}_3)}))^{\ell_2}}}}}{\cdot; \ell_0 \Vdash_{\mathsf{A}} (\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; (\mathsf{app}\{\mathcal{U}\}\; v_0\; (\mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\; (\!(v_1)\!)^{\ell_4 rev(\overline{\ell}_3)}))^{\ell_2})^{\overline{\ell}_3 \ell_4}}$$

5. CASE $(\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; (\!(v_0)\!)^{\overline{\ell}_2})^{\ell_3} \rhd_{\mathsf{A}} e_2$
   by lemma 7.37

□

LEMMA 7.29. *If* $\cdot \vdash_{\mathsf{A}} e_0 : \mathcal{U}$ *and* $\cdot; \ell_0 \Vdash_{\mathsf{A}} e_0$ *and* $e_0 \blacktriangleright_{\mathsf{A}} e_1$ *then* $\cdot; \ell_0 \Vdash_{\mathsf{A}} e_1$

PROOF. By case analysis of $\blacktriangleright_{\mathsf{A}}$.

1. CASE $v_0 \notin \mathsf{mon}\,(\ell \blacktriangleleft \tau \times \tau \blacktriangleleft \ell)\; v$
   and $\delta_A(unop, v_0)$ is not defined
   and $(unop\{\mathcal{U}\}\; v_0)^{\ell_0} \blacktriangleright_{\mathsf{A}} (\mathsf{TagErr}\,\bullet)^{\ell_0}$
   1.1. QED
2. CASE $\delta_A(unop, v_0)$ is defined
   and $(unop\{\mathcal{U}\}\; v_0)^{\ell_0} \blacktriangleright_{\mathsf{A}} (\delta_A(unop, v_0))^{\ell_0}$
   2.1. QED
   by lemma 7.34
3. CASE $(\mathsf{fst}\{\mathcal{U}\}\; (\!(\mathsf{trace}^{?}_{\mathsf{v}}\; \overline{b}_0\; (\!(\mathsf{mon}\,(\ell_1 \blacktriangleleft (\tau_0 \times \tau_1)^{\ell_2} \blacktriangleleft \ell_2)\; (v_1)^{\ell_3})\!)^{\overline{\ell}_4})\!)^{\overline{\ell}_5})^{\ell_6} \blacktriangleright_{\mathsf{A}}$
   $(\mathsf{trace}\; \overline{b}_0\; (\!(\mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\; (\mathsf{fst}\{\tau_0\}\; v_1)^{\ell_3})\!)^{\overline{\ell}_4})^{\overline{\ell}_5 \ell_6}$
   3.1. $\overline{\ell}_5 = \ell_6 \cdots \ell_6$
   and $\overline{b}_0 \simeq \overline{\ell}_4$
   and $last(\overline{\ell}_4) = \ell_1$
   and $\ell_2 = \ell_3$
   by inversion $\Vdash_{\mathsf{A}}$
   3.2. QED

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{by inversion } \Vdash_{\mathsf{A}}}{\cdot; \ell_2 \Vdash_{\mathsf{A}} v_1}}{\cdot; \ell_2 \Vdash_{\mathsf{A}} \mathsf{fst}\{\tau_0\}\, v_1}}{\cdot; \ell_2 \Vdash_{\mathsf{A}} (\mathsf{fst}\{\tau_0\}\, v_1)^{\ell_3}}}{\cdot; \mathit{last}(\overline{\ell}_4) \Vdash_{\mathsf{A}} \mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\ (\mathsf{fst}\{\tau_0\}\, v_1)^{\ell_3}}}{\cdot; \ell_6 \Vdash_{\mathsf{A}} \mathsf{trace}\, \overline{b}_0\ (\!(\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\ (\mathsf{fst}\{\tau_0\}\, v_1)^{\ell_3})\!)^{\overline{\ell}_4}}}{\cdot; \ell_6 \Vdash_{\mathsf{A}} (\mathsf{trace}\, \overline{b}_0\ (\!(\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\ (\mathsf{fst}\{\tau_0\}\, v_1)^{\ell_3})\!)^{\overline{\ell}_4})^{\overline{\ell}_5 \ell_6}}$$

4. CASE $(\mathsf{snd}\{\mathcal{U}\}\ (\!(\mathsf{trace}^?_{\mathsf{v}}\, \overline{b}_0\ (\!(\mathsf{mon}\ (\ell_1 \blacktriangleleft (\tau_0 \times \tau_1)^{\ell_2} \blacktriangleleft \ell_2)\ (v_1)^{\ell_3})\!)^{\overline{\ell}_4})\!)^{\overline{\ell}_5})^{\ell_6} \blacktriangleright_{\mathsf{A}}$

   $(\mathsf{trace}\, \overline{b}_0\ (\!(\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)\ (\mathsf{snd}\{\tau_1\}\, v_1)^{\ell_3})\!)^{\overline{\ell}_4})^{\overline{\ell}_5 \ell_6}$

   4.1. QED

   similar to fst

5. CASE $\delta_A(\mathit{binop}, v_0, v_1)$ is not defined

   and $(\mathit{binop}\{\mathcal{U}\}\ v_0\ v_1)^{\ell_0} \blacktriangleright_{\mathsf{A}} (\mathsf{TagErr}\, \bullet)^{\ell_0}$

   5.1. QED

6. CASE $\delta_A(\mathit{binop}, v_0, v_1)$ is defined

   and $(\mathit{binop}\{\mathcal{U}\}\ v_0\ v_1)^{\ell_0} \blacktriangleright_{\mathsf{A}} (\delta_A(\mathit{binop}, v_0, v_1))^{\ell_0}$

   6.1. QED

   by lemma 7.34

7. CASE $(\mathsf{app}\{\mathcal{U}\}\ (\!(\mathsf{trace}^?_{\mathsf{v}}\, \overline{b}_0\ (\!(\lambda x_0.\, e_0)\!)^{\overline{\ell}_1})\!)^{\overline{\ell}_2}\ v_1)^{\ell_3} \blacktriangleright_{\mathsf{A}}$

   $(\mathsf{trace}\, \overline{b}_0\ (\!(e_0[x_0 \leftarrow \mathit{add\text{-}trace}(\mathit{rev}(\overline{b}_0), (\!(v_1)\!)^{\ell_3 \mathit{rev}(\overline{\ell}_2) \mathit{rev}(\overline{\ell}_1)})])\!)^{\overline{\ell}_1})^{\overline{\ell}_2 \ell_3}$

   7.1. $\overline{\ell}_2 = \ell_3 \cdots \ell_3$

   and $\overline{b}_0 \simeq \overline{\ell}_1$

   and $\cdot; \mathit{last}(\overline{\ell}_1) \Vdash_{\mathsf{A}} \lambda x_0.\, e_0$

   7.2. $\cdot; \mathit{last}(\overline{\ell}_1) \Vdash_{\mathsf{A}} \mathit{add\text{-}trace}(\mathit{rev}(\overline{b}_0), (\!(v_1)\!)^{\ell_3 \mathit{rev}(\overline{\ell}_2) \mathit{rev}(\overline{\ell}_1)})$

   7.3. QED

$$\dfrac{\dfrac{\dfrac{\text{by lemma 7.40}}{\cdot; \mathit{last}(\overline{\ell}_1) \Vdash_{\mathsf{A}} e_0[x_0 \leftarrow \mathit{add\text{-}trace}(\mathit{rev}(\overline{b}_0), (\!(v_1)\!)^{\ell_3 \mathit{rev}(\overline{\ell}_2) \mathit{rev}(\overline{\ell}_1)})]}}{\cdot; \ell_3 \Vdash_{\mathsf{A}} \mathsf{trace}\, \overline{b}_0\ (\!(e_0[x_0 \leftarrow \mathit{add\text{-}trace}(\mathit{rev}(\overline{b}_0), (\!(v_1)\!)^{\ell_3 \mathit{rev}(\overline{\ell}_2) \mathit{rev}(\overline{\ell}_1)})])\!)^{\overline{\ell}_1}}}{\cdot; \ell_3 \Vdash_{\mathsf{A}} (\mathsf{trace}\, \overline{b}_0\ (\!(e_0[x_0 \leftarrow \mathit{add\text{-}trace}(\mathit{rev}(\overline{b}_0), (\!(v_1)\!)^{\ell_3 \mathit{rev}(\overline{\ell}_2) \mathit{rev}(\overline{\ell}_1)})])\!)^{\overline{\ell}_1})^{\overline{\ell}_2 \ell_3}}$$

8. CASE $(\mathsf{app}\{\mathcal{U}\}\ (\!(\mathsf{trace}^?_{\mathsf{v}}\, \overline{b}_0\ (\!(\mathsf{mon}\ (\ell_1 \blacktriangleleft (\tau_0 \Rightarrow \tau_1)^{\ell_2} \blacktriangleleft \ell_2)\ (v_0)^{\ell_3})\!)^{\overline{\ell}_4})\!)^{\overline{\ell}_5}\ v_1)^{\ell_6} \blacktriangleright_{\mathsf{A}}$

   $(\mathsf{trace}\, \overline{b}_0\ (\!(\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)\ (\mathsf{app}\{\tau_1\}\ v_0\ (\mathsf{dyn}\ (\ell_2 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (v_2)^{\mathit{last}(\overline{\ell}_4)}))^{\ell_3})\!)^{\overline{\ell}_4})^{\overline{\ell}_5 \ell_6}$

8.1. $\overline{\ell}_5 = \ell_6 \cdots \ell_6$
and $\overline{b}_0 \simeq \overline{\ell}_4$
and $\ell_1 = \mathit{last}(\overline{\ell}_4)$
and $\ell_2 = \ell_3$
by inversion $\Vdash_A$

8.2. QED

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{by inversion } \Vdash_A}{\cdot; \ell_2 \Vdash_A v_0} \quad \dfrac{\dfrac{\text{by inversion } \Vdash_A}{\cdot; \ell_1 \Vdash_A v_2}}{\cdot; \ell_2 \Vdash_A \mathsf{dyn}\ (\ell_2 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (v_2)^{\mathit{last}(\overline{\ell}_4)}}}{\cdot; \ell_2 \Vdash_A \mathsf{app}\{\tau_1\}\ v_0\ (\mathsf{dyn}\ (\ell_2 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (v_2)^{\mathit{last}(\overline{\ell}_4)})}}{\cdot; \mathit{last}(\overline{\ell}_4) \Vdash_A \mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)\ (\mathsf{app}\{\tau_1\}\ v_0\ (\mathsf{dyn}\ (\ell_2 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (v_2)^{\mathit{last}(\overline{\ell}_4)}))^{\ell_3}}}{\cdot; \ell_6 \Vdash_A \mathsf{trace}\ \overline{b}_0\ (\!(\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)\ (\mathsf{app}\{\tau_1\}\ v_0\ (\mathsf{dyn}\ (\ell_2 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (v_2)^{\mathit{last}(\overline{\ell}_4)}))^{\ell_3})\!)^{\overline{\ell}_4}}}{\cdot; \ell_6 \Vdash_A (\mathsf{trace}\ \overline{b}_0\ (\!(\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)\ (\mathsf{app}\{\tau_1\}\ v_0\ (\mathsf{dyn}\ (\ell_2 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (v_2)^{\mathit{last}(\overline{\ell}_4)}))^{\ell_3})\!)^{\overline{\ell}_4})^{\overline{\ell}_5 \ell_6}}$$

9. CASE $(\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\ (\!(v_0)\!)^{\ell_3})^{\ell_0} \blacktriangleright_A e_2$

9.1. QED
by lemma 7.38

10. CASE $(\mathsf{trace}\ \overline{b}_0\ v_0)^{\ell_1}$
$\blacktriangleright_A (\mathit{add\text{-}trace}(\overline{b}_0, v_0))^{\ell_1}$

10.1. QED
by lemma 7.39

□

LEMMA 7.30. *If* $\cdot \vdash_A e_0 : \tau?$ *and* $\cdot; \ell_0 \Vdash_A e_0$ *then either:*

- $e_0 \in (\!(v)\!)^{\overline{\ell}}$
- $e_0 = E_0[\mathsf{app}\{\tau?\}\ (v_0)^{\overline{\ell}_0}\ (v_1)^{\ell_1}]^{\ell_2}$
- $e_0 = E_0[\mathit{unop}\{\tau?\}\ (\!(v_0)\!)^{\overline{\ell}_0}]^{\ell_1}$
- $e_0 = E_0[\mathit{binop}\{\tau?\}\ (\!(v_0)\!)^{\overline{\ell}_0}\ (\!(v_1)\!)^{\overline{\ell}_1}]^{\ell_2}$
- $e_0 = E_0[\mathsf{dyn}\ b_1\ (\!(v_1)\!)^{\overline{\ell}_0}]^{\ell_1}$
- $e_0 = E_0[\mathsf{stat}\ b_1\ (\!(v_1)\!)^{\overline{\ell}_0}]^{\ell_1}$
- $e_0 = E_0[\mathsf{stat}\ b_1\ (\!(v_1)\!)^{\overline{\ell}_0}]^{\ell_1}$
- $e_0 = E_0[\mathsf{trace}\ \overline{b}_1\ (\!(v_1)\!)^{\overline{\ell}_0}]^{\ell_1}$
- $e_0 = E_0[\mathsf{Err}]^{\ell_0}$

PROOF. By induction on the structure of e_0.

1. CASE $e_0 = x_0$

1.1. CONTRADICTION:
$\cdot; \ell_0 \Vdash_A e_0$

2. CASE $e_0 \in (\!|v|\!)^{\overline{\ell}}$
   2.1. QED
3. CASE $e_0 = \langle e_1, e_2 \rangle$
   3.1. $\cdot \vdash_{\mathsf{A}} e_1 : \tau?$ and $\cdot \vdash_{\mathsf{A}} e_2 : \tau?$
      by inversion $\vdash_{\mathsf{A}}$
   3.2. $\cdot; \ell_0 \Vdash_{\mathsf{A}} e_1$ and $\cdot; \ell_0 \Vdash_{\mathsf{A}} e_2$
      by inversion $\Vdash_{\mathsf{A}}$
   3.3. SCASE $e_1 \notin (\!|v|\!)^{\overline{\ell}}$
      3.3.1. QED
         by the induction hypothesis
   3.4. SCASE $e_1 \in (\!|v|\!)^{\overline{\ell}}$ and $e_2 \notin (\!|v|\!)^{\overline{\ell}}$
      3.4.1. QED
         by the induction hypothesis
   3.5. SCASE $e_1 \in (\!|v|\!)^{\overline{\ell}}$ and $e_2 \in (\!|v|\!)^{\overline{\ell}}$
      3.5.1. QED
         $e_0 \in v$
4. CASE $e_0 = \mathsf{app}\{\tau?\}\, e_1\, e_2$
   4.1. QED
      by the induction hypothesis
5. CASE $e_0 = \mathit{unop}\{\tau?\}\, e_1$
   5.1. QED
      by the induction hypothesis
6. CASE $e_0 = \mathit{binop}\{\tau?\}\, e_1\, e_2$
   6.1. QED
      by the induction hypothesis
7. CASE $e_0 = \mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, (e_1)^{\ell_1}$
   7.1. $\cdot; \ell_1 \Vdash_{\mathsf{A}} e_1$
      by inversion $\Vdash_{\mathsf{A}}$
   7.2. QED
      by the induction hypothesis
8. CASE $e_0 = \mathsf{stat}\, b_1\, e_1$
   8.1. QED
      by the induction hypothesis
9. CASE $e_0 = \mathsf{trace}\, \overline{b}_1\, (\!|e_1|\!)^{\overline{\ell}_2}$
   9.1. QED
      by the induction hypothesis

□

LEMMA 7.31. *If* $\cdot; \ell_0 \Vdash_{\mathsf{A}} E_0[e_0]$ *then* $\exists\, \ell_1$ *such that* $\cdot; \ell_1 \Vdash_{\mathsf{A}} e_0$

PROOF. By induction on the structure of E_0.

1. $E_0 = [\,]$

1.1. QED

2. $E_0 = \langle E_1, e_2 \rangle$

2.1. QED

by the induction hypothesis

3. $E_0 = \langle e_1, E_2 \rangle$

3.1. QED

by the induction hypothesis

4. $E_0 = \mathit{unop}\{\tau?\}\, E_1$

4.1. QED

by the induction hypothesis

5. $E_0 = \mathit{binop}\{\tau?\}\, E_1\, e_2$

5.1. QED

by the induction hypothesis

6. $E_0 = \mathit{binop}\{\tau?\}\, e_1\, E_2$

6.1. QED

by the induction hypothesis

7. $E_0 = \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\,(E_1)^{\ell_3}$

7.1. $\cdot; \ell_2 \Vdash_{\mathsf{A}} E_1[e_0]$

7.2. QED

by the induction hypothesis

8. $E_0 = \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\,(E_1)^{\ell_3}$

8.1. QED

by the induction hypothesis

9. $E_0 = (E_1)^{\ell_0}$

9.1. QED

by the induction hypothesis

10. $E_0 = \mathsf{trace}\,\overline{b}_1\,(\!(E_1)\!)^{\overline{\ell}_1}$

10.1. QED

by the induction hypothesis

□

Lemma 7.32 ($\Vdash_{\mathsf{A}}$ replacement). *If* $\cdot; \ell_0 \Vdash_{\mathsf{A}} E_0[e_0]$ *and the derivation contains a proof of* $\cdot; \ell_1 \Vdash_{\mathsf{A}} e_0$ *and* $\cdot; \ell_1 \Vdash_{\mathsf{A}} e_1$ *then* $L_0; \ell_0 \Vdash_{\mathsf{A}} E_0[e_1]$

Proof. By induction on the structure of E_0.

1. $E_0 = [\,]$

1.1. QED

2. $E_0 = \langle E_1, e_2 \rangle$

2.1. QED

by the induction hypothesis

3. $E_0 = \langle e_1, E_2 \rangle$

3.1. QED

by the induction hypothesis

4. $E_0 = unop\{\tau?\}\, E_1$

4.1. QED

by the induction hypothesis

5. $E_0 = binop\{\tau?\}\, E_1\, e_2$

5.1. QED

by the induction hypothesis

6. $E_0 = binop\{\tau?\}\, e_1\, E_2$

6.1. QED

by the induction hypothesis

7. $E_0 = \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\,(E_1)^{\ell_3}$

7.1. $\cdot; \ell_2 \Vdash_{\mathsf{A}} E_1[e_0]$

7.2. QED

by the induction hypothesis

8. $E_0 = \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\,(E_1)^{\ell_3}$

8.1. QED

by the induction hypothesis

9. $E_0 = (E_1)^{\ell_0}$

9.1. QED

by the induction hypothesis

10. $E_0 = \mathsf{trace}\,\overline{b}_1\,(\!(E_1)\!)^{\overline{\ell}_1}$

10.1. QED

by the induction hypothesis

□

Lemma 7.33 (δ_A label progress).

- *If* $\cdot \vdash_{\mathsf{A}} unop\{\tau_1\}\, v_0 : \tau_0$ *and* $\cdot; \ell_0 \Vdash_{\mathsf{A}} unop\{\tau_1\}\, v_0$ *and* $(unop\{\tau_1\}\, v_0)^{\ell_0} \rhd_{\mathsf{A}} (e_1)^{\ell_0}$.
- *if* $\cdot \vdash_{\mathsf{A}} binop\{\tau_1\}\, v_0\, v_1 : \tau_0$ *and* $\cdot; \ell_0 \Vdash_{\mathsf{A}} binop\{\tau_1\}\, v_0\, v_1$ *and* $(binop\{\tau_1\}\, v_0\, v_1)^{\ell_0} \rhd_{\mathsf{A}} (e_2)^{\ell_0}$.
- *If* $\cdot \vdash_{\mathsf{A}} unop\{\mathcal{U}\}\, v_0 : \mathcal{U}$ *and* $\cdot; \ell_0 \Vdash_{\mathsf{A}} unop\{\mathcal{U}\}\, v_0$ *then* $(unop\{\mathcal{U}\}\, v_0)^{\ell_0} \blacktriangleright_{\mathsf{A}} (e_1)^{\ell_0}$.
- *if* $\cdot \vdash_{\mathsf{A}} binop\{\mathcal{U}\}\, v_0\, v_1 : \mathcal{U}$ *and* $\cdot; \ell_0 \Vdash_{\mathsf{A}} binop\{\mathcal{U}\}\, v_0\, v_1$ *then* $(binop\{\mathcal{U}\}\, v_0\, v_1)^{\ell_0} \blacktriangleright_{\mathsf{A}} (e_2)^{\ell_0}$.

Proof. By case analysis of δ_A, $\vdash_{\mathsf{A}}$, $\Vdash_{\mathsf{A}}$, and $\blacktriangleright_{\mathsf{A}}$.

1. CASE $\cdot \vdash_{\mathsf{A}} unop\{\tau_1\}\, v_0 : \tau_0$

1.1. $v_0 \in ((\!(\langle v, v\rangle)\!)^{\overline{\ell}}) \cup ((\!(\mathsf{mon}\,(\ell \blacktriangleleft (\tau\times\tau)^{\ell} \blacktriangleleft \ell)\, v)\!)^{\overline{\ell}})$

by $\vdash_{\mathsf{A}}$ inversion and canonical forms

1.2. SCASE $v_0 = (\!(\langle v_1, v_2\rangle)\!)^{\overline{\ell}_0}$

1.2.1. QED

$(unop\{\tau_1\}\, v_0)^{\ell_0} \rhd_{\mathsf{A}} (\delta_A(unop, v_0))^{\ell_0}$

1.3. SCASE $v_0 = (\!(\mathsf{mon}\,(\ell_1 \blacktriangleleft (\tau_1\times\tau_2)^{\ell_2} \blacktriangleleft \ell_2)\, v_1)\!)^{\overline{\ell}_3}$

1.3.1. QED

$(\mathsf{fst}\{\tau_0\}\ v_0)^{\ell_0} \rhd_{\mathsf{A}} (\!(\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\ (\mathsf{fst}\{\mathcal{U}\}\ v_1))\!)^{\overline{\ell_3}\ell_1}$

(and similarly for snd)

2. CASE $\cdot \vdash_{\mathsf{A}} binop\{\tau_1\}\ v_0\ v_1 : \tau_0$
   2.1. $v_0 \in (\!(i)\!)^{\overline{\ell}}$ and $v_1 \in (\!(i)\!)^{\overline{\ell}}$
   by $\vdash_{\mathsf{A}}$ inversion and canonical forms
   2.2. QED
   $(binop\{\tau_1\}\ v_0\ v_1)^{\ell_0} \rhd_{\mathsf{A}} (\delta_A(binop, v_0, v_1))^{\ell_0}$
3. CASE $\cdot \vdash_{\mathsf{A}} unop\{\mathcal{U}\}\ v_0 : \mathcal{U}$
   3.1. SCASE $v_0 \in \mathsf{trace}^?_{\mathsf{v}}\ \overline{b}\ (\!(\mathsf{mon}\ b\ (v)^{\ell})\!)^{\overline{\ell}}$
      3.1.1. QED
      by definition $\blacktriangleright_{\mathsf{A}}$
   3.2. SCASE $v_0 \in \mathsf{trace}^?_{\mathsf{v}}\ \overline{b}\ (\!(\langle v, v\rangle)\!)^{\overline{\ell}}$
      3.2.1. QED
      $(unop\{\mathcal{U}\}\ v_0)^{\ell_0} \blacktriangleright_{\mathsf{A}} (\delta_A(unop, v_0))^{\ell_0}$
   3.3. SCASE $rem\text{-}trace(v_0) \notin \langle v, v\rangle \cup (\mathsf{mon}\ b\ (v)^{\ell})$
      3.3.1. QED
      $(unop\{\mathcal{U}\}\ v_0)^{\ell_0} \blacktriangleright_{\mathsf{A}} (\mathsf{TagErr}\ \bullet)^{\ell_0}$
4. CASE $\cdot \vdash_{\mathsf{A}} binop\{\mathcal{U}\}\ v_0\ v_1 : \mathcal{U}$
   4.1. QED
   by definition $\blacktriangleright_{\mathsf{A}}$

□

Lemma 7.34 (δ_A label preservation).

- *If* $\cdot; \ell_0 \Vdash_{\mathsf{A}} unop\{\tau?\}\ v_0$ *and* $(unop\{\tau?\}\ v_0)^{\ell_0}\ (\rhd_{\mathsf{A}} \cup \blacktriangleright_{\mathsf{A}})\ (e_1)^{\ell_0}$ *then* $\cdot; \ell_0 \Vdash_{\mathsf{A}} e_1$.
- *If* $\cdot; \ell_0 \Vdash_{\mathsf{A}} binop\{\tau?\}\ v_0\ v_1$ *and* $(binop\{\tau?\}\ v_0\ v_1)^{\ell_0}\ (\rhd_{\mathsf{A}} \cup \blacktriangleright_{\mathsf{A}})\ (e_1)^{\ell_0}$ *then* $\cdot; \ell_0 \Vdash_{\mathsf{A}} e_1$.

Proof. By case analysis of $(\rhd_{\mathsf{A}} \cup \blacktriangleright_{\mathsf{A}})$.

1. $(\mathsf{fst}\{\tau_0\}\ (\!(\langle v_1, v_2\rangle)\!)^{\overline{\ell_1}})^{\ell_0} \rhd_{\mathsf{A}} ((\!(v_1)\!)^{\overline{\ell_1}})^{\ell_0}$
   1.1. $\cdot; \ell_0 \Vdash_{\mathsf{A}} v_0$
   and $\overline{\ell_1} = \ell_0 \cdots \ell_0$
   by inversion $\Vdash_{\mathsf{A}}$
   1.2. QED
2. $(\mathsf{fst}\{\tau_0\}\ (\!(\mathsf{mon}\ (\ell_1 \blacktriangleleft (\tau_1 \times \tau_2)^{\ell_2} \blacktriangleleft \ell_2)\ (v_1)^{\ell_2})\!)^{\overline{\ell_3}})^{\ell_0} \rhd_{\mathsf{A}} (\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\ (\mathsf{fst}\{\mathcal{U}\}\ v_0)^{\ell_2})^{\overline{\ell_3}\ell_0}$
   2.1. $\overline{\ell_3} = \ell_0 \cdots \ell_0$
   and $\ell_0 = \ell_1$
   by inversion $\Vdash_{\mathsf{A}}$
   2.2. QED

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{by inversion } \Vdash_{\mathsf{A}}}{\cdot; \ell_0 \Vdash_{\mathsf{A}} v_0}}{\cdot; \ell_0 \Vdash_{\mathsf{A}} \mathsf{fst}\{\mathcal{U}\}\, v_0}}{\cdot; \ell_0 \Vdash_{\mathsf{A}} \mathsf{dyn}\,(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\,(\mathsf{fst}\{\mathcal{U}\}\, v_0)^{\ell_2}}}{\cdot; \ell_0 \Vdash_{\mathsf{A}} (\mathsf{dyn}\,(\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\,(\mathsf{fst}\{\mathcal{U}\}\, v_0)^{\ell_2})^{\overline{\ell}_3 \ell_0}}$$

3. $(\mathsf{fst}\{\mathcal{U}\}\, (\!(\mathsf{trace}^?_{\mathsf{v}}\, \overline{b}_0\, (\!(\langle v_1, v_2\rangle)\!)^{\overline{\ell}_1})\!)^{\overline{\ell}_2})^{\ell_0} \blacktriangleright_{\mathsf{A}} (\mathit{add\text{-}trace}(\overline{b}_0, (\!(v_1)\!)^{\overline{\ell}_1}))^{\overline{\ell}_2 \ell_0}$
   3.1. $\overline{\ell}_2 = \ell_0 \cdots \ell_0$
        and $\overline{b}_0 \simeq \overline{\ell}_1$
        and $\cdot; \mathit{last}(\overline{\ell}_1) \Vdash_{\mathsf{A}} v_1$
      by inversion $\Vdash_{\mathsf{A}}$
   3.2. QED
      by lemma 7.39
4. $(\mathsf{fst}\{\mathcal{U}\}\, (\!(\mathsf{trace}^?_{\mathsf{v}}\, \overline{b}_0\, (\!(\mathsf{mon}\,(\ell_1 \blacktriangleleft (\tau_1 \times \tau_2)^{\ell_2} \blacktriangleleft \ell_2)\,(v_1)^{\ell_3})\!)^{\overline{\ell}_4})\!)^{\overline{\ell}_5})^{\ell_0} \blacktriangleright_{\mathsf{A}}$
   $(\mathsf{trace}\, \overline{b}_0\, (\!(\mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)\,(\mathsf{fst}\{\mathit{forget}(\tau_1)\}\, v_1)^{\ell_3})\!)^{\overline{\ell}_4})^{\overline{\ell}_5 \ell_0}$
   4.1. $\overline{\ell}_5 = \ell_0 \cdots \ell_0$
        and $\overline{b}_0 \simeq \overline{\ell}_4$
        and $\mathit{last}(\overline{\ell}_4) = \ell_1$
        and $\cdot; \ell_2 \Vdash_{\mathsf{A}} v_1$
      by inversion $\Vdash_{\mathsf{A}}$
   4.2. QED

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{by inversion } \Vdash_{\mathsf{A}}}{\cdot; \ell_0 \Vdash_{\mathsf{A}} v_1}}{\cdot; \ell_0 \Vdash_{\mathsf{A}} \mathsf{fst}\{\mathit{forget}(\tau_1)\}\, v_1}}{\cdot; \ell_0 \Vdash_{\mathsf{A}} \mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)\,(\mathsf{fst}\{\mathit{forget}(\tau_1)\}\, v_1)^{\ell_3}}}{\cdot; \ell_0 \Vdash_{\mathsf{A}} \mathsf{trace}\, \overline{b}_0\, (\!(\mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)\,(\mathsf{fst}\{\mathit{forget}(\tau_1)\}\, v_1)^{\ell_3})\!)^{\overline{\ell}_4}}}{\cdot; \ell_0 \Vdash_{\mathsf{A}} (\mathsf{trace}\, \overline{b}_0\, (\!(\mathsf{stat}\,(\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_2)\,(\mathsf{fst}\{\mathit{forget}(\tau_1)\}\, v_1)^{\ell_3})\!)^{\overline{\ell}_4})^{\overline{\ell}_5 \ell_0}}$$

5. $(\mathsf{snd}\{\tau?\}\, v_0)^{\ell_0}\ (\rhd_{\mathsf{A}} \cup \blacktriangleright_{\mathsf{A}})\ (\!(e_2)\!)^{\ell_0}$
   5.1. QED
      similar to fst cases
6. $(\mathsf{sum}\{\tau?\}\, v_0\, v_1)^{\ell_0}\ (\rhd_{\mathsf{A}} \cup \blacktriangleright_{\mathsf{A}})\ (i_2)^{\ell_0}$
   6.1. QED
7. $(\mathsf{quotient}\{\tau?\}\, v_0\, v_1)^{\ell_0}\ (\rhd_{\mathsf{A}} \cup \blacktriangleright_{\mathsf{A}})\ (\mathsf{DivErr})^{\ell_0}$
   7.1. QED
8. $(\mathsf{quotient}\{\tau?\}\, (\!(i_1)\!)^{\ell_1}\, (\!(i_2)\!)^{\ell_2})^{\ell_0}\ (\rhd_{\mathsf{A}} \cup \blacktriangleright_{\mathsf{A}})\ (\lfloor i_1 / i_2 \rfloor)^{\ell_0}$

8.1. QED

□

Lemma 7.35. *If* $\cdot \vdash_A \mathsf{dyn}\ b_0\ v_0 : \tau_0$ *and* $\cdot; \ell_0 \Vdash_A \mathsf{dyn}\ b_0\ v_0$ *then* $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_A (e_1)^{\ell_0}$.

Proof. By inversion of $\vdash_A$ and case analysis of $\mathit{tag\text{-}match}(\lfloor\tau_0\rfloor, v_0)$.

1. $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
   and $\ell_0; \ell_1 \Vdash \tau_0$
   and $\cdot; \ell_1 \Vdash_A v_0$
   and $v_0 = (\!(v_1)\!)^{\ell_1}$
   by inversion $\Vdash_A$
2. CASE $\mathit{tag\text{-}match}(\lfloor\tau_0\rfloor, v_1)$
   and $\mathit{rem\text{-}trace}(v_1) \in (\!(\lambda x.\, e)\!)^{\overline{\ell}} \cup (\!(\langle v, v\rangle)\!)^{\overline{\ell}} \cup (\!(\mathsf{mon}\ b\ v)\!)^{\overline{\ell}}$
   2.1. QED
   $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_A (\mathsf{mon}\ b_0\ v_0)^{\ell_0}$
3. CASE $v_1 \in i$ and $\mathit{tag\text{-}match}(\lfloor\mathsf{Int}\rfloor, v_1)$
   3.1. QED
   $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_A (v_1)^{\ell_0}$
4. CASE $v_1 \in n$ and $\mathit{tag\text{-}match}(\lfloor\mathsf{Nat}\rfloor, v_1)$
   4.1. QED
   $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_A (v_1)^{\ell_0}$
5. CASE $\neg\mathit{tag\text{-}match}(\lfloor\tau_0\rfloor, v_1)$
   5.1. QED
   $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_A (\mathsf{BndryErr}\,(b_0, v_0))^{\ell_0}$

□

Lemma 7.36. *If* $\cdot \vdash_A \mathsf{stat}\ b_0\ v_0 : \mathcal{U}$ *and* $\cdot; \ell_0 \Vdash_A \mathsf{stat}\ b_0\ v_0$ *then* $(\mathsf{stat}\ b_0\ v_0)^{\ell_0} \blacktriangleright_A (e_1)^{\ell_0}$.

Proof. By case analysis on v_0.

1. $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
   and $\ell_0; \ell_1 \Vdash \tau_0$
   and $\cdot; \ell_1 \Vdash_A v_0$
   and $v_0 = (\!(v_1)\!)^{\overline{\ell}_2}$
   by inversion $\Vdash_A$
2. CASE $v_1 \in \lambda x.\, e$
   2.1. Contradiction:
   $\cdot \vdash_A \mathsf{stat}\ b_0\ v_0 : \mathcal{U}$
3. CASE $v_1 \in \lambda(x\!:\!\tau).\, e$
   3.1. QED
   $(\mathsf{stat}\ b_0\ v_0)^{\ell_0} \blacktriangleright_A (\mathsf{mon}\ b_0\ v_0)^{\ell_0}$
4. CASE $v_1 \in \langle v, v\rangle$
   4.1. QED

$(\mathsf{stat}\ b_0\ v_0)^{\ell_0} \blacktriangleright_{\mathsf{A}} (\mathsf{mon}\ b_0\ v_0)^{\ell_0}$

5. CASE $v_1 = \mathsf{mon}\ b_1\ (\!(\mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b}_2\ (\!(v_2)\!)^{\overline{\ell}_3})\!)^{\overline{\ell}_4}$
   5.1. SCASE $v_2 \in (\lambda x.\, e) \cup (\langle v, v \rangle)$
      5.1.1. QED
         $(\mathsf{stat}\ b_0\ v_0)^{\ell_0} \blacktriangleright_{\mathsf{A}} (\mathsf{trace}\ b_0 b_1 \overline{b}_2\ (\!(v_2)\!)^{\overline{\ell}_3 \overline{\ell}_4 \overline{\ell}_2})^{\ell_0}$
   5.2. SCASE $v_2 \in (\lambda(x\,{:}\,\tau).\, e)$
      5.2.1. CONTRADICTION:
         $\cdot \vdash_{\mathsf{A}} v_0 : \tau_0$
   5.3. SCASE $v_2 = (\mathsf{mon}\ b_5\ (\!(v_3)\!)^{\overline{\ell}_6})$
      5.3.1. SSCASE $v_3 \in (\lambda(x\,{:}\,e).) \cup \langle v, v \rangle$
         $(\mathsf{stat}\ b_0\ v_0)^{\ell_0} \blacktriangleright_{\mathsf{A}} (\mathsf{trace}\ b_0 b_1 \overline{b}_2\ (\!(v_2)\!)^{\overline{\ell}_3 \overline{\ell}_4 \overline{\ell}_2})^{\ell_0}$
      5.3.2. SSCASE $v_3 \notin (\lambda(x\,{:}\,e).) \cup \langle v, v \rangle$
         5.3.2.1. CONTRADICTION:
            $\cdot \vdash_{\mathsf{A}} v_1 : \tau_0$
   5.4. SCASE otherwise
      5.4.1. CONTRADICTION:
         $\cdot \vdash_{\mathsf{A}} v_1 : \tau_0$
6. CASE $v_1 = \mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b}_0\ (\!(i_1)\!)^{\overline{\ell}_1}$
   6.1. QED
      $(\mathsf{stat}\ b_0\ v_0)^{\ell_0} \blacktriangleright_{\mathsf{A}} (i_1)^{\ell_0}$

□

LEMMA 7.37. *If* $\cdot \vdash_{\mathsf{A}} \mathsf{dyn}\ b_0\ v_0 : \tau_0$ *and* $\cdot; \ell_0 \Vdash_{\mathsf{A}} \mathsf{dyn}\ b_0\ v_0$ *and* $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_{\mathsf{A}} (e_1)^{\ell_0}$ *then* $\cdot; \ell_0 \Vdash_{\mathsf{A}} e_1$.

PROOF. By case analysis of $\rhd_{\mathsf{A}}$.

1. $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
   and $\ell_0; \ell_1 \Vdash \tau_0$
   and $\cdot; \ell_1 \Vdash_{\mathsf{A}} v_0$
   by inversion $\Vdash_{\mathsf{A}}$
2. CASE $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_{\mathsf{A}} (\mathsf{mon}\ b_0\ v_0)^{\ell_0}$
   2.1. QED

$$\dfrac{\dfrac{\text{by inversion } \Vdash_{\mathsf{A}}}{\cdot; \ell_1 \Vdash_{\mathsf{A}} v_0}}{\cdot; \ell_0 \Vdash_{\mathsf{A}} \mathsf{mon}\ b_0\ v_0}$$

3. CASE $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_{\mathsf{A}} (i_1)^{\ell_0}$
   3.1. QED
4. CASE $(\mathsf{dyn}\ b_0\ v_0)^{\ell_0} \rhd_{\mathsf{A}} (\mathsf{BndryErr}\ (b_0, v_0))^{\ell_0}$
   4.1. QED

□

Lemma 7.38 (A-stat preservation). *If* $\cdot \vdash_{\mathsf{A}} \mathsf{stat}\ b_0\ v_0 : \mathcal{U}$ *and* $\cdot; \ell_0 \mathrel{\overline{\Vdash}_{\mathsf{A}}} \mathsf{stat}\ b_0\ v_0$ *and* $(\mathsf{stat}\ b_0\ v_0)^{\ell_0} \blacktriangleright_{\mathsf{A}} (e_1)^{\ell_0}$ *then* $\cdot; \ell_0 \mathrel{\overline{\Vdash}_{\mathsf{A}}} e_1$.

Proof. By case analysis of $\blacktriangleright_{\mathsf{A}}$.

1. $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
   and $\cdot; \ell_1 \mathrel{\overline{\Vdash}_{\mathsf{A}}} v_0$
   by inversion $\overline{\Vdash}_{\mathsf{A}}$
2. case $(\mathsf{stat}\ b_0\ v_0)^{\ell_0} \blacktriangleright_{\mathsf{A}} (\mathsf{mon}\ b_0\ v_0)^{\ell_0}$
   2.1. qed

$$\cfrac{\cfrac{\text{by inversion } \overline{\Vdash}_{\mathsf{A}}}{\cdot; \ell_1 \mathrel{\overline{\Vdash}_{\mathsf{A}}} v_0}}{\cdot; \ell_0 \mathrel{\overline{\Vdash}_{\mathsf{A}}} \mathsf{mon}\ b_0\ v_0}$$

3. case $(\mathsf{stat}\ b_0\ (\!(\mathsf{mon}\ b_1\ (\!(\mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b}_2\ v_2)\!)^{\overline{\ell}_4})\!)^{\overline{\ell}_5})^{\ell_0} \blacktriangleright_{\mathsf{A}} (\mathsf{trace}\ b_0 b_1 \overline{b}_2\ (\!(v_2)\!)^{\overline{\ell}_4 \overline{\ell}_5 \ell_0})^{\ell_0}$
   3.1. $\overline{b}_2 \simeq \overline{\ell}_4$
        by inversion $\overline{\Vdash}_{\mathsf{A}}$
   3.2. qed

$$\cfrac{\cfrac{\cfrac{\text{by inversion } \overline{\Vdash}_{\mathsf{A}}}{\cdot; \mathit{last}(\overline{\ell}_4) \mathrel{\overline{\Vdash}_{\mathsf{A}}} v_2}}{\cdot; \ell_0 \mathrel{\overline{\Vdash}_{\mathsf{A}}} \mathsf{trace}\ b_0 b_1 \overline{b}_2\ (\!(v_2)\!)^{\overline{\ell}_4 \overline{\ell}_5 \ell_0}}}{\cdot; \ell_0 \mathrel{\overline{\Vdash}_{\mathsf{A}}} (\mathsf{trace}\ b_0 b_1 \overline{b}_2\ (\!(v_2)\!)^{\overline{\ell}_4 \overline{\ell}_5 \ell_0})^{\ell_0}}$$

4. case $(\mathsf{stat}\ b_0\ v_0)^{\ell_0} \blacktriangleright_{\mathsf{A}} (i_1)^{\ell_0}$
   qedstep

□

Lemma 7.39. *If* $\cdot \vdash_{\mathsf{A}} \mathsf{trace}\ \overline{b}_0\ v_0 : \mathcal{U}$ *and* $\cdot; \ell_0 \mathrel{\overline{\Vdash}_{\mathsf{A}}} \mathsf{trace}\ \overline{b}_0\ v_0$ *then* $\cdot; \ell_0 \mathrel{\overline{\Vdash}_{\mathsf{A}}} \mathit{add\text{-}trace}(\overline{b}_0, v_0)$.

Proof. By case analysis of *add-trace*.

1. case $\mathit{add\text{-}trace}(\cdot, v_0) = v_0$
   1.1. qed
2. case $\mathit{add\text{-}trace}(\overline{b}_0, (\!(\mathsf{trace}_{\mathsf{v}}\ \overline{b}_1\ v_1)\!)^{\overline{\ell}_2}) = \mathsf{trace}_{\mathsf{v}}\ \overline{b}_0 \overline{b}_1\ (\!(v_1)\!)^{\overline{\ell}_2}$
   2.1. qed
      2.1.1. $\overline{b}_0 \simeq \overline{\ell}_2$
             by inversion $\overline{\Vdash}_{\mathsf{A}}$
      2.1.2. qed
3. case $\mathit{add\text{-}trace}(\overline{b}_0, v_1) = \mathsf{trace}_{\mathsf{v}}\ \overline{b}_0\ v_1$
   and $v_0 \notin \mathsf{trace}_{\mathsf{v}}\ \overline{b}\ v$
   3.1. $v_1 = (\!(v_2)\!)^{\overline{\ell}_2}$
        and $\overline{b}_0 \simeq \overline{\ell}_2$

by inversion $\cdot; \ell_0 \Vdash_{\mathsf{A}} \mathsf{trace}\,\overline{b}_0\, v_1$

3.2. QED

□

LEMMA 7.40. *If* $(x_0 : \tau_0), \Gamma_0 \vdash_{\mathsf{A}} e_1 : \tau?$ *and* $(x_0 : \ell_0), L_0; \ell_1 \Vdash_{\mathsf{A}} e_1$ *and* $\cdot \vdash_{\mathsf{A}} v_0 : \tau?'$ *and* $\cdot; \ell_0 \Vdash_{\mathsf{A}} v_0$ *then* $\Gamma_0 \vdash_{\mathsf{A}} e_1[x_0 \leftarrow v_0] : \tau?$ *and* $L_0 \Vdash_{\mathsf{A}} e_1[x_0 \leftarrow v_0]$.

PROOF. By induction on the structure of e_0.

1. $e_0 = x_2$
   1.1. SCASE $x_0 = x_2$
      1.1.1. QED
   1.2. SCASE $x_0 \neq x_2$
      1.2.1. QED
         $e_1[x_0 \leftarrow v_0] = e_1$
2. CASE $e_0 \in i$
   2.1. QED
      $e_1[x_0 \leftarrow v_0] = e_1$
3. CASE $e_0 = \lambda x_2.\, e_2$
   or$e_0 = \lambda(x_2 : \tau_2).\, e_2$
   3.1. SCASE $x_0 = x_2$
      3.1.1. QED
         $e_1[x_0 \leftarrow v_0] = e_1$
   3.2. SCASE $x_0 \neq x_2$
      3.2.1. QED
         by the induction hypothesis
4. CASE $e_0 = \langle e_1, e_2 \rangle$
   4.1. QED
      by the induction hypothesis
5. CASE $e_0 = \mathsf{app}\{\tau?\}\, e_1\, e_2$
   5.1. QED
      by the induction hypothesis
6. CASE $e_0 = \mathit{unop}\{\tau?\}\, e_1$
   6.1. QED
      by the induction hypothesis
7. CASE $e_0 = \mathit{binop}\{\tau?\}\, e_1\, e_2$
   7.1. QED
      by the induction hypothesis
8. CASE $e_0 = \mathsf{dyn}\, b_1\, e_1$
   8.1. QED
      by the induction hypothesis
9. CASE $e_0 = \mathsf{stat}\, b_1\, e_1$
   9.1. QED

by the induction hypothesis

10. CASE $e_0 = (e_1)^{\ell_1}$

10.1. $\ell_0 = \ell_1$

by inversion $\Vdash_{\mathsf{A}}$

10.2. QED

by the induction hypothesis

11. CASE $e_0 = \mathsf{trace}\,\overline{b}_1\,(\!(e_1)\!)^{\overline{\ell}_1}$

11.1. $\cdot; \mathit{last}(\overline{\ell}_1) \Vdash_{\mathsf{A}} e_1$

by inversion $\Vdash_{\mathsf{A}}$

11.2. QED

by the induction hypothesis

12. CASE $e_0 = \mathsf{trace}_{\mathsf{v}}\,\overline{b}_1\,(\!(e_1)\!)^{\overline{\ell}_1}$

12.1. QED

by the induction hypothesis

□

Lemma 7.41 (boundary preservation). *If* $e_0 : \tau?\ \overline{\mathbf{wf}}$ *and* $e_0 \rightarrow^*_{\mathsf{A}} E_0[\mathsf{dyn}\ b_1\ v_1]$ *then either* $\mathit{has\text{-}boundary}(e_0, b_1)$ *or* $\mathit{has\text{-}boundary}(e_0, \mathit{flip}(b_1))$.

Proof. By case analysis of $\rhd_{\mathsf{A}}$ and $\blacktriangleright_{\mathsf{A}}$, evaluation does not create new labels and only creates a new boundary by flipping an existing boundary.

□

Lemma 7.42. *If* $e_0 : \tau?\ \overline{\mathbf{wf}}$ *and* $e_0 \rightarrow^*_{\mathsf{A}} E[\mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_2)\ v_0]$ *and* $\mathit{tag\text{-}match}(\lfloor \tau_1 \times \tau_2 \rfloor, v_0)$ *then* $\tau_0 \in (\tau \times \tau)^{\ell}$

Proof. Surface expressions do not contain monitors, and $\blacktriangleright_{\mathsf{A}}$ and $\rhd_{\mathsf{A}}$ only create monitors with compatible types and values.

□

8 N/A SIMULATION

$\boxed{v \lesssim v}$

$$\frac{}{i_0 \lesssim i_0} \qquad \frac{}{i_0 \lesssim \mathsf{trace}_{\mathsf{v}}\,\overline{b}_0\; i_0} \qquad \frac{v_0 \lesssim v_2 \qquad v_1 \lesssim v_3}{\langle v_0, v_1\rangle \lesssim \langle v_2, v_3\rangle} \qquad \frac{v_0 \lesssim \mathsf{mon}^{?}\;\mathsf{fst}\,(b_0)\; v_2 \qquad v_1 \lesssim \mathsf{mon}^{?}\;\mathsf{snd}\,(b_0)\; v_3}{\langle v_0, v_1\rangle \lesssim \mathsf{mon}\; b_0\; \langle v_2, v_3\rangle}$$

$$\frac{v_0 \lesssim \textit{add-trace}\,(\mathsf{fst}\,(\overline{b}_0), v_2) \qquad v_1 \lesssim \textit{add-trace}\,(\mathsf{snd}\,(\overline{b}_0), v_3)}{\langle v_0, v_1\rangle \lesssim \mathsf{trace}_{\mathsf{v}}\;\overline{b}_0\;\langle v_2, v_3\rangle} \qquad \frac{\begin{array}{c} v_0 \lesssim \textit{add-trace}\,(\mathsf{fst}\,(\overline{b}_0), (\mathsf{mon}^{?}\;\mathsf{fst}\,(b_1)\; v_2)) \\ v_1 \lesssim \textit{add-trace}\,(\mathsf{snd}\,(\overline{b}_0), (\mathsf{mon}^{?}\;\mathsf{snd}\,(b_1)\; v_3))\end{array}}{\langle v_0, v_1\rangle \lesssim \mathsf{trace}_{\mathsf{v}}\;\overline{b}_0\;(\mathsf{mon}\; b_1\;\langle v_2, v_3\rangle)}$$

$$\frac{v_0 \lesssim \mathsf{mon}^{?}\;\mathsf{fst}\,(b_0)\;(\textit{add-trace}\,(\mathsf{fst}\,(\overline{b}_1), v_2)) \qquad v_1 \lesssim \mathsf{mon}^{?}\;\mathsf{snd}\,(b_1)\;(\textit{add-trace}\,(\mathsf{snd}\,(\overline{b}_1), v_3))}{\langle v_0, v_1\rangle \lesssim \mathsf{mon}\; b_0\;(\mathsf{trace}_{\mathsf{v}}\;\overline{b}_1\;\langle v_2, v_3\rangle)}$$

$$\frac{v_0 \lesssim \mathsf{mon}^{?}\;\mathsf{fst}\,(b_0)\;(\mathsf{mon}^{?}\;\mathsf{fst}\,(b_1)\; v_2) \qquad v_1 \lesssim \mathsf{mon}^{?}\;\mathsf{snd}\,(b_0)\;(\mathsf{mon}^{?}\;\mathsf{snd}\,(b_1)\; v_3)}{\langle v_0, v_1\rangle \lesssim \mathsf{mon}\; b_0\;(\mathsf{mon}\; b_1\;\langle v_2, v_3\rangle)}$$

$$\frac{\begin{array}{c} v_0 \lesssim \mathsf{mon}^{?}\;\mathsf{fst}\,(b_0)\;(\textit{add-trace}\,(\mathsf{fst}\,(\overline{b}_1), (\mathsf{mon}\;\mathsf{fst}\,(b_2)\; v_2))) \\ v_1 \lesssim \mathsf{mon}^{?}\;\mathsf{snd}\,(b_0)\;(\textit{add-trace}\,(\mathsf{snd}\,(\overline{b}_1), (\mathsf{mon}\;\mathsf{snd}\,(b_2)\; v_3)))\end{array}}{\langle v_0, v_1\rangle \lesssim \mathsf{mon}\; b_0\;(\mathsf{trace}_{\mathsf{v}}\;\overline{b}_1\;(\mathsf{mon}\; b_2\;\langle v_2, v_3\rangle))}$$

$$\frac{v_0 \lesssim \mathsf{mon}^{+?}\;\mathsf{fst}\,(\overline{b}_0)\;(\mathsf{mon}^{?}\;\mathsf{fst}\,(b_1)\;(\mathsf{mon}^{?}\;\mathsf{fst}\,(b_2)\; v_2)) \qquad v_1 \lesssim \mathsf{mon}^{+?}\;\mathsf{snd}\,(\overline{b}_0)\;(\mathsf{mon}^{?}\;\mathsf{snd}\,(b_1)\;(\mathsf{mon}^{?}\;\mathsf{snd}\,(b_2)\; v_3))}{\langle v_0, v_1\rangle \lesssim \mathsf{mon}^{+}\;\overline{b}_0\;(\mathsf{mon}\; b_1\;(\mathsf{mon}\; b_2\;\langle v_2, v_3\rangle))}$$

$$\frac{\begin{array}{c} v_0 \lesssim \mathsf{mon}^{+?}\;\mathsf{fst}\,(\overline{b}_0)\;(\mathsf{mon}^{?}\;\mathsf{fst}\,(b_1)\;(\textit{add-trace}\,(\mathsf{fst}\,(\overline{b}_2), (\mathsf{mon}^{?}\;\mathsf{fst}\,(b_3)\; v_2)))) \\ v_1 \lesssim \mathsf{mon}^{+?}\;\mathsf{snd}\,(\overline{b}_0)\;(\mathsf{mon}^{?}\;\mathsf{snd}\,(b_1)\;(\textit{add-trace}\,(\mathsf{snd}\,(\overline{b}_2), (\mathsf{mon}^{?}\;\mathsf{snd}\,(b_3)\; v_3))))\end{array}}{\langle v_0, v_1\rangle \lesssim \mathsf{mon}^{+}\;\overline{b}_0\;(\mathsf{mon}\; b_1\;(\mathsf{trace}_{\mathsf{v}}\;\overline{b}_2\;(\mathsf{mon}\; b_3\;\langle v_2, v_3\rangle)))} \qquad \frac{e_0 \lesssim e_1}{\lambda x_0.\, e_0 \lesssim \lambda x_0.\, e_1}$$

$$\frac{e_0 \lesssim e_1}{\lambda(x_0\!:\!\tau_0).\, e_0 \lesssim \lambda(x_0\!:\!\tau_0).\, e_1} \qquad \frac{\begin{array}{c} v_1 \in \lambda x.\, e \cup \mathsf{mon}\; b\;(\lambda(x\!:\!\tau).\, e) \\ b_0 \leqslant\colon b_2 \qquad b_1 \leqslant\colon b_3 \qquad v_0 \lesssim v_1\end{array}}{\mathsf{mon}\; b_0\;(\mathsf{mon}\; b_1\; v_0) \lesssim \mathsf{trace}_{\mathsf{v}}\; b_2 b_3 \cdot v_1}$$

$$\frac{\begin{array}{c} v_1 \in \lambda x.\, e \cup \mathsf{mon}\; b\;(\lambda(x\!:\!\tau).\, e) \\ b_0 \leqslant\colon b_2 \qquad b_1 \leqslant\colon b_3 \qquad v_0 \lesssim \mathsf{trace}_{\mathsf{v}}\;\overline{b}_4\; v_1\end{array}}{\mathsf{mon}\; b_0\;(\mathsf{mon}\; b_1\; v_0) \lesssim \mathsf{trace}_{\mathsf{v}}\; b_2 b_3 \overline{b}_4\; v_1} \qquad \frac{b_0 \leqslant\colon b_1 \qquad v_0 \lesssim v_1}{\mathsf{mon}\; b_0\; v_0 \lesssim \mathsf{mon}\; b_1\; v_1}$$

$\boxed{e \lesssim e}$

$$\frac{}{x_0 \lesssim x_0} \qquad \frac{e_0 \lesssim e_2 \qquad e_1 \lesssim e_3}{\langle e_0, e_1 \rangle \lesssim \langle e_2, e_3 \rangle} \qquad \frac{e_0 \lesssim e_2 \qquad e_1 \lesssim e_3}{\mathsf{app}\{\tau?_0\}\ e_0\ e_1 \lesssim \mathsf{app}\{\tau?_0\}\ e_2\ e_3} \qquad \frac{e_0 \lesssim e_1}{\mathit{unop}\{\tau?\}\ e_0 \lesssim \mathit{unop}\{\tau?\}\ e_1}$$

$$\frac{e_0 \lesssim e_2 \qquad e_1 \lesssim e_3}{\mathit{binop}\{\tau?\}\ e_0\ e_1 \lesssim \mathit{binop}\{\tau?\}\ e_2\ e_3} \qquad \frac{b_0 \leqslant: b_1 \qquad e_0 \lesssim e_1}{\mathsf{dyn}\ b_0\ e_0 \lesssim \mathsf{dyn}\ b_1\ e_1} \qquad \frac{b_0 \leqslant: b_1 \qquad e_0 \lesssim e_1}{\mathsf{stat}\ b_0\ e_0 \lesssim \mathsf{stat}\ b_1\ e_1}$$

$$\frac{\begin{array}{c} b_0 \leqslant: b_2 \qquad b_1 \leqslant: b_3 \\ e_0 \lesssim \mathsf{trace}\ \overline{b}_4\ e_1 \end{array}}{\mathsf{stat}\ b_0\ (\mathsf{dyn}\ b_1\ e_0) \lesssim \mathsf{trace}\ b_2 b_3 \overline{b}_4\ e_1} \qquad \frac{}{\mathsf{TagErr} \circ \lesssim \mathsf{TagErr} \circ} \qquad \frac{}{\mathsf{TagErr} \bullet \lesssim \mathsf{TagErr} \bullet} \qquad \frac{}{\mathsf{DivErr} \lesssim \mathsf{DivErr}}$$

$$\frac{}{\mathsf{BndryErr}\,(b_0, v_0) \lesssim e_1}$$

$\boxed{E \lesssim E}$

$$\frac{}{[\,] \lesssim [\,]} \qquad \frac{E_0 \lesssim E_2 \qquad e_1 \lesssim e_3}{\langle E_0, e_1 \rangle \lesssim \langle E_2, e_3 \rangle} \qquad \frac{v_0 \lesssim v_2 \qquad E_1 \lesssim E_3}{\langle v_0, E_1 \rangle \lesssim \langle v_2, E_3 \rangle} \qquad \frac{E_0 \lesssim E_2 \qquad e_1 \lesssim e_3}{\mathsf{app}\{\tau?_0\}\ E_0\ e_1 \lesssim \mathsf{app}\{\tau?_0\}\ E_2\ e_3}$$

$$\frac{v_0 \lesssim v_2 \qquad E_1 \lesssim E_3}{\mathsf{app}\{\tau?_0\}\ v_0\ E_1 \lesssim \mathsf{app}\{\tau?_0\}\ v_2\ E_3} \qquad \frac{E_0 \lesssim E_1}{\mathit{unop}\{\tau?\}\ E_0 \lesssim \mathit{unop}\{\tau?\}\ E_1} \qquad \frac{E_0 \lesssim E_2 \qquad e_1 \lesssim e_3}{\mathit{binop}\{\tau?\}\ E_0\ e_1 \lesssim \mathit{binop}\{\tau?\}\ E_2\ e_3}$$

$$\frac{v_0 \lesssim v_2 \qquad E_1 \lesssim E_3}{\mathit{binop}\{\tau?\}\ v_0\ E_1 \lesssim \mathit{binop}\{\tau?\}\ v_2\ E_3} \qquad \frac{b_0 \leqslant: b_1 \qquad E_0 \lesssim E_1}{\mathsf{dyn}\ b_0\ E_0 \lesssim \mathsf{dyn}\ b_1\ E_1} \qquad \frac{b_0 \leqslant: b_1 \qquad E_0 \lesssim E_1}{\mathsf{stat}\ b_0\ E_0 \lesssim \mathsf{stat}\ b_1\ E_1}$$

$$\frac{b_0 \leqslant: b_2 \qquad b_1 \leqslant: b_3 \qquad E_0 \lesssim \mathsf{trace}\ \overline{b}_4\ E_1}{\mathsf{stat}\ b_0\ (\mathsf{dyn}\ b_1\ E_0) \lesssim \mathsf{trace}\ b_2 b_3 \overline{b}_4\ E_1}$$

Corollary 8.1. *If* $e_0 : \tau?$ **wf** *and* $e_0 \rightarrow^*_{\mathsf{A}} \mathsf{BndryErr}\,(\overline{b}_2, v_2)$ *then* $e_0 \rightarrow^*_{\mathsf{N}} \mathsf{BndryErr}\,(b_1, v_1)$

Proof. By lemma 8.6 and the fact that $e \lesssim \mathsf{BndryErr}\,(\overline{b}_2, v_2)$ implies $e \in \mathsf{BndryErr}\,(b, v)$

□

Example 8.2. There exists $e_0 : \tau?$ **wf** such that $e_0 \rightarrow^*_{\mathsf{A}} \mathsf{BndryErr}\,(\overline{b}_2, v_2)$ and $e_0 \rightarrow^*_{\mathsf{N}} \mathsf{BndryErr}\,(b_1, v_1)$ and $b_1 \notin \overline{b}_2$

Proof. Choose any e_0 where Natural detects an error that Amnesic misses, and then Amnesic detects an error at a boundary between two different components, e.g.:

$$e_0 = \mathsf{sum}\,(\mathsf{fst}\{\mathsf{Nat}\}\,(\mathsf{dyn}\,(\ell_0 \blacktriangleleft (\mathsf{Nat}{\times}\mathsf{Nat})^{\ell_1} \blacktriangleleft \ell_1)\,\langle 0, -1\rangle))$$
$$(\mathsf{dyn}\,(\ell_0 \blacktriangleleft (\mathsf{Nat})^{\ell_1} \blacktriangleleft \ell_1)\,(\mathsf{stat}\,(\ell_1 \blacktriangleleft (\mathsf{Nat})^{\ell_2} \blacktriangleleft \ell_2)\,(\mathsf{dyn}\,(\ell_2 \blacktriangleleft (\mathsf{Nat})^{\ell_3} \blacktriangleleft \ell_3)\,-1)))$$

□

Theorem 8.3. *If* $e_0 : \tau?$ **wf** *and* $e_0 \rightarrow^*_{\mathsf{N}} e_1 \rightarrow_{\mathsf{N}} \mathsf{BndryErr}\,(b_1, v_1)$ *then* $\exists\, e_2$ *such that* $e_0 \rightarrow^*_{\mathsf{A}} e_2$ *and one of the following holds:*

- $e_1 \lesssim e_2$, *or*
- $e_1 = E_1[\mathsf{app}\{\tau_2\}\, v_0\, (\mathsf{dyn}\; b_1\; v_1)]$ *and* $e_2 = E_2[\mathsf{app}\{\tau_2\}\,(\mathsf{trace}_{\mathsf{v}}\, \overline{b}_2\, v_2)\, v_3]$ *and* $E_1 \lesssim E_2$ *and* $\mathit{flip}(b_1) \in \overline{b}_2$

Proof. by lemma 8.6 and lemma 8.26.

□

Example 8.4. There exists $e_0 : \tau?$ **wf** such that $e_0 \rightarrow^*_{\mathsf{N}} \mathsf{BndryErr}\,(b_1, v_1)$ and $e_0 \rightarrow^*_{\mathsf{A}} v_1$

Proof. $e_0 = \mathsf{dyn}\,(\ell_0 \blacktriangleleft \mathsf{Nat}{\times}\mathsf{Nat} \blacktriangleleft \ell_1)\,\langle -1, -2\rangle$

□

Lemma 8.5 ($\lesssim$ surface reflexivity). *If* $e_0 : \tau?$ **wf** *then* $e_0 \lesssim e_0$

Proof. By induction on e_0.

1. case $e_0 = x_0$
   1.1. qed
   $x_0 \lesssim x_0$
2. case $e_0 = i_0$
   2.1. qed
   $i_0 \lesssim i_0$
3. case $e_0 = \lambda x_1.\, e_1$
   3.1. $e_1 \lesssim e_1$
   by the induction hypothesis
   3.2. qed
   $\lambda x_1.\, e_1 \lesssim \lambda x_1.\, e_1$
4. case $e_0 = \langle e_1, e_2\rangle$
   4.1. qed
   by the induction hypothesis
5. case $e_0 = \mathsf{app}\{\tau?\}\; e_1\; e_2$

5.1. QED

by the induction hypothesis

6. CASE $e_0 = \mathit{unop}\{\tau?\}\ e_1$

6.1. QED

by the induction hypothesis

7. CASE $e_0 = \mathit{binop}\{\tau?\}\ e_1\ e_2$

7.1. QED

by the induction hypothesis

8. CASE $e_0 = \mathsf{dyn}\ b_0\ e_1$

8.1. $b_0 \leqslant: b_0$

by reflexivity $\leqslant:$

8.2. QED

by the induction hypothesis

9. CASE $e_0 = \mathsf{stat}\ b_0\ e_1$

9.1. QED

by the induction hypothesis

□

LEMMA 8.6 (SIMULATION).

- *if* $e_0 \lesssim e_2$ *and* $e_0 \rightarrow_{\mathsf{N}} e_1$ *then* $\exists\, e_3, e_4$ *such that* $e_1 \rightarrow^*_{\mathsf{N}} e_3$ *and* $e_2 \rightarrow^*_{\mathsf{A}} e_4$ *and* $e_3 \lesssim e_4$
- *if* $e_0 \lesssim e_2$ *and* $e_2 \rightarrow_{\mathsf{A}} e_3$ *then* $\exists\, e_1, e_4$ *such that* $e_3 \rightarrow^*_{\mathsf{A}} e_4$ *and* $e_0 \rightarrow^*_{\mathsf{N}} e_1$ *and* $e_1 \lesssim e_4$

PROOF. By lemma 8.8 and lemma 8.9.

□

Definition 8.7 (reduced WF expressions). Expressions e_1 and e_2 are *reduced WF expressions* if $e_0 : \tau?$ **wf** and $e_0 \rightarrow^*_{\mathsf{N}} e_1$ and $e_0 \rightarrow^*_{\mathsf{A}} e_2$

LEMMA 8.8 (H-STEP SIMULATION). *If* e_0 *and* e_2 *are reduced WF expressions and* $e_0 \lesssim e_2$ *and* $e_0 \rightarrow_{\mathsf{N}} e_1$ *then* $\exists\, e_3, e_4$ *such that* $e_1 \rightarrow^*_{\mathsf{N}} e_3$ *and* $e_2 \rightarrow^*_{\mathsf{A}} e_4$ *and* $e_3 \lesssim e_4$

PROOF. By case analysis of $e_0 \rightarrow_{\mathsf{N}} e_1$

1. CASE $e_0 = E_0[\mathsf{BndryErr}\,(b_0, v_0)]$ and $e_0 \rightarrow_{\mathsf{N}} \mathsf{BndryErr}\,(b_0, v_0)$

1.1. QED

trivial, $\mathsf{BndryErr}\,(b_0, v_0) \lesssim e_2$

2. CASE $e_0 = E_0[\mathsf{Err}_0]$ and $\mathsf{Err}_0 \notin \mathsf{BndryErr}\,(\overline{b}, v)$ and $e_0 \rightarrow_{\mathsf{N}} \mathsf{Err}_0$

2.1. $e_2 = E_2[\mathsf{Err}_1]$ and $E_0 \lesssim E_2$ and $\mathsf{Err}_0 \lesssim \mathsf{Err}_1$

by lemma 8.10

2.2. $e_2 \rightarrow_{\mathsf{A}} \mathsf{Err}_1$

by definition $\rightarrow_{\mathsf{A}}$

2.3. QED

$\mathsf{Err}_0 \lesssim \mathsf{Err}_1$

3. CASE $e_0 = E_0[\mathsf{dyn}\ b_0\ v_0]$ and $\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} e_3$

3.1. $e_2 = E_2[\mathsf{dyn}\ b_2\ v_2]$ and $E_0 \lesssim E_2$ and $\mathsf{dyn}\ b_0\ v_0 \lesssim \mathsf{dyn}\ b_2\ v_2$
by lemma 8.10

3.2. $\mathsf{dyn}\ b_0\ v_0 \rightarrow^*_{\mathsf{N}} e_3$ and $\mathsf{dyn}\ b_2\ v_2 \rightarrow^*_{\mathsf{A}} e_4$ and $e_3 \lesssim e_4$
by lemma 8.12

3.3. QED
by lemma 8.31

4. CASE $e_0 = E_0[\mathsf{stat}\ b_0\ v_0]$ and $\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{N}} e_3$

4.1. $e_2 = E_2[\mathsf{stat}\ b_2\ v_2]$ and $E_0 \lesssim E_2$ and $\mathsf{stat}\ b_0\ v_0 \lesssim \mathsf{stat}\ b_2\ v_2$
by lemma 8.10

4.2. $\mathsf{stat}\ b_0\ v_0 \rightarrow^*_{\mathsf{N}} e_3$ and $\mathsf{stat}\ b_2\ v_2 \rightarrow^*_{\mathsf{A}} e_4$ and $e_3 \lesssim e_4$
by lemma 8.14

4.3. QED
by lemma 8.31

5. CASE $e_0 = E_0[unop\{\tau?\}\ v_0]$ and $unop\{\tau?\}\ v_0 \triangleright_{\mathsf{N}} \delta_N(unop, v_0)$

5.1. $e_2 = E_2[unop\{\tau?\}\ v_2]$ and $E_0 \lesssim E_2$ and $unop\{\tau?\}\ v_0 \lesssim unop\{\tau?\}\ v_2$
by lemma 8.10

5.2. $\cdot \vdash_{\mathsf{N}} unop\{\tau?\}\ v_0 : \tau_0$ and $\cdot \vdash_{\mathsf{A}} unop\{\tau?\}\ v_2 : \tau_0$
by type soundness

5.3. $unop\{\tau?\}\ v_2 \rightarrow^*_{\mathsf{A}} v_3$ and $\delta_N(unop, v_0) \lesssim v_3$
by lemma 8.22

5.4. QED
by lemma 8.31

6. CASE $e_0 = E_0[unop\{\tau?\}\ v_0]$ and $unop\{\tau?\}\ v_0 \blacktriangleright_{\mathsf{N}} e_1$

6.1. QED
by lemma 8.23

7. CASE $e_0 = E_0[\mathsf{app}\{\tau_0\}\ (\mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2) \blacktriangleleft \ell_1)\ v_0)\ v_1]$
and $\mathsf{app}\{\tau_0\}\ (\mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2) \blacktriangleleft \ell_1)\ v_0)\ v_1 \triangleright_{\mathsf{N}} \mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\mathcal{U}\}\ v_0\ (\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ v_1))$

7.1. $e_2 = E_2[\mathsf{app}\{\tau_0\}\ v_3\ v_4]$
by lemma 8.10

7.2. SCASE $v_3 = \mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_3 \Rightarrow \tau_4) \blacktriangleleft \ell_1)\ v_5$

7.2.1. $\mathsf{app}\{\tau_0\}\ (\mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_3 \Rightarrow \tau_4) \blacktriangleleft \ell_1)\ v_5)\ v_4 \triangleright_{\mathsf{A}} \mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\mathcal{U}\}\ v_5\ (\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_3 \blacktriangleleft \ell_0)\ v_4))$
by definition $\triangleright_{\mathsf{A}}$

7.2.2. QED
by lemma 8.31

7.3. SCASE $v_3 = \mathsf{trace}_{\mathsf{v}}\ \overline{b}_0\ v_5$

7.3.1. CONTRADICTION:
$\cdot \vdash_{\mathsf{A}} \mathsf{app}\{\tau_0\}\ v_3\ v_4 : \tau_0$

8. CASE $e_0 = E_0[\mathsf{app}\{\mathcal{U}\}\ (\mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1)\ v_0)\ v_1]$
and $\mathsf{app}\{\mathcal{U}\}\ (\mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1)\ v_0)\ v_1 \blacktriangleright_{\mathsf{N}} \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\tau_1\}\ v_0\ (\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_0 \blacktriangleleft \ell_0)\ v_1))$

8.1. $e_2 = E_2[\mathsf{app}\{\mathcal{U}\}\ v_3\ v_4]$
by lemma 8.10

8.2. SCASE $v_3 = \text{mon}\,(\ell_0 \blacktriangleleft (\tau_3 \Rightarrow \tau_4) \blacktriangleleft \ell_1)\; v_3$ and $v_0 \lesssim v_3$

8.2.1. QED

by definition $\blacktriangleright_{\mathsf{A}}$ and $\lesssim$

8.3. SCASE $v_3 = \text{trace}_{\mathsf{v}}\, \overline{b}_2\, v_5$

8.3.1. QED

by lemma 8.26

9. CASE $e_0 = E_0[\text{app}\{\tau_0\}\,(\lambda(x_0 : \tau_1).\, e_0)\; v_1]$ and $e_0 \rhd_{\mathsf{N}} E_0[e_0[x_0 \leftarrow v_1]]$

9.1. $e_2 = E_2[\text{app}\{\tau_0\}\; v_2\; v_3]$

by lemma 8.10

9.2. $v_2 = \lambda(x_0 : \tau_1).\, e_4$ and $e_0 \lesssim e_4$

by inversion $\lesssim$

9.3. QED

by lemma 8.30

10. CASE $e_0 = E_0[\text{app}\{\mathcal{U}\}\,(\lambda x_0.\, e_0)\; v_1]$ and $e_0 \blacktriangleright_{\mathsf{N}} E_0[e_0[x_0 \leftarrow v_1]]$

10.1. $e_2 = E_2[\text{app}\{\tau_0\}\; v_2\; v_3]$

by lemma 8.10

10.2. $v_2 = \lambda x_0.\, e_4$ and $e_0 \lesssim e_4$

by inversion $\lesssim$

10.3. QED

by lemma 8.30

11. CASE $e_0 = E_0[\mathit{binop}\{\tau?\}\; v_0\; v_1]$ and $e_0 \rightarrow_{\mathsf{N}} E_0[e_1]$

11.1. $e_2 = E_2[\mathit{binop}\{\tau?\}\; v_2\; v_3]$

by lemma 8.10

11.2. QED

by lemma 8.29

□

LEMMA 8.9 (A-STEP SIMULATION). *If e_0 and e_2 are reduced WF expressions and $e_0 \lesssim e_2$ and $e_2 \rightarrow_{\mathsf{A}} e_3$ then $\exists\, e_1, e_4$ such that $e_3 \rightarrow^*_{\mathsf{A}} e_4$ and $e_0 \rightarrow^*_{\mathsf{N}} e_1$ and $e_1 \lesssim e_4$*

PROOF. By case analysis of $e_2 \rightarrow_{\mathsf{A}} e_3$

1. CASE e_0 contains a boundary error; $e_0 = E_0[\text{BndryErr}\,(b_0, v_0)]$

1.1. $e_0 \rightarrow_{\mathsf{N}} \text{BndryErr}\,(b_0, v_0)$

by definition $\rightarrow_{\mathsf{N}}$

1.2. QED

$\text{BndryErr}\,(b_0, v_0) \lesssim e_3$

2. CASE $e_2 = E_2[\text{Err}_2]$ and $e_2 \rightarrow_{\mathsf{A}} \text{Err}_2$

2.1. $e_0 = E_0[\text{Err}_0]$ and $E_0 \lesssim E_2$ and $\text{Err}_0 \lesssim \text{Err}_2$

by lemma 8.11

2.2. QED

$e_0 \rightarrow_{\mathsf{N}} \text{Err}_0 \lesssim \text{Err}_2$

3. CASE $e_2 = E_2[\text{dyn}\; b_2\; v_2]$ and $\text{dyn}\; b_2\; v_2 \rhd_{\mathsf{A}} e_3$

3.1. $e_0 = E_0[\mathsf{dyn}\ b_0\ v_0]$ and $\mathsf{dyn}\ b_0\ v_0 \lesssim \mathsf{dyn}\ b_2\ v_2$
by lemma 8.11
3.2. $\mathsf{dyn}\ b_0\ v_0 \rightarrow^*_{\mathsf{N}} e_1$ and $\mathsf{dyn}\ b_2\ v_2 \rightarrow^*_{\mathsf{A}} e_4$ and $e_1 \lesssim e_4$
by lemma 8.12
3.3. QED
by lemma 8.31

4. CASE $e_2 = E_2[\mathsf{stat}\ b_2\ v_2]$ and $\mathsf{stat}\ b_2\ v_2 \blacktriangleright_{\mathsf{A}} e_3$
4.1. $e_0 = E_0[\mathsf{stat}\ b_0\ v_0]$ and $\mathsf{stat}\ b_0\ v_0 \lesssim \mathsf{stat}\ b_2\ v_2$
by lemma 8.11
4.2. $\mathsf{stat}\ b_0\ v_0 \rightarrow^*_{\mathsf{N}} e_1$ and $\mathsf{stat}\ b_2\ v_2 \rightarrow^*_{\mathsf{A}} e_4$ and $e_1 \lesssim e_4$
by lemma 8.14
4.3. QED
by lemma 8.31

5. CASE $e_2 = E_2[\mathit{unop}\{\tau?\}\ v_2]$ and $\mathit{unop}\{\tau?\}\ v_2 \vartriangleright_{\mathsf{A}} e_3$
5.1. $e_0 = E_0[\mathit{unop}\{\tau?\}\ v_0]$ and $E_0 \lesssim E_2$ and $v_0 \lesssim v_2$
by lemma 8.11
5.2. $\cdot \vdash_{\mathsf{N}} \mathit{unop}\{\tau?\}\ v_0 : \tau_0$ and $\cdot \vdash_{\mathsf{A}} \mathit{unop}\{\tau?\}\ v_2 : \tau_0$
by type soundness
5.3. $\mathit{unop}\{\tau?\}\ v_2 \rightarrow^*_{\mathsf{A}} v_4$ and $\delta_N(\mathit{unop}, v_0) \lesssim v_4$
by lemma 8.22
5.4. QED
by lemma 8.31

6. CASE $e_2 = E_2[\mathit{unop}\{\tau?\}\ v_2]$ and $\mathit{unop}\{\tau?\}\ v_2 \blacktriangleright_{\mathsf{A}} e_3$
6.1. QED
by lemma 8.23

7. CASE $e_2 = E_2[\mathsf{app}\{\tau_2\}\ (\mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_3 \Rightarrow \tau_4) \blacktriangleleft \ell_1)\ v_2)\ v_3]$
and $e_2 \rightarrow_{\mathsf{A}} E_2[\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\mathcal{U}\}\ v_2\ (\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_3 \blacktriangleleft \ell_0)\ v_3))]$
7.1. $\tau_4 \leqslant: \tau_2$
by inversion $\vdash_{\mathsf{A}}$
7.2. $e_0 = E_0[\mathsf{app}\{\tau_2\}\ v_0\ v_1]$
by lemma 8.11
7.3. $v_0 = \mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_0 \Rightarrow \tau_1) \blacktriangleleft \ell_1)\ v_4$
by inversion $\lesssim$
7.4. QED
$\mathsf{app}\{\tau_2\}\ v_0\ v_1 \vartriangleright_{\mathsf{N}} \mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\mathcal{U}\}\ v_4\ (\mathsf{stat}\ (\ell_1 \blacktriangleleft \tau_3 \blacktriangleleft \ell_0)\ v_1))$

8. CASE $e_2 = E_2[\mathsf{app}\{\mathcal{U}\}\ (\mathsf{trace}_{\mathsf{v}}\ b_0\overline{b}_1\ v_2)\ v_3]$
8.1. $e_0 = E_0[\mathsf{app}\{\mathcal{U}\}\ v_0\ v_1]$
by lemma 8.11
8.2. QED
by lemma 8.26

9. CASE $e_2 = E_2[\mathsf{app}\{\tau_0\}\ (\lambda(x_1 : \tau_1).\ e_2)\ v_3]$

9.1. $e_0 = E_0[\mathrm{app}\{\tau_0\}\ v_0\ v_1]$
by lemma 8.11
9.2. $v_0 = \lambda x_0.\, e_1$ and $e_1 \lesssim e_2$
by inversion $\lesssim$
9.3. QED
by lemma 8.30

10. CASE $e_2 = E_2[\mathit{binop}\{\tau?\}\ v_2\ v_3]$ and $e_2 \rightarrow_{\mathsf{A}} E_2[e_4]$
10.1. $e_0 = E_0[\mathit{binop}\{\tau?\}\ v_0\ v_1]$
by lemma 8.11
10.2. QED
by lemma 8.29

□

Lemma 8.10 (H context-matching). *If $E_0[e_0] \lesssim e_1$ and $e_0 \notin v$ then $\exists\, E_2, e_2$ such that $e_1 = E_2[e_2]$ and $E_0 \lesssim E_2$ and $e_0 \lesssim e_2$*

Proof. By induction on the structure of E_0

1. CASE $E_0 = [\,]$
trivial, $E_2 = [\,]$ and $e_2 = e_1$
2. CASE $E_0 = \langle E_3, e_4\rangle$
2.1. $E[e_0] = \langle E_3[e_0], e_4\rangle \lesssim e_1$
by assumption
2.2. $e_1 = \langle e_5, e_6\rangle$ and $E_3[e_0] \lesssim e_5$ and $e_4 \lesssim e_6$
by inversion $\lesssim$
2.3. $e_4 = E_7[e_8]$ and $E_3 \lesssim E_7$ and $e_0 \lesssim e_8$
by the induction hypothesis
2.4. QED
$E_1 = \langle E_7, e_6\rangle$ and $e_2 = e_8$
3. CASE $E_0 = \langle v_3, E_4\rangle$
3.1. $e_1 = \langle e_5, e_6\rangle$ and $v_3 \lesssim e_5$ and $E_4[e_0] \lesssim e_6$
by inversion $\lesssim$
3.2. QED
by the induction hypothesis
4. CASE $E_0 = \mathrm{app}\{\tau?_0\}\ E_3\ e_4$
4.1. $e_1 = \mathrm{app}\{\tau?_0\}\ e_5\ e_6$ and $E_3[e_0] \lesssim e_5$ and $e_4 \lesssim e_6$
by inversion $\lesssim$
4.2. QED
by the induction hypothesis
5. CASE $E_0 = \mathrm{app}\{\tau?_0\}\ v_3\ E_4$
5.1. $e_1 = \mathrm{app}\{\tau?_0\}\ e_5\ e_6$ and $v_3 \lesssim e_5$ and $E_4[e_0] \lesssim e_6$
by inversion $\lesssim$
5.2. QED

by the induction hypothesis

6. CASE $E_0 = \mathit{unop}\{\tau?\}\, E_3$

6.1. $e_1 = \mathit{unop}\{\tau?\}\, e_4$ and $E_3[e_0] \lesssim e_4$

by inversion $\lesssim$

6.2. QED

by the induction hypothesis

7. CASE $E_0 = \mathit{binop}\{\tau?\}\, E_3\, e_4$

7.1. $e_1 = \mathit{binop}\{\tau?\}\, e_5\, e_6$ and $E_3[e_0] \lesssim e_5$ and $e_4 \lesssim e_6$

by inversion $\lesssim$

7.2. QED

by the induction hypothesis

8. CASE $E_0 = \mathit{binop}\{\tau?\}\, v_3\, E_4$

8.1. $e_1 = \mathit{binop}\{\tau?\}\, v_5\, e_6$ and $v_3 \lesssim v_5$ and $E_4[e_0] \lesssim e_6$

by inversion $\lesssim$

8.2. QED

by the induction hypothesis

9. CASE $E_0 = \mathsf{dyn}\; b_0\; E_3$

9.1. $e_1 = \mathsf{dyn}\; b_1\; e_4$ and $E_3[e_0] \lesssim e_4$

by inversion $\lesssim$

9.2. QED

by the induction hypothesis

10. CASE $E_0 = \mathsf{stat}\; b_0\; E_3$

10.1. $e_1 = \mathsf{stat}\; b_1\; e_4$ and $E_3[e_0] \lesssim e_4$

by inversion $\lesssim$

10.2. QED

by the induction hypothesis

□

Lemma 8.11 (A context-matching). *If $e_0 \lesssim E_2[e_2]$ and $e_2 \notin v$ and e_0 does not contain a subterm* $\mathsf{BndryErr}\,(b, v)$ *then $\exists\, E_1, e_1$ such that $e_0 = E_1[e_1]$ and $E_1 \lesssim E_2$ and $e_1 \lesssim e_2$*

Proof. By induction on the structure of E_2

1. CASE $E_2 = [\,]$

trivial, $E_1 = [\,]$ and $e_1 = e_0$

2. CASE $E_2 = \langle E_3, e_3 \rangle$

2.1. $e_0 = \langle e_4, e_5 \rangle \lesssim \langle E_3[e_2], e_3 \rangle$

by inversion $\lesssim$

2.2. QED

by the induction hypothesis

3. CASE $E_2 = \langle v_3, E_3 \rangle$

3.1. $e_0 = \langle e_4, e_5 \rangle \lesssim \langle v_3, E_3[e_2] \rangle$

by inversion $\lesssim$

3.2. QED
by the induction hypothesis
4. CASE $E_2 = \mathsf{app}\{\tau?_0\}\, E_3\; e_3$
4.1. $e_0 = \mathsf{app}\{\tau?_0\}\, e_4\; e_5 \lesssim \mathsf{app}\{\tau?_0\}\, E_3[e_2]\; e_3$
by inversion $\lesssim$
4.2. QED
by the induction hypothesis
5. CASE $E_2 = \mathsf{app}\{\tau?_0\}\, v_3\; E_3$
5.1. $e_0 = \mathsf{app}\{\tau?_0\}\, e_4\; e_5 \lesssim \mathsf{app}\{\tau?_0\}\, v_3\; E_3[e_2]$
by inversion $\lesssim$
5.2. QED
by the induction hypothesis
6. CASE $E_2 = \mathit{unop}\{\tau?\}\, E_3$
6.1. $e_0 = \mathit{unop}\{\tau?\}\, e_4 \lesssim \mathit{unop}\{\tau?\}\, E_3[e_2]$
by inversion $\lesssim$
6.2. QED
by the induction hypothesis
7. CASE $E_2 = \mathit{binop}\{\tau?\}\, E_3\; e_3$
7.1. $e_0 = \mathit{binop}\{\tau?\}\, e_4\; e_5 \lesssim \mathit{binop}\{\tau?\}\, E_3[e_2]\; e_3$
by inversion $\lesssim$
7.2. QED
by the induction hypothesis
8. CASE $E_2 = \mathit{binop}\{\tau?\}\, v_3\; E_3$
8.1. $e_0 = \mathit{binop}\{\tau?\}\, e_4\; e_5 \lesssim \mathit{binop}\{\tau?\}\, v_3\; E_3[e_2]$
by inversion $\lesssim$
8.2. QED
by the induction hypothesis
9. CASE $E_2 = \mathsf{dyn}\; b_0\; E_3$
9.1. $e_0 = \mathsf{dyn}\; b_0\; e_4 \lesssim \mathsf{dyn}\; b_2\; E_3[e_2]$
by inversion $\lesssim$
9.2. QED
by the induction hypothesis
10. CASE $E_2 = \mathsf{stat}\; b_0\; E_3$
10.1. $e_0 = \mathsf{stat}\; b_0\; e_4 \lesssim \mathsf{stat}\; b_1\; E_3[e_2]$
by inversion $\lesssim$
10.2. QED
by the induction hypothesis
11. CASE $E_2 = \mathsf{trace}\, b_0 b_1 \overline{b_2}\; E_3$
11.1. $e_0 = \mathsf{stat}\; b_3\; \mathsf{dyn}\; b_4\; e_4$ and $e_4 \lesssim E_3[e_0]$
by inversion $\lesssim$
11.2. QED

by the induction hypothesis

□

Lemma 8.12. *If* (dyn b_0 v_0) *and* (dyn b_1 v_1) *are reduced WF expressions and* dyn b_0 $v_0 \lesssim$ dyn b_1 v_1 *then* dyn b_0 $v_0 \rightarrow^*_{\mathsf{N}}$ e_2 *and* dyn b_1 $v_1 \rightarrow^*_{\mathsf{A}}$ e_3 *then* $e_2 \lesssim e_3$

Proof. By case analysis of $v_0 \lesssim v_1$.

1. case $v_0 = i_0 \lesssim i_0 = v_1$
   1.1. scase *tag-match*$(\lfloor\tau_0\rfloor, v_0)$
      1.1.1. dyn b_0 $v_0 \rhd_{\mathsf{N}} v_0$
         by definition $\rhd_{\mathsf{N}}$
      1.1.2. *tag-match*$(\lfloor\tau_1\rfloor, v_1)$
         by lemma 8.17
      1.1.3. qed
         dyn b_1 $v_1 \rhd_{\mathsf{A}} v_1$
   1.2. scase $\neg$*tag-match*$(\lfloor\tau_0\rfloor, v_0)$
      1.2.1. qed
         dyn b_0 $v_0 \rhd_{\mathsf{N}}$ BndryErr$(b_0, v_0) \lesssim$ dyn b_1 v_1
2. case $v_0 = i_0 \lesssim \mathsf{trace}_{\mathsf{v}}\ \overline{b_2}\ i_0 = v_1$
   2.1. scase *tag-match*$(\lfloor\tau_0\rfloor, v_0)$
      2.1.1. dyn b_0 $v_0 \rhd_{\mathsf{N}} v_0$
         by definition $\rhd_{\mathsf{N}}$
      2.1.2. *tag-match*$(\lfloor\tau_1\rfloor, v_1)$
         by lemma 8.17
      2.1.3. qed
         dyn b_1 $v_1 \rhd_{\mathsf{A}} v_1$
   2.2. scase $\neg$*tag-match*$(\lfloor\tau_0\rfloor, v_0)$
      2.2.1. qed
         dyn b_0 $v_0 \rhd_{\mathsf{N}}$ BndryErr$(b_0, v_0) \lesssim$ dyn b_1 v_1
3. case $v_0 = \langle v_2, v_3\rangle \lesssim \langle v_4, v_5\rangle = v_1$
   3.1. scase *tag-match*$(\lfloor\tau_0\rfloor, v_0)$
      3.1.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)$
      3.1.2. $v_2 \lesssim v_4$ and $v_3 \lesssim v_5$
         by inversion $\lesssim$
      3.1.3. $\tau_0 = \tau_2 \times \tau_3 \leqslant: \tau_4 \times \tau_5 = \tau_1$
         by 3.1 and inversion $\leqslant:$
      3.1.4. dyn $(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ $v_0 \rhd_{\mathsf{N}} \langle$dyn $(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)$ v_2, dyn $(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)$ $v_3\rangle$
         by definition $\rhd_{\mathsf{N}}$
      3.1.5. *tag-match*$(\lfloor\tau_1\rfloor, v_1)$
         by lemma 8.17
      3.1.6. dyn b_1 $v_1 \rhd_{\mathsf{A}}$ mon b_1 v_1
         by definition $\rhd_{\mathsf{A}}$

3.1.7. $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\; v_2 \rightarrow^*_{\mathsf{N}} e_6 \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\; v_4$
and $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\; v_3 \rightarrow^*_{\mathsf{N}} e_7 \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\; v_5$
by lemma 8.13

3.1.8. QED
either e_6 or $e_7 \in \mathsf{BndryErr}\,(b, v)$ or:

$$\frac{e_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\; v_4 \qquad e_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\; v_5}{\langle e_6, e_7\rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\; \langle v_4, v_5\rangle}$$

3.2. SCASE $\neg \mathit{tag\text{-}match}\,(\lfloor \tau_0 \rfloor, v_0)$

3.2.1. QED
$\mathsf{dyn}\; b_0\; v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}\,(b_0, v_0) \lesssim \mathsf{dyn}\; b_1\; v_1$

4. CASE $v_0 = \langle v_2, v_3\rangle \lesssim \mathsf{mon}\,(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3)\; \langle v_4, v_5\rangle = v_1$

4.1. SCASE $\mathit{tag\text{-}match}\,(\lfloor \tau_0 \rfloor, v_0)$

4.1.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
and $b_1 = (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)$

4.1.2. $v_2 \lesssim \mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\; v_4$ and $v_3 \lesssim \mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\; v_5$
by inversion $\lesssim$

4.1.3. $\tau_0 = \tau_2 \times \tau_3 \leqslant: \tau_4 \times \tau_5 = \tau_1$
by 4.1 and inversion $\leqslant:$

4.1.4. $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_0 \rhd_{\mathsf{N}} \langle \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\; v_2, \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\; v_3\rangle$
by definition $\rhd_{\mathsf{N}}$

4.1.5. $\mathit{tag\text{-}match}\,(\lfloor \tau_1 \rfloor, v_1)$
by lemma 8.17

4.1.6. $\mathsf{dyn}\; b_1\; v_1 \rhd_{\mathsf{A}} \mathsf{mon}\; b_1\; v_1$
by definition $\rhd_{\mathsf{A}}$

4.1.7. $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\; v_2 \rightarrow^*_{\mathsf{N}} e_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\; v_4)$
and $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\; v_3 \rightarrow^*_{\mathsf{N}} e_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\; v_5)$
by lemma 8.13

4.1.8. QED
either e_6 or $e_7 \in \mathsf{BndryErr}\,(b, v)$ or:

$$\frac{e_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\; v_4) \qquad e_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\; v_5)}{\langle e_6, e_7\rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\,(\mathsf{mon}\,(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3)\; \langle v_4, v_5\rangle)}$$

4.2. SCASE $\neg \mathit{tag\text{-}match}\,(\lfloor \tau_0 \rfloor, v_0)$

4.2.1. QED
$\mathsf{dyn}\; b_0\; v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}\,(b_0, v_0) \lesssim \mathsf{dyn}\; b_1\; v_1$

5. CASE $v_0 = \langle v_2, v_3\rangle \lesssim \mathsf{trace}_{\mathsf{v}}\; \overline{b}_2\; \langle v_4, v_5\rangle = v_1$

5.1. SCASE $\mathit{tag\text{-}match}\,(\lfloor \tau_0 \rfloor, v_0)$

5.1.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
and $b_1 = (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)$

5.1.2. $v_2 \lesssim \mathit{add\text{-}trace}\,(\overline{b}_2, v_4)$ and $v_3 \lesssim \mathit{add\text{-}trace}\,(\overline{b}_2, v_5)$
by inversion $\lesssim$

5.1.3. $\tau_0 = \tau_2 \times \tau_3 \leqslant: \tau_4 \times \tau_5 = \tau_1$
by 5.1 and inversion $\leqslant:$
5.1.4. $\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0 \rhd_{\mathsf{N}} \langle \mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2, \mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3 \rangle$
by definition $\rhd_{\mathsf{N}}$
5.1.5. $\textit{tag-match}(\lfloor \tau_1 \rfloor, v_1)$
by lemma 8.17
5.1.6. $\mathsf{dyn}\ b_1\ v_1 \rhd_{\mathsf{A}} \mathsf{mon}\ b_1\ v_1$
by definition $\rhd_{\mathsf{A}}$
5.1.7. $\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2 \rightarrow^*_{\mathsf{N}} e_6 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\ (\textit{add-trace}(\overline{b}_2, v_4))$
and $\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3 \rightarrow^*_{\mathsf{N}} e_7 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\ (\textit{add-trace}(\overline{b}_2, v_5))$
by lemma 8.13
5.1.8. QED
either e_6 or $e_7 \in \mathsf{BndryErr}\,(b, v)$ or:

$$\frac{e_6 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\ (\textit{add-trace}(\overline{b}_2, v_4)) \qquad e_7 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\ (\textit{add-trace}(\overline{b}_2, v_5))}{\langle e_6, e_7 \rangle \lesssim \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (\mathsf{trace_v}\ \overline{b}_2\ \langle v_4, v_5 \rangle)}$$

5.2. SCASE $\neg \textit{tag-match}(\lfloor \tau_0 \rfloor, v_0)$
5.2.1. QED
$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}\,(b_0, v_0) \lesssim \mathsf{dyn}\ b_1\ v_1$

6. CASE $v_0 = \langle v_2, v_3 \rangle \lesssim \mathsf{trace_v}\ \overline{b}_2\ (\mathsf{mon}\ (\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3)\ \langle v_4, v_5 \rangle) = v_1$
6.1. SCASE $\textit{tag-match}(\lfloor \tau_0 \rfloor, v_0)$
6.1.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$
and $b_1 = (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)$
6.1.2. $v_2 \lesssim \textit{add-trace}(\overline{b}_2, (\mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\ v_4))$ and $v_3 \lesssim \textit{add-trace}(\overline{b}_2, (\mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\ v_5))$
by inversion $\lesssim$
6.1.3. $\tau_0 = \tau_2 \times \tau_3 \leqslant: \tau_4 \times \tau_5 = \tau_1$
by 6.1 and inversion $\leqslant:$
6.1.4. $\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0 \rhd_{\mathsf{N}} \langle \mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2, \mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3 \rangle$
by definition $\rhd_{\mathsf{N}}$
6.1.5. $\textit{tag-match}(\lfloor \tau_1 \rfloor, v_1)$
by lemma 8.17
6.1.6. $\mathsf{dyn}\ b_1\ v_1 \rhd_{\mathsf{A}} \mathsf{mon}\ b_1\ v_1$
by definition $\rhd_{\mathsf{A}}$
6.1.7. $\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2 \rightarrow^*_{\mathsf{N}} e_6 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\ (\textit{add-trace}(\overline{b}_2, (\mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\ v_4)))$
and $\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3 \rightarrow^*_{\mathsf{N}} e_7 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\ (\textit{add-trace}(\overline{b}_2, (\mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\ v_5)))$
by lemma 8.13
6.1.8. QED
either e_6 or $e_7 \in \mathsf{BndryErr}\,(b, v)$ or:

$$\frac{\begin{array}{c} e_6 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\ (\textit{add-trace}(\overline{b}_2, (\mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\ v_4))) \\ e_7 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\ (\textit{add-trace}(\overline{b}_2, (\mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\ v_5))) \end{array}}{\langle e_6, e_7 \rangle \lesssim \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (\mathsf{trace_v}\ \overline{b}_2\ (\mathsf{mon}\ (\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3)\ \langle v_4, v_5 \rangle))}$$

6.2. SCASE $\neg\textit{tag-match}(\lfloor\tau_0\rfloor, v_0)$

6.2.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}(b_0, v_0) \lesssim \mathsf{dyn}\ b_1\ v_1$

7. CASE $v_0 = \langle v_2, v_3\rangle \lesssim \mathsf{mon}\ b_2\ (\mathsf{trace_v}\ \overline{b}_3\ \langle v_4, v_5\rangle) = v_1$

7.1. CONTRADICTION:

$\cdot \vdash_{\mathsf{A}} v_1 : \mathcal{U}$

8. CASE $v_0 = \langle v_2, v_3\rangle \lesssim \mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ \langle v_4, v_5\rangle) = v_1$

8.1. CONTRADICTION:

by lemma 7.10

9. CASE $v_0 = \langle v_2, v_3\rangle \lesssim \mathsf{mon}\ b_2\ (\mathsf{trace_v}\ \overline{b}_3\ (\mathsf{mon}\ b_4\ \langle v_4, v_5\rangle)) = v_1$

9.1. CONTRADICTION:

by lemma 7.10

10. CASE $v_0 = \langle v_2, v_3\rangle \lesssim \mathsf{mon}^+\ \overline{b}_2\ (\mathsf{mon}\ b_3\ (\mathsf{mon}\ b_4\ \langle v_4, v_5\rangle)) = v_1$

10.1. CONTRADICTION:

by lemma 7.10

11. CASE $v_0 = \langle v_2, v_3\rangle \lesssim \mathsf{mon}^+\ \overline{b}_2\ (\mathsf{mon}\ b_3\ (\mathsf{trace_v}\ \overline{b}_4\ (\mathsf{mon}\ b_5\ \langle v_4, v_5\rangle))) = v_1$

11.1. CONTRADICTION:

by lemma 7.10

12. CASE $v_0 = \lambda x_2.\, e_2 \lesssim \lambda x_2.\, e_3 = v_1$

12.1. SCASE $\textit{tag-match}(\lfloor\tau_0\rfloor, v_0)$

12.1.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{mon}\ b_0\ v_0$ and $\mathsf{dyn}\ b_1\ v_1 \rhd_{\mathsf{A}} \mathsf{mon}\ b_1\ v_1$

12.2. SCASE $\neg\textit{tag-match}(\lfloor\tau_0\rfloor, v_0)$

12.2.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}(b_0, v_0) \lesssim \mathsf{dyn}\ b_1\ v_1$

13. CASE $v_0 = \lambda(x_2 : \tau_2).\, e_2 \lesssim \lambda(x_2 : \tau_2).\, e_3 = v_1$

13.1. CONTRADICTION:

$\cdot \vdash_{\mathsf{N}} v_0 : \mathcal{U}$

14. CASE $v_0 = \mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ v_2) \lesssim \mathsf{trace_v}\ b_4 b_5\ (\lambda x_0.\, e_3) = v_1$

14.1. SCASE $\textit{tag-match}(\lfloor\tau_0\rfloor, v_0)$

14.1.1. $\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{mon}\ b_0\ v_0$

by definition $\rhd_{\mathsf{N}}$

14.1.2. $\textit{tag-match}(\lfloor\tau_1\rfloor, v_1)$

by lemma 8.17

14.1.3. $\mathsf{dyn}\ b_1\ v_1 \rhd_{\mathsf{A}} \mathsf{mon}\ b_1\ v_1$

by definition $\rhd_{\mathsf{A}}$

14.1.4. QED

$$\frac{\mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ v_2) \lesssim \mathsf{trace_v}\ b_4 b_5\ (\lambda x_0.\, e_3)}{\mathsf{mon}\ b_0\ (\mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ v_2)) \lesssim \mathsf{mon}\ b_1\ (\mathsf{trace_v}\ b_4 b_5\ (\lambda x_0.\, e_3))}$$

14.2. SCASE $\neg$*tag-match* $(\lfloor\tau_0\rfloor, v_0)$

14.2.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}\,(b_0, v_0) \lesssim \mathsf{dyn}\ b_1\ v_1$

15. CASE $v_0 = \mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ v_2) \lesssim (\mathsf{trace}_{\mathsf{v}}\ b_4 b_5 \overline{b}_6\ (\lambda x_3.\, e_3)) = v_1$

15.1. *tag-match* $(\lfloor\tau_0\rfloor, v_0)$

15.1.1. $\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{mon}\ b_0\ v_0$

by definition $\rhd_{\mathsf{N}}$

15.1.2. *tag-match* $(\lfloor\tau_1\rfloor, v_1)$

by lemma 8.17

15.1.3. $\mathsf{dyn}\ b_1\ v_1 \rhd_{\mathsf{A}} \mathsf{mon}\ b_1\ v_1$

by definition $\rhd_{\mathsf{A}}$

15.1.4. QED

$$\frac{\mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ v_2) \lesssim (\mathsf{trace}_{\mathsf{v}}\ b_4 b_5 \overline{b}_6\ (\lambda x_3.\, e_3))}{\mathsf{mon}\ b_0\ (\mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ v_2)) \lesssim \mathsf{mon}\ b_1\ (\mathsf{trace}_{\mathsf{v}}\ b_4 b_5 \overline{b}_6\ (\lambda x_3.\, e_3))}$$

15.2. $\neg$*tag-match* $(\lfloor\tau_0\rfloor, v_0)$

15.2.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}\,(b_0, v_0) \lesssim \mathsf{dyn}\ b_1\ v_1$

16. CASE $v_0 = \mathsf{mon}\ b_2\ v_2 \lesssim \mathsf{mon}\ b_3\ v_3 = v_1$

16.1. SCASE *tag-match* $(\lfloor\tau_0\rfloor, v_0)$

16.1.1. $\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{mon}\ b_0\ v_0$

by definition $\rhd_{\mathsf{N}}$

16.1.2. *tag-match* $(\lfloor\tau_1\rfloor, v_1)$

by lemma 8.17

16.1.3. $\mathsf{dyn}\ b_1\ v_1 \rhd_{\mathsf{A}} \mathsf{mon}\ b_1\ v_1$

by definition $\rhd_{\mathsf{A}}$

16.1.4. QED

$$\frac{\mathsf{mon}\ b_2\ v_2 \lesssim \mathsf{mon}\ b_3\ v_3}{\mathsf{mon}\ b_0\ (\mathsf{mon}\ b_2\ v_2) \lesssim \mathsf{mon}\ b_1\ (\mathsf{mon}\ b_3\ v_3)}$$

16.2. SCASE $\neg$*tag-match* $(\lfloor\tau_0\rfloor, v_0)$

16.2.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}\,(b_0, v_0) \lesssim \mathsf{dyn}\ b_1\ v_1$

□

LEMMA 8.13. *If* $\cdot \vdash_{\mathsf{N}} v_0 : \mathcal{U}$ *and* $\cdot \vdash_{\mathsf{A}} v_1 : \mathcal{U}$ *and* $v_0 \lesssim v_1$ *and* $b_0 \leqslant: b_1$ *then* $\mathsf{dyn}\ b_0\ v_0 \rightarrow^*_{\mathsf{N}} e_2$ *and* $e_2 \lesssim \mathsf{mon}^?\ b_1\ v_1$.

PROOF. By induction on v_0 via case analysis of $v_0 \lesssim v_1$.

1. CASE $i_0 \lesssim i_0$

1.1. SCASE *tag-match* $(\lfloor\tau_0\rfloor, v_0)$

1.1.1. $\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} v_0$

by definition $\rhd_{\mathsf{N}}$

1.1.2. QED

$\mathsf{mon}^?\ b_1\ v_1 = v_1$

1.2. SCASE $\neg\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

1.2.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}(b_0, v_0) \lesssim \mathsf{mon}^?\ b_1\ v_1$

2. CASE $i_0 \lesssim \mathsf{trace_v}\ \overline{b_2}\ i_0$

2.1. SCASE $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

2.1.1. $\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} v_0$

by definition $\rhd_{\mathsf{N}}$

2.1.2. QED

$\mathsf{mon}^?\ b_1\ v_1 = v_1$

2.2. SCASE $\neg\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

2.2.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}(b_0, v_0) \lesssim \mathsf{mon}^?\ b_1\ v_1$

3. CASE $\langle v_2, v_3 \rangle \lesssim \langle v_4, v_5 \rangle$

3.1. SCASE $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

3.1.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_4 \times \tau_5 \blacktriangleleft \ell_1)$

3.1.2. $\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \langle \mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2, \mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3 \rangle$

by definition $\rhd_{\mathsf{N}}$

3.1.3. $v_2 \lesssim v_4$ and $v_3 \lesssim v_5$

by inversion $\lesssim$

3.1.4. $\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2 \rightarrow^*_{\mathsf{N}} e_6 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\ v_4$
and $\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3 \rightarrow^*_{\mathsf{N}} e_7 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\ v_5$

by the induction hypothesis

3.1.5. QED

either e_6 or $e_7 \in \mathsf{BndryErr}(b, v)$, or:

$$\frac{e_6 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\ v_4 \qquad e_7 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\ v_5}{\langle e_6, e_7 \rangle \lesssim \mathsf{mon}\ b_1\ \langle v_4, v_5 \rangle}$$

3.2. SCASE $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

3.2.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}(b_0, v_0) \lesssim \mathsf{mon}^?\ b_1\ v_1$

4. CASE $\langle v_2, v_3 \rangle \lesssim \mathsf{mon}\ (\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3)\ \langle v_4, v_5 \rangle$

4.1. SCASE $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

4.1.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_4 \times \tau_5 \blacktriangleleft \ell_1)$

4.1.2. $\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \langle \mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2, \mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3 \rangle$

by definition $\rhd_{\mathsf{N}}$

4.1.3. $v_2 \lesssim \mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\ v_4$ and $v_3 \lesssim \mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\ v_5$

by inversion $\lesssim$

4.1.4. $\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2 \rightarrow^*_{\mathsf{N}} e_6 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\ (\mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\ v_4)$
and $\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3 \rightarrow^*_{\mathsf{N}} e_7 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\ (\mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\ v_5)$

by the induction hypothesis

4.1.5. QED

either e_6 or $e_7 \in \mathsf{BndryErr}\,(b, v)$, or:

$$\frac{e_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\,v_4) \qquad e_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\,v_5)}{\langle e_6, e_7\rangle \lesssim \mathsf{mon}\,b_1\,(\mathsf{mon}\,(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3)\,\langle v_4, v_5\rangle)}$$

4.2. SCASE $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

4.2.1. QED

$\mathsf{dyn}\,b_0\,v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}\,(b_0, v_0) \lesssim \mathsf{mon}^?\,b_1\,v_1$

5. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{trace}_{\mathsf{v}}\,\overline{b}_2\,\langle v_4, v_5\rangle$
   - 5.1. SCASE $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$
     - 5.1.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_4 \times \tau_5 \blacktriangleleft \ell_1)$
     - 5.1.2. $\mathsf{dyn}\,b_0\,v_0 \rhd_{\mathsf{N}} \langle \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\,v_2, \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\,v_3\rangle$
       by definition $\rhd_{\mathsf{N}}$
     - 5.1.3. $v_2 \lesssim \mathit{add\text{-}trace}(\overline{b}_2, v_4)$ and $v_3 \lesssim \mathit{add\text{-}trace}(\overline{b}_2, v_5)$
       by inversion $\lesssim$
     - 5.1.4. $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\,v_2 \rightarrow^*_{\mathsf{N}} e_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathit{add\text{-}trace}(\overline{b}_2, v_4))$
       and $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\,v_3 \rightarrow^*_{\mathsf{N}} e_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathit{add\text{-}trace}(\overline{b}_2, v_5))$
       by the induction hypothesis
     - 5.1.5. QED
       either e_6 or $e_7 \in \mathsf{BndryErr}\,(b, v)$, or:
       $$\frac{e_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathit{add\text{-}trace}(\overline{b}_2, v_4)) \qquad e_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathit{add\text{-}trace}(\overline{b}_2, v_5))}{\langle e_6, e_7\rangle \lesssim \mathsf{mon}\,b_1\,(\mathsf{trace}_{\mathsf{v}}\,\overline{b}_2\,\langle v_4, v_5\rangle)}$$
   - 5.2. SCASE $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$
     - 5.2.1. QED
       $\mathsf{dyn}\,b_0\,v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}\,(b_0, v_0) \lesssim \mathsf{mon}^?\,b_1\,v_1$
6. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{trace}_{\mathsf{v}}\,\overline{b}_2\,\langle v_4, v_5\rangle$
   - 6.1. SCASE $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$
     - 6.1.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_4 \times \tau_5 \blacktriangleleft \ell_1)$
     - 6.1.2. $\mathsf{dyn}\,b_0\,v_0 \rhd_{\mathsf{N}} \langle \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\,v_2, \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\,v_3\rangle$
       by definition $\rhd_{\mathsf{N}}$
     - 6.1.3. $v_2 \lesssim \mathit{add\text{-}trace}(\overline{b}_2, v_4)$ and $v_3 \lesssim \mathit{add\text{-}trace}(\overline{b}_2, v_5)$
       by inversion $\lesssim$
     - 6.1.4. $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\,v_2 \rightarrow^*_{\mathsf{N}} e_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathit{add\text{-}trace}(\overline{b}_2, v_4))$
       and $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\,v_3 \rightarrow^*_{\mathsf{N}} e_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathit{add\text{-}trace}(\overline{b}_2, v_5))$
       by the induction hypothesis
     - 6.1.5. QED
       either e_6 or $e_7 \in \mathsf{BndryErr}\,(b, v)$, or:
       $$\frac{e_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathit{add\text{-}trace}(\overline{b}_2, v_4)) \qquad e_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathit{add\text{-}trace}(\overline{b}_2, v_5))}{\langle e_6, e_7\rangle \lesssim \mathsf{mon}\,b_1\,(\mathsf{trace}_{\mathsf{v}}\,\overline{b}_2\,\langle v_4, v_5\rangle)}$$
   - 6.2. SCASE $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

6.2.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}\,(b_0, v_0) \lesssim \mathsf{mon}^?\ b_1\ v_1$

7. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{trace}_{\mathsf{v}}\ \overline{b}_2\,(\mathsf{mon}\,(\ell_2 \blacktriangleleft \tau_6{\times}\tau_7 \blacktriangleleft \ell_1)\ \langle v_4, v_5\rangle)$

7.1. SCASE $\mathit{tag\text{-}match}\,(\lfloor\tau_0\rfloor, v_0)$

7.1.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2{\times}\tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_4{\times}\tau_5 \blacktriangleleft \ell_1)$

7.1.2. $\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \langle \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2, \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3\rangle$

by definition $\rhd_{\mathsf{N}}$

7.1.3. $v_2 \lesssim \mathit{add\text{-}trace}\,(\overline{b}_2, (\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_7)\ v_4))$ and $v_3 \lesssim \mathit{add\text{-}trace}\,(\overline{b}_2, (\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_7)\ v_5))$

by inversion $\lesssim$

7.1.4. $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2 \rightarrow^*_{\mathsf{N}} e_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathit{add\text{-}trace}\,(\overline{b}_2, (\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_7)\ v_4)))$

and $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3 \rightarrow^*_{\mathsf{N}} e_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathit{add\text{-}trace}\,(\overline{b}_2, (\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_7)\ v_5)))$

by the induction hypothesis

7.1.5. QED

either e_6 or $e_7 \in \mathsf{BndryErr}\,(b, v)$, or:

$$\frac{\begin{array}{c} e_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathit{add\text{-}trace}\,(\overline{b}_2, (\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_7)\ v_4))) \\ e_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathit{add\text{-}trace}\,(\overline{b}_2, (\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_7)\ v_5)))\end{array}}{\langle e_6, e_7\rangle \lesssim \mathsf{mon}\ b_1\ (\mathsf{trace}_{\mathsf{v}}\ \overline{b}_2\,(\mathsf{mon}\,(\ell_2 \blacktriangleleft \tau_6{\times}\tau_7 \blacktriangleleft \ell_3)\ \langle v_4, v_5\rangle))}$$

7.2. SCASE $\mathit{tag\text{-}match}\,(\lfloor\tau_0\rfloor, v_0)$

7.2.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}\,(b_0, v_0) \lesssim \mathsf{mon}^?\ b_1\ v_1$

8. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}\ b_2\ (\mathsf{trace}_{\mathsf{v}}\ \overline{b}_3\ \langle v_4, v_5\rangle)$

8.1. CONTRADICTION:

$\cdot \vdash_{\mathsf{A}} v_1 : \mathcal{U}$

9. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ \langle v_4, v_5\rangle)$

9.1. $v_2 \lesssim \mathsf{mon}^?\ b_2\ (\mathsf{mon}^?\ b_3\ v_4)$

and $v_3 \lesssim \mathsf{mon}^?\ b_2\ (\mathsf{mon}^?\ b_3\ v_5)$

by inversion $\lesssim$

9.2. $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0 \blacktriangleright_{\mathsf{N}} \langle \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2, \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3\rangle$

by definition $\blacktriangleright_{\mathsf{N}}$

9.3. $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2 \rightarrow^*_{\mathsf{N}} v_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\ b_2\ (\mathsf{mon}^?\ b_3\ v_4))$

and $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3 \rightarrow^*_{\mathsf{N}} v_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\ b_2\ (\mathsf{mon}^?\ b_3\ v_5))$

by the induction hypothesis

9.4. QED

$$\frac{v_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\ b_2\ (\mathsf{mon}^?\ b_3\ v_4)) \qquad v_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\ b_2\ (\mathsf{mon}^?\ b_3\ v_5))}{\langle v_6, v_7\rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\,(\mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ \langle v_4, v_5\rangle))}$$

10. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}\ b_2\ (\mathsf{trace}_{\mathsf{v}}\ \overline{b}_3\,(\mathsf{mon}\ b_4\ \langle v_4, v_5\rangle))$

10.1. $v_2 \lesssim \mathsf{mon}^?\ b_2\ (\mathsf{trace}_{\mathsf{v}}\ \overline{b}_3\,(\mathsf{mon}^?\ b_4\ v_4))$

and $v_3 \lesssim \mathsf{mon}^?\ b_2\ (\mathsf{trace}_{\mathsf{v}}\ \overline{b}_3\,(\mathsf{mon}^?\ b_4\ v_5))$

by inversion $\lesssim$

10.2. $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_0 \blacktriangleright_{\mathsf{N}} \langle \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\; v_2, \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\; v_3 \rangle$
by definition $\blacktriangleright_{\mathsf{N}}$

10.3. $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\; v_2 \rightarrow^*_{\mathsf{N}} v_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\; (\mathsf{mon}^?\; b_2\; (\mathsf{trace_v}\; \overline{b}_3\; (\mathsf{mon}^?\; b_4\; v_4)))$
and $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\; v_3 \rightarrow^*_{\mathsf{N}} v_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\; (\mathsf{mon}^?\; b_2\; (\mathsf{trace_v}\; \overline{b}_3\; (\mathsf{mon}^?\; b_4\; v_5)))$
by the induction hypothesis

10.4. QED

$$\frac{v_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\; (\mathsf{mon}^?\; b_2\; (\mathsf{trace_v}\; \overline{b}_3\; (\mathsf{mon}^?\; b_4\; v_4))) \qquad v_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\; (\mathsf{mon}^?\; b_2\; (\mathsf{trace_v}\; \overline{b}_3\; (\mathsf{mon}^?\; b_4\; v_5)))}{\langle v_6, v_7 \rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\; (\mathsf{mon}\; b_2\; (\mathsf{trace_v}\; \overline{b}_3\; (\mathsf{mon}\; b_4\; \langle v_4, v_5 \rangle)))}$$

11. CASE $\langle v_2, v_3 \rangle \lesssim \mathsf{mon}^+\; \overline{b}_2\; (\mathsf{mon}\; b_3\; (\mathsf{mon}\; b_4\; \langle v_4, v_5 \rangle))$

11.1. $v_2 \lesssim \mathsf{mon}^{+?}\; \overline{b}_2\; (\mathsf{mon}^?\; b_3\; (\mathsf{mon}^?\; b_4\; v_4))$
and $v_3 \lesssim \mathsf{mon}^{+?}\; \overline{b}_2\; (\mathsf{mon}^?\; b_3\; (\mathsf{mon}^?\; b_4\; v_5))$
by inversion $\lesssim$

11.2. $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_0 \blacktriangleright_{\mathsf{N}} \langle \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\; v_2, \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\; v_3 \rangle$
by definition $\blacktriangleright_{\mathsf{N}}$

11.3. $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\; v_2 \rightarrow^*_{\mathsf{N}} v_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\; (\mathsf{mon}^{+?}\; \overline{b}_2\; (\mathsf{mon}^?\; b_3\; (\mathsf{mon}^?\; b_4\; v_4)))$
and $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\; v_3 \rightarrow^*_{\mathsf{N}} v_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\; (\mathsf{mon}^{+?}\; \overline{b}_2\; (\mathsf{mon}^?\; b_3\; (\mathsf{mon}^?\; b_4\; v_5)))$
by the induction hypothesis

11.4. QED

$$\frac{v_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\; (\mathsf{mon}^{+?}\; \overline{b}_2\; (\mathsf{mon}^?\; b_3\; (\mathsf{mon}^?\; b_4\; v_4))) \qquad v_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\; (\mathsf{mon}^{+?}\; \overline{b}_2\; (\mathsf{mon}^?\; b_3\; (\mathsf{mon}^?\; b_4\; v_5)))}{\langle v_6, v_7 \rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\; (\mathsf{mon}^+\; \overline{b}_2\; (\mathsf{mon}\; b_3\; (\mathsf{mon}\; b_4\; \langle v_4, v_5 \rangle)))}$$

12. CASE $\langle v_2, v_3 \rangle \lesssim \mathsf{mon}^+\; \overline{b}_2\; (\mathsf{mon}\; b_3\; (\mathsf{trace_v}\; \overline{b}_4\; (\mathsf{mon}\; b_5\; \langle v_4, v_5 \rangle)))$

12.1. $v_2 \lesssim \mathsf{mon}^+\; \overline{b}_2\; (\mathsf{mon}\; b_3\; (\mathsf{trace_v}\; \overline{b}_4\; (\mathsf{mon}\; b_5\; v_4)))$
and $v_3 \lesssim \mathsf{mon}^+\; \overline{b}_2\; (\mathsf{mon}\; b_3\; (\mathsf{trace_v}\; \overline{b}_4\; (\mathsf{mon}\; b_5\; v_5)))$
by inversion $\lesssim$

12.2. $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_0 \blacktriangleright_{\mathsf{N}} \langle \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\; v_2, \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\; v_3 \rangle$
by definition $\blacktriangleright_{\mathsf{N}}$

12.3. $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\; v_2 \rightarrow^*_{\mathsf{N}} v_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\; (\mathsf{mon}^+\; \overline{b}_2\; (\mathsf{mon}\; b_3\; (\mathsf{trace_v}\; \overline{b}_4\; (\mathsf{mon}\; b_5\; v_4))))$
and $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\; v_3 \rightarrow^*_{\mathsf{N}} v_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\; (\mathsf{mon}^+\; \overline{b}_2\; (\mathsf{mon}\; b_3\; (\mathsf{trace_v}\; \overline{b}_4\; (\mathsf{mon}\; b_5\; v_5))))$
by the induction hypothesis

12.4. QED

$$\frac{v_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\; (\mathsf{mon}^+\; \overline{b}_2\; (\mathsf{mon}\; b_3\; (\mathsf{trace_v}\; \overline{b}_4\; (\mathsf{mon}\; b_5\; v_4)))) \qquad v_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\; (\mathsf{mon}^+\; \overline{b}_2\; (\mathsf{mon}\; b_3\; (\mathsf{trace_v}\; \overline{b}_4\; (\mathsf{mon}\; b_5\; v_5))))}{\langle v_6, v_7 \rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\; (\mathsf{mon}^+\; \overline{b}_2\; (\mathsf{mon}\; b_3\; (\mathsf{trace_v}\; \overline{b}_4\; (\mathsf{mon}\; b_5\; \langle v_4, v_5 \rangle))))}$$

13. CASE $\lambda x_2.\, e_2 \lesssim \lambda x_2.\, e_3$

13.1. SCASE $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

13.1.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{mon}\ b_0\ v_0 \lesssim \mathsf{mon}\ b_1\ v_1$

13.2. SCASE $\neg \mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

13.2.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}(b_0, v_0) \lesssim \mathsf{dyn}\ b_1\ v_1$

14. CASE $\lambda(x_2 : \tau_2).\, e_2 \lesssim \lambda(x_2 : \tau_2).\, e_3$

14.1. CONTRADICTION:

$\cdot \vdash_{\mathsf{N}} v_0 : \mathcal{U}$

15. CASE $\mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ v_2) \lesssim \mathsf{trace}_{\mathsf{v}}\ b_4 b_5\ (\lambda x_0.\, e_3)$

15.1. SCASE $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

15.1.1. $\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{mon}\ b_0\ v_0$

by definition $\rhd_{\mathsf{N}}$

15.1.2. QED

$$\frac{\mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ v_2) \lesssim \mathsf{trace}_{\mathsf{v}}\ b_4 b_5\ (\lambda x_0.\, e_3)}{\mathsf{mon}\ b_0\ (\mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ v_2)) \lesssim \mathsf{mon}\ b_1\ (\mathsf{trace}_{\mathsf{v}}\ b_4 b_5\ (\lambda x_0.\, e_3))}$$

15.2. SCASE $\neg \mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

15.2.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}(b_0, v_0) \lesssim \mathsf{dyn}\ b_1\ v_1$

16. CASE $\mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ v_2) \lesssim (\mathsf{trace}_{\mathsf{v}}\ b_4 b_5 \overline{b_6}\ (\lambda x_3.\, e_3))$

16.1. $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

16.1.1. $\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{mon}\ b_0\ v_0$

by definition $\rhd_{\mathsf{N}}$

16.1.2. QED

$$\frac{\mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ v_2) \lesssim (\mathsf{trace}_{\mathsf{v}}\ b_4 b_5 \overline{b_6}\ (\lambda x_3.\, e_3))}{\mathsf{mon}\ b_0\ (\mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ v_2)) \lesssim \mathsf{mon}\ b_1\ (\mathsf{trace}_{\mathsf{v}}\ b_4 b_5 \overline{b_6}\ (\lambda x_3.\, e_3))}$$

16.2. $\neg \mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

16.2.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}(b_0, v_0) \lesssim \mathsf{dyn}\ b_1\ v_1$

17. CASE $v_0 = \mathsf{mon}\ b_2\ v_2 \lesssim \mathsf{mon}\ b_3\ v_3 = v_1$

17.1. SCASE $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

17.1.1. $\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{mon}\ b_0\ v_0$

by definition $\rhd_{\mathsf{N}}$

17.1.2. QED

$$\frac{\mathsf{mon}\ b_2\ v_2 \lesssim \mathsf{mon}\ b_3\ v_3}{\mathsf{mon}\ b_0\ (\mathsf{mon}\ b_2\ v_2) \lesssim \mathsf{mon}\ b_1\ (\mathsf{mon}\ b_3\ v_3)}$$

17.2. SCASE $\neg \mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$

17.2.1. QED

$\mathsf{dyn}\ b_0\ v_0 \rhd_{\mathsf{N}} \mathsf{BndryErr}\,(b_0, v_0) \lesssim \mathsf{dyn}\ b_1\ v_1$

□

Lemma 8.14. *If* $\mathsf{stat}\ b_0\ v_0$ *and* $\mathsf{stat}\ b_1\ v_1$ *are reduced WF expressions and* $\mathsf{stat}\ b_0\ v_0 \lesssim \mathsf{stat}\ b_1\ v_1$ *and* $\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{N}} e_2$ *and* $\mathsf{stat}\ b_1\ v_1 \blacktriangleright_{\mathsf{A}} e_3$ *then* $e_2 \lesssim e_3$

Proof. By case analysis of $v_0 \lesssim v_1$.

1. case $i_0 \lesssim i_0$
   1.1. $\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{N}} v_0$ and $\mathsf{stat}\ b_1\ v_1 \blacktriangleright_{\mathsf{A}} v_1$
   by definition $\blacktriangleright_{\mathsf{N}}$, $\blacktriangleright_{\mathsf{A}}$
   1.2. qed
2. case $i_0 \lesssim \mathsf{trace}_{\mathsf{v}}\ \overline{b}_2\ i_0$
   2.1. Contradiction:
   $\cdot \vdash_{\mathsf{A}} v_1 : \tau_1$
3. case $\langle v_2, v_3\rangle \lesssim \langle v_4, v_5\rangle$
   3.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2{\times}\tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_2{\times}\tau_3 \blacktriangleleft \ell_1)$
   3.2. $v_2 \lesssim v_4$ and $v_3 \lesssim v_5$
   by inversion $\lesssim$
   3.3. $\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{N}} \langle \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2, \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3\rangle$
   and $\mathsf{stat}\ b_1\ v_1 \blacktriangleright_{\mathsf{A}} \mathsf{mon}\ b_1\ v_1$
   by definition $\blacktriangleright_{\mathsf{N}}$ and $\blacktriangleright_{\mathsf{A}}$
   3.4. $\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2 \rightarrow^*_{\mathsf{N}} v_6 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\ v_4$
   and $\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3 \rightarrow^*_{\mathsf{N}} v_7 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\ v_5$
   by lemma 8.15
   3.5. qed

$$\frac{v_6 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\ v_4 \qquad v_7 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\ v_5}{\langle v_6, v_7\rangle \lesssim \mathsf{mon}\ b_1\ \langle v_4, v_5\rangle}$$

4. case $\langle v_2, v_3\rangle \lesssim \mathsf{mon}\ (\ell_2 \blacktriangleleft \tau_6{\times}\tau_7 \blacktriangleleft \ell_3)\ \langle v_4, v_5\rangle$
   4.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2{\times}\tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_2{\times}\tau_3 \blacktriangleleft \ell_1)$
   4.2. $v_2 \lesssim \mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\ v_4$
   and $v_3 \lesssim \mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\ v_5$
   by inversion $\lesssim$
   4.3. $\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{N}} \langle \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2, \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3\rangle$
   and $\mathsf{stat}\ b_1\ v_1 \rightarrow^*_{\mathsf{A}} \mathsf{trace}_{\mathsf{v}}\ b_1 (\ell_2 \blacktriangleleft \tau_6{\times}\tau_7 \blacktriangleleft \ell_3)\ \langle v_4, v_5\rangle$
   by definition $\blacktriangleright_{\mathsf{N}}$ and $\blacktriangleright_{\mathsf{A}}$
   4.4. $\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2 \rightarrow^*_{\mathsf{N}} v_6 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\ (\mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\ v_4)$
   and $\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3 \rightarrow^*_{\mathsf{N}} v_7 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\ (\mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\ v_5)$
   by lemma 8.15
   4.5. $v_6 \lesssim \textit{add-trace}\,((\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3), v_4)$
   and $v_7 \lesssim \textit{add-trace}\,((\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3), v_5)$
   by lemma 8.16

4.6. QED

$$\frac{v_6 \lesssim \textit{add-trace}((\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3), v_4) \qquad v_7 \lesssim \textit{add-trace}((\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3), v_5)}{\langle v_6, v_7 \rangle \lesssim \mathsf{trace}_{\mathsf{v}}\, b_1(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3)\, \langle v_4, v_5 \rangle}$$

5. CASE $\langle v_2, v_3 \rangle \lesssim \mathsf{trace}_{\mathsf{v}}\, \overline{b}_2\, \langle v_4, v_5 \rangle$
   5.1. CONTRADICTION:
   $\cdot \vdash_{\mathsf{A}} v_1 : \tau_1$
6. CASE $\langle v_2, v_3 \rangle \lesssim \mathsf{trace}_{\mathsf{v}}\, \overline{b}_2\, (\mathsf{mon}\ b_3\ \langle v_4, v_5 \rangle)$
   6.1. CONTRADICTION:
   $\cdot \vdash_{\mathsf{A}} v_1 : \tau_1$
7. CASE $\langle v_2, v_3 \rangle \lesssim \mathsf{mon}\ (\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3)\ (\mathsf{trace}_{\mathsf{v}}\, \overline{b}_4\, \langle v_4, v_5 \rangle)$
   7.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$
   7.2. $v_2 \lesssim \mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\ (\textit{add-trace}(\overline{b}_4, v_4))$
   and $v_3 \lesssim \mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\ (\textit{add-trace}(\overline{b}_4, v_5))$
   by inversion $\lesssim$
   7.3. $\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{N}} \langle \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2, \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3 \rangle$
   and $\mathsf{stat}\ b_1\ v_1 \rightarrow^*_{\mathsf{A}} \mathsf{trace}_{\mathsf{v}}\, b_1(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3)\overline{b}_4\, \langle v_4, v_5 \rangle$
   by definition $\blacktriangleright_{\mathsf{N}}$ and $\blacktriangleright_{\mathsf{A}}$
   7.4. $\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2 \rightarrow^*_{\mathsf{N}} v_6 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\ (\mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\ (\textit{add-trace}(\overline{b}_4, v_4)))$
   and $\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3 \rightarrow^*_{\mathsf{N}} v_7 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\ (\mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\ (\textit{add-trace}(\overline{b}_4, v_5)))$
   by lemma 8.15
   7.5. $v_6 \lesssim \textit{add-trace}((\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3), (\textit{add-trace}(\overline{b}_4, v_4)))$
   and $v_7 \lesssim \textit{add-trace}((\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3), (\textit{add-trace}(\overline{b}_4, v_5)))$
   by lemma 8.16
   7.6. QED

$$\frac{v_6 \lesssim \textit{add-trace}((\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\overline{b}_4, v_4) \qquad v_7 \lesssim \textit{add-trace}((\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\overline{b}_4, v_5)}{\langle v_6, v_7 \rangle \lesssim \mathsf{trace}_{\mathsf{v}}\, b_1(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3)\overline{b}_4\, \langle v_4, v_5 \rangle}$$

8. CASE $\langle v_2, v_3 \rangle \lesssim \mathsf{mon}\ (\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3)\ (\mathsf{mon}\ (\ell_4 \blacktriangleleft \tau_8 \times \tau_9 \blacktriangleleft \ell_5)\ \langle v_4, v_5 \rangle)$
   8.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)$
   8.2. $v_2 \lesssim \mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\ (\mathsf{mon}^?\ (\ell_4 \blacktriangleleft \tau_8 \blacktriangleleft \ell_5)\ v_4)$
   and $v_3 \lesssim \mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\ (\mathsf{mon}^?\ (\ell_4 \blacktriangleleft \tau_9 \blacktriangleleft \ell_5)\ v_5)$
   by inversion $\lesssim$
   8.3. $\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{N}} \langle \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2, \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3 \rangle$
   and $\mathsf{stat}\ b_1\ v_1 \rightarrow^*_{\mathsf{A}} \mathsf{trace}_{\mathsf{v}}\, b_1(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3)\,(\mathsf{mon}\ (\ell_4 \blacktriangleleft \tau_8 \times \tau_9 \blacktriangleleft \ell_5)\ \langle v_4, v_5 \rangle)$
   by definition $\blacktriangleright_{\mathsf{N}}$ and $\blacktriangleright_{\mathsf{A}}$
   8.4. $\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2 \rightarrow^*_{\mathsf{N}} v_6 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\ (\mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\ (\mathsf{mon}^?\ (\ell_4 \blacktriangleleft \tau_8 \blacktriangleleft \ell_5)\ v_4))$
   and $\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3 \rightarrow^*_{\mathsf{N}} v_7 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\ (\mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\ (\mathsf{mon}^?\ (\ell_4 \blacktriangleleft \tau_9 \blacktriangleleft \ell_5)\ v_5))$
   by lemma 8.15

8.5. $v_6 \lesssim \mathit{add\text{-}trace}((\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3), (\mathsf{mon}^? \, (\ell_4 \blacktriangleleft \tau_8 \blacktriangleleft \ell_5) \; v_4))$
and $v_7 \lesssim \mathit{add\text{-}trace}((\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3), (\mathsf{mon}^? \, (\ell_4 \blacktriangleleft \tau_9 \blacktriangleleft \ell_5) \; v_5))$
by lemma 8.16

8.6. QED

$$\frac{v_6 \lesssim \mathit{add\text{-}trace}((\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3), (\mathsf{mon}^? \, (\ell_4 \blacktriangleleft \tau_8 \blacktriangleleft \ell_5) \; v_4)) \qquad v_7 \lesssim \mathit{add\text{-}trace}((\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3), (\mathsf{mon}^? \, (\ell_4 \blacktriangleleft \tau_9 \blacktriangleleft \ell_5) \; v_5))}{\langle v_6, v_7 \rangle \lesssim \mathsf{trace_v} \; b_1(\ell_2 \blacktriangleleft \tau_6 {\times} \tau_7 \blacktriangleleft \ell_3) \, (\mathsf{mon} \, (\ell_4 \blacktriangleleft \tau_8 {\times} \tau_9 \blacktriangleleft \ell_5) \; \langle v_4, v_5 \rangle)}$$

9. CASE $\langle v_2, v_3 \rangle \lesssim \mathsf{mon} \, (\ell_2 \blacktriangleleft \tau_6 {\times} \tau_7 \blacktriangleleft \ell_3) \, (\mathsf{trace_v} \; \overline{b}_4 \, (\mathsf{mon} \, (\ell_5 \blacktriangleleft \tau_8 {\times} \tau_9 \blacktriangleleft \ell_6) \; \langle v_4, v_5 \rangle))$

9.1. let $b_0 = (\ell_0 \blacktriangleleft \tau_2 {\times} \tau_3 \blacktriangleleft \ell_1)$ and $b_1 = (\ell_0 \blacktriangleleft \tau_2 {\times} \tau_3 \blacktriangleleft \ell_1)$

9.2. $v_2 \lesssim \mathsf{mon}^? \, (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) \, (\mathsf{trace}^?_\mathsf{v} \; \overline{b}_4 \, (\mathsf{mon}^? \, (\ell_5 \blacktriangleleft \tau_8 \blacktriangleleft \ell_6) \; v_4))$
and $v_3 \lesssim \mathsf{mon}^? \, (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) \, (\mathsf{trace}^?_\mathsf{v} \; \overline{b}_4 \, (\mathsf{mon}^? \, (\ell_5 \blacktriangleleft \tau_9 \blacktriangleleft \ell_6) \; v_5))$
by inversion $\lesssim$

9.3. $\mathsf{stat} \; b_0 \; v_0 \blacktriangleright_\mathsf{N} \langle \mathsf{stat} \, (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) \; v_2, \mathsf{stat} \, (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) \; v_3 \rangle$
and $\mathsf{stat} \; b_1 \; v_1 \rightarrow^*_\mathsf{A} \mathsf{trace_v} \; b_1(\ell_2 \blacktriangleleft \tau_6 {\times} \tau_7 \blacktriangleleft \ell_3)\overline{b}_4 \, (\mathsf{mon} \, (\ell_5 \blacktriangleleft \tau_8 {\times} \tau_9 \blacktriangleleft \ell_6) \; \langle v_4, v_5 \rangle)$
by definition $\blacktriangleright_\mathsf{N}$ and $\blacktriangleright_\mathsf{A}$

9.4. $\mathsf{stat} \, (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1) \; v_2 \rightarrow^*_\mathsf{N} v_6 \lesssim \mathsf{mon}^? \, (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1) \, (\mathsf{mon}^? \, (\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3) \, (\mathsf{trace}^?_\mathsf{v} \; \overline{b}_4 \, (\mathsf{mon}^? \, (\ell_5 \blacktriangleleft \tau_8 \blacktriangleleft \ell_6) \; v_4)))$
and $\mathsf{stat} \, (\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1) \; v_3 \rightarrow^*_\mathsf{N} v_7 \lesssim \mathsf{mon}^? \, (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1) \, (\mathsf{mon}^? \, (\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3) \, (\mathsf{trace}^?_\mathsf{v} \; \overline{b}_4 \, (\mathsf{mon}^? \, (\ell_5 \blacktriangleleft \tau_9 \blacktriangleleft \ell_6) \; v_5)))$
by lemma 8.15

9.5. $v_6 \lesssim \mathit{add\text{-}trace}((\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3), (\mathsf{trace}^?_\mathsf{v} \; \overline{b}_4 \, (\mathsf{mon}^? \, (\ell_5 \blacktriangleleft \tau_8 \blacktriangleleft \ell_6) \; v_4)))$
and $v_7 \lesssim \mathit{add\text{-}trace}((\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3), (\mathsf{trace}^?_\mathsf{v} \; \overline{b}_4 \, (\mathsf{mon}^? \, (\ell_5 \blacktriangleleft \tau_9 \blacktriangleleft \ell_6) \; v_5)))$
by lemma 8.16

9.6. QED

$$\frac{v_6 \lesssim \mathit{add\text{-}trace}(b_1(\ell_2 \blacktriangleleft \tau_6 {\times} \tau_7 \blacktriangleleft \ell_3)\overline{b}_4, (\mathsf{mon}^? \, (\ell_5 \blacktriangleleft \tau_8 \blacktriangleleft \ell_6) \; v_4)) \qquad v_7 \lesssim \mathit{add\text{-}trace}(b_1(\ell_2 \blacktriangleleft \tau_6 {\times} \tau_7 \blacktriangleleft \ell_3)\overline{b}_4, (\mathsf{mon}^? \, (\ell_5 \blacktriangleleft \tau_9 \blacktriangleleft \ell_6) \; v_5))}{\langle v_6, v_7 \rangle \lesssim \mathsf{trace_v} \; b_1(\ell_2 \blacktriangleleft \tau_6 {\times} \tau_7 \blacktriangleleft \ell_3)\overline{b}_4 \, (\mathsf{mon} \, (\ell_5 \blacktriangleleft \tau_8 {\times} \tau_9 \blacktriangleleft \ell_6) \; \langle v_4, v_5 \rangle)}$$

10. CASE $\langle v_2, v_3 \rangle \lesssim \mathsf{mon}^+ \; \overline{b}_2 \, (\mathsf{mon} \; b_3 \, (\mathsf{mon} \; b_4 \; \langle v_4, v_5 \rangle))$

10.1. CONTRADICTION:
by lemma 7.9

11. CASE $\langle v_2, v_3 \rangle \lesssim \mathsf{mon}^+ \; \overline{b}_2 \, (\mathsf{mon} \; b_3 \, (\mathsf{trace_v} \; \overline{b}_4 \, (\mathsf{mon} \; b_5 \; \langle v_4, v_5 \rangle)))$

11.1. CONTRADICTION:
by lemma 7.9

12. CASE $\lambda x_2.\, e_2 \lesssim \lambda x_2.\, e_3$

12.1. CONTRADICTION:
$\cdot \vdash_\mathsf{N} v_0 : \tau_0$

13. CASE $\lambda(x_2 \!:\! \tau_2).\, e_2 \lesssim \lambda(x_2 \!:\! \tau_2).\, e_3$

13.1. QED
$\mathsf{stat} \; b_0 \; v_0 \blacktriangleright_\mathsf{N} \mathsf{mon} \; b_0 \; v_0$ and $\mathsf{stat} \; b_1 \; v_1 \blacktriangleright_\mathsf{N} \mathsf{mon} \; b_1 \; v_1$

14. CASE $\mathsf{mon} \; b_2 \, (\mathsf{mon} \; b_3 \; v_2) \lesssim \mathsf{trace_v} \; \overline{b}_4 \; v_3$

14.1. CONTRADICTION:

$\cdot \vdash_A v_1 : \tau_1$

15. CASE mon $b_2\ v_2 \lesssim$ mon $b_3\ v_3$

15.1. stat $b_0\ v_0 \blacktriangleright_N$ mon $b_0\ v_0$

by definition $\blacktriangleright_N$

15.2. stat $b_1\ v_1 \blacktriangleright_A$ *add-trace* $(b_1 b_3$ *get-trace* (v_3), *rem-trace* $(v_3))$

by definition $\blacktriangleright_A$

15.3. QED

$$\frac{b_0 \leqslant: b_1 \qquad b_1 \leqslant: b_3 \qquad v_2 \lesssim \mathsf{trace_v}\ \textit{get-trace}(v_3)\ \textit{rem-trace}(v_3)}{\mathsf{mon}\ b_0\ (\mathsf{mon}\ b_1\ v_2) \lesssim \mathsf{trace_v}\ b_1 b_3\ \textit{get-trace}(v_3)\ \textit{rem-trace}(v_3)}$$

□

LEMMA 8.15. *If v_0 and v_1 are reduced WF expressions and $v_0 \lesssim v_1$ and $\cdot \vdash_N v_0 : \tau_0$ and $\cdot \vdash_A v_1 : \tau_1$ and $\tau_0 \leqslant: \tau_1$ then* stat $(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0 \rightarrow_N^* v_2$ *and* $v_2 \lesssim$ mon$^?$ $(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ v_3$

PROOF. By induction on v_0 via case analysis of $v_0 \lesssim v_1$.

1. CASE $i_0 \lesssim i_0$

1.1. QED

stat $(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0 \blacktriangleright_N v_0 \lesssim$ mon$^?$ $(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ v_1$

1.2. QED

2. CASE $i_0 \lesssim$ trace$_v$ $\overline{b}_2\ i_0$

2.1. CONTRADICTION:

$\cdot \vdash_A v_1 : \tau_1$

3. CASE $\langle v_2, v_3\rangle \lesssim \langle v_4, v_5\rangle$

3.1. $v_2 \lesssim v_4$ and $v_3 \lesssim v_5$

by inversion $\lesssim$

3.2. stat $(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0 \blacktriangleright_N \langle$stat $(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2$, stat $(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3\rangle$

by definition $\blacktriangleright_N$

3.3. stat $(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2 \rightarrow_N^* v_6 \lesssim$ mon$^?$ $(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\ v_4$

and stat $(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3 \rightarrow_N^* v_7 \lesssim$ mon$^?$ $(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\ v_5$

by the induction hypothesis

3.4. QED

$$\frac{v_6 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\ v_4 \qquad v_7 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\ v_5}{\langle v_6, v_7\rangle \lesssim \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ \langle v_4, v_5\rangle}$$

4. CASE $\langle v_2, v_3\rangle \lesssim$ mon $(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3)\ \langle v_4, v_5\rangle$

4.1. $v_2 \lesssim$ mon$^?$ $(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\ v_4$

and $v_3 \lesssim$ mon$^?$ $(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\ v_5$

by inversion $\lesssim$

4.2. stat $(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0 \blacktriangleright_N \langle$stat $(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ v_2$, stat $(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\ v_3\rangle$

by definition $\blacktriangleright_N$

4.3. $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\; v_2 \rightarrow^{*}_{\mathsf{N}} v_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\; v_4)$
and $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\; v_3 \rightarrow^{*}_{\mathsf{N}} v_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\; v_5)$
by the induction hypothesis

4.4. QED

$$\frac{v_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\; v_4) \qquad v_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\; v_5)}{\langle v_6, v_7\rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_4 \times \tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}\,(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3)\; \langle v_4, v_5\rangle)}$$

5. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{trace}_{\mathsf{v}}\; \overline{b}_2\; \langle v_4, v_5\rangle$

5.1. CONTRADICTION:
$\cdot \vdash_{\mathsf{A}} v_1 : \tau_1$

6. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{trace}_{\mathsf{v}}\; \overline{b}_2\; (\mathsf{mon}\; b_3\; \langle v_4, v_5\rangle)$

6.1. CONTRADICTION:
$\cdot \vdash_{\mathsf{A}} v_1 : \tau_1$

7. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}\,(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3)\,(\mathsf{trace}_{\mathsf{v}}\; \overline{b}_4\; \langle v_4, v_5\rangle)$

7.1. $v_2 \lesssim \mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\,(\textit{add-trace}(\overline{b}_4, v_4))$
and $v_3 \lesssim \mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\,(\textit{add-trace}(\overline{b}_4, v_5))$
by inversion $\lesssim$

7.2. $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_0 \blacktriangleright_{\mathsf{N}} \langle \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\; v_2, \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\; v_3\rangle$
by definition $\blacktriangleright_{\mathsf{N}}$

7.3. $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\; v_2 \rightarrow^{*}_{\mathsf{N}} v_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\,(\textit{add-trace}(\overline{b}_4, v_4)))$
and $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\; v_3 \rightarrow^{*}_{\mathsf{N}} v_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\,(\textit{add-trace}(\overline{b}_4, v_5)))$
by the induction hypothesis

7.4. QED

$$\frac{\begin{array}{c} v_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\,(\textit{add-trace}(\overline{b}_4, v_4))) \\ v_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\,(\textit{add-trace}(\overline{b}_4, v_5))) \end{array}}{\langle v_6, v_7\rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_4 \times \tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}\,(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3)\,(\mathsf{trace}_{\mathsf{v}}\; \overline{b}_4\; \langle v_4, v_5\rangle))}$$

8. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}\,(\ell_2 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_3)\,(\mathsf{mon}\,(\ell_4 \blacktriangleleft \tau_8 \times \tau_9 \blacktriangleleft \ell_5)\; \langle v_4, v_5\rangle)$

8.1. $v_2 \lesssim \mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\,(\mathsf{mon}^?\,(\ell_4 \blacktriangleleft \tau_8 \blacktriangleleft \ell_5)\; v_4)$
and $v_3 \lesssim \mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\,(\mathsf{mon}^?\,(\ell_4 \blacktriangleleft \tau_9 \blacktriangleleft \ell_5)\; v_5)$
by inversion $\lesssim$

8.2. $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_0 \blacktriangleright_{\mathsf{N}} \langle \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\; v_2, \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\; v_3\rangle$
by definition $\blacktriangleright_{\mathsf{N}}$

8.3. $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\; v_2 \rightarrow^{*}_{\mathsf{N}} v_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\,(\mathsf{mon}^?\,(\ell_4 \blacktriangleleft \tau_8 \blacktriangleleft \ell_5)\; v_4))$
and $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\; v_3 \rightarrow^{*}_{\mathsf{N}} v_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\,(\mathsf{mon}^?\,(\ell_4 \blacktriangleleft \tau_9 \blacktriangleleft \ell_5)\; v_5))$
by the induction hypothesis

8.4. QED

$$\frac{\begin{array}{c} v_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\,(\mathsf{mon}^?\,(\ell_4 \blacktriangleleft \tau_8 \blacktriangleleft \ell_5)\,v_4)) \\ v_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\,(\mathsf{mon}^?\,(\ell_4 \blacktriangleleft \tau_9 \blacktriangleleft \ell_5)\,v_5)) \end{array}}{\langle v_6, v_7\rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_4{\times}\tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}\,(\ell_2 \blacktriangleleft \tau_6{\times}\tau_7 \blacktriangleleft \ell_3)\,(\mathsf{mon}\,(\ell_4 \blacktriangleleft \tau_8{\times}\tau_9 \blacktriangleleft \ell_5)\,\langle v_4, v_5\rangle))}$$

9. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}\,(\ell_2 \blacktriangleleft \tau_6{\times}\tau_7 \blacktriangleleft \ell_3)\,(\mathsf{trace}_\mathsf{v}\,\overline{b}_4\,(\mathsf{mon}\,(\ell_5 \blacktriangleleft \tau_8{\times}\tau_9 \blacktriangleleft \ell_6)\,\langle v_4, v_5\rangle))$
   9.1. $v_2 \lesssim \mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\,(\mathsf{trace}^?_\mathsf{v}\,\overline{b}_4\,(\mathsf{mon}^?\,(\ell_5 \blacktriangleleft \tau_8 \blacktriangleleft \ell_6)\,v_4))$
   and $v_3 \lesssim \mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\,(\mathsf{trace}^?_\mathsf{v}\,\overline{b}_4\,(\mathsf{mon}^?\,(\ell_5 \blacktriangleleft \tau_9 \blacktriangleleft \ell_6)\,v_5))$
   by inversion $\lesssim$
   9.2. $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\,v_0 \blacktriangleright_\mathsf{N} \langle \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\,v_2, \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\,v_3\rangle$
   by definition $\blacktriangleright_\mathsf{N}$
   9.3. $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\,v_2 \rightarrow^*_\mathsf{N} v_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\,(\mathsf{trace}^?_\mathsf{v}\,\overline{b}_4\,(\mathsf{mon}^?\,(\ell_5 \blacktriangleleft \tau_8 \blacktriangleleft \ell_6)\,v_4)))$
   and $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\,v_3 \rightarrow^*_\mathsf{N} v_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\,(\mathsf{trace}^?_\mathsf{v}\,\overline{b}_4\,(\mathsf{mon}^?\,(\ell_5 \blacktriangleleft \tau_9 \blacktriangleleft \ell_6)\,v_5)))$
   by the induction hypothesis
   9.4. QED

$$\frac{\begin{array}{c} v_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_6 \blacktriangleleft \ell_3)\,(\mathsf{trace}^?_\mathsf{v}\,\overline{b}_4\,(\mathsf{mon}^?\,(\ell_5 \blacktriangleleft \tau_8 \blacktriangleleft \ell_6)\,v_4))) \\ v_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}^?\,(\ell_2 \blacktriangleleft \tau_7 \blacktriangleleft \ell_3)\,(\mathsf{trace}^?_\mathsf{v}\,\overline{b}_4\,(\mathsf{mon}^?\,(\ell_5 \blacktriangleleft \tau_9 \blacktriangleleft \ell_6)\,v_5))) \end{array}}{\langle v_6, v_7\rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\,(\mathsf{mon}\,(\ell_2 \blacktriangleleft \tau_6{\times}\tau_7 \blacktriangleleft \ell_3)\,(\mathsf{trace}_\mathsf{v}\,\overline{b}_4\,(\mathsf{mon}\,(\ell_5 \blacktriangleleft \tau_8{\times}\tau_9 \blacktriangleleft \ell_6)\,\langle v_4, v_5\rangle)))}$$

10. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}^+\,\overline{b}_2\,(\mathsf{mon}\,b_3\,(\mathsf{mon}\,b_4\,\langle v_4, v_5\rangle))$
   10.1. $v_2 \lesssim \mathsf{mon}^{+?}\,\overline{b}_2\,(\mathsf{mon}^?\,b_3\,(\mathsf{mon}^?\,b_4\,v_4))$
   and $v_3 \lesssim \mathsf{mon}^{+?}\,\overline{b}_2\,(\mathsf{mon}^?\,b_3\,(\mathsf{mon}^?\,b_4\,v_5))$
   by inversion $\lesssim$
   10.2. $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\,v_0 \blacktriangleright_\mathsf{N} \langle \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\,v_2, \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\,v_3\rangle$
   by definition $\blacktriangleright_\mathsf{N}$
   10.3. $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\,v_2 \rightarrow^*_\mathsf{N} v_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathsf{mon}^{+?}\,\overline{b}_2\,(\mathsf{mon}^?\,b_3\,(\mathsf{mon}^?\,b_4\,v_4)))$
   and $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\,v_3 \rightarrow^*_\mathsf{N} v_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}^{+?}\,\overline{b}_2\,(\mathsf{mon}^?\,b_3\,(\mathsf{mon}^?\,b_4\,v_5)))$
   by the induction hypothesis
   10.4. QED

$$\frac{\begin{array}{c} v_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathsf{mon}^{+?}\,\overline{b}_2\,(\mathsf{mon}^?\,b_3\,(\mathsf{mon}^?\,b_4\,v_4))) \\ v_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}^{+?}\,\overline{b}_2\,(\mathsf{mon}^?\,b_3\,(\mathsf{mon}^?\,b_4\,v_5))) \end{array}}{\langle v_6, v_7\rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\,(\mathsf{mon}^+\,\overline{b}_2\,(\mathsf{mon}\,b_3\,(\mathsf{mon}\,b_4\,\langle v_4, v_5\rangle)))}$$

11. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}^+\,\overline{b}_2\,(\mathsf{mon}\,b_3\,(\mathsf{trace}_\mathsf{v}\,\overline{b}_4\,(\mathsf{mon}\,b_5\,\langle v_4, v_5\rangle)))$
   11.1. $v_2 \lesssim \mathsf{mon}^+\,\overline{b}_2\,(\mathsf{mon}\,b_3\,(\mathsf{trace}_\mathsf{v}\,\overline{b}_4\,(\mathsf{mon}\,b_5\,v_4)))$
   and $v_3 \lesssim \mathsf{mon}^+\,\overline{b}_2\,(\mathsf{mon}\,b_3\,(\mathsf{trace}_\mathsf{v}\,\overline{b}_4\,(\mathsf{mon}\,b_5\,v_5)))$
   by inversion $\lesssim$
   11.2. $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\,v_0 \blacktriangleright_\mathsf{N} \langle \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\,v_2, \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\,v_3\rangle$
   by definition $\blacktriangleright_\mathsf{N}$
   11.3. $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\,v_2 \rightarrow^*_\mathsf{N} v_6 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathsf{mon}^+\,\overline{b}_2\,(\mathsf{mon}\,b_3\,(\mathsf{trace}_\mathsf{v}\,\overline{b}_4\,(\mathsf{mon}\,b_5\,v_4))))$
   and $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_3 \blacktriangleleft \ell_1)\,v_3 \rightarrow^*_\mathsf{N} v_7 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}^+\,\overline{b}_2\,(\mathsf{mon}\,b_3\,(\mathsf{trace}_\mathsf{v}\,\overline{b}_4\,(\mathsf{mon}\,b_5\,v_5))))$

by the induction hypothesis

11.4. QED

$$\frac{v_6 \lesssim \mathsf{mon}^{?}\,(\ell_0 \blacktriangleleft \tau_4 \blacktriangleleft \ell_1)\,(\mathsf{mon}^{+}\,\overline{b}_2\,(\mathsf{mon}\,b_3\,(\mathsf{trace_v}\,\overline{b}_4\,(\mathsf{mon}\,b_5\,v_4)))) \quad v_7 \lesssim \mathsf{mon}^{?}\,(\ell_0 \blacktriangleleft \tau_5 \blacktriangleleft \ell_1)\,(\mathsf{mon}^{+}\,\overline{b}_2\,(\mathsf{mon}\,b_3\,(\mathsf{trace_v}\,\overline{b}_4\,(\mathsf{mon}\,b_5\,v_5))))}{\langle v_6, v_7\rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\,(\mathsf{mon}^{+}\,\overline{b}_2\,(\mathsf{mon}\,b_3\,(\mathsf{trace_v}\,\overline{b}_4\,(\mathsf{mon}\,b_5\,\langle v_4, v_5\rangle))))}$$

12. CASE $\lambda x_2.\,e_2 \lesssim \lambda x_2.\,e_3$

12.1. CONTRADICTION:

$\cdot \vdash_{\mathsf{N}} v_0 : \tau_0$

13. CASE $\lambda(x_2 : \tau_2).\,e_2 \lesssim \lambda(x_2 : \tau_2).\,e_3$

13.1. QED

$\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\,v_0 \blacktriangleright_{\mathsf{N}} \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\,v_0 \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\,v_1$

14. CASE $\mathsf{mon}\,b_2\,(\mathsf{mon}\,b_3\,v_2) \lesssim \mathsf{trace_v}\,\overline{b}_4\,v_3$

14.1. CONTRADICTION:

$\cdot \vdash_{\mathsf{A}} v_1 : \tau_1$

15. CASE $\mathsf{mon}\,b_2\,v_2 \lesssim \mathsf{mon}\,b_3\,v_3$

15.1. QED

$\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\,v_0 \blacktriangleright_{\mathsf{N}} \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\,v_0 \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\,v_1$

□

LEMMA 8.16. *If* $\cdot \vdash_{\mathsf{N}} v_0 : \mathcal{U}$ *and* $\cdot \vdash_{\mathsf{A}} \mathsf{mon}^{?}\,b_0\,(\mathsf{mon}^{?}\,b_1\,v_1) : \mathcal{U}$ *and* $v_0 \lesssim \mathsf{mon}^{?}\,b_0\,(\mathsf{mon}^{?}\,b_1\,v_1)$ *then* $v_0 \lesssim$ *add-trace*$(b_0 b_1, v_1)$.

PROOF. By induction on v_0 via case analysis of $v_0 \lesssim \mathsf{mon}^{?}\,b_0\,(\mathsf{mon}^{?}\,b_1\,v_1)$.

1. CASE $i_0 \lesssim i_0$

1.1. QED

$i_0 \lesssim \mathsf{trace_v}\,b_0 b_1\,i_0$

2. CASE $i_0 \lesssim \mathsf{trace_v}\,\overline{b}_2\,i_0$

2.1. QED

$i_0 \lesssim \mathsf{trace_v}\,b_0 b_1 \overline{b}_2\,i_0$

3. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}\,b_0\,(\mathsf{mon}\,b_1\,\langle v_4, v_5\rangle)$

3.1. $v_2 \lesssim \mathsf{mon}^{?}\,b_0\,(\mathsf{mon}^{?}\,b_1\,v_4)$
and $v_3 \lesssim \mathsf{mon}^{?}\,b_0\,(\mathsf{mon}^{?}\,b_1\,v_5)$

by inversion $\lesssim$

3.2. $v_2 \lesssim$ *add-trace*$(b_0 b_1, v_4)$
and $v_3 \lesssim$ *add-trace*$(b_0 b_1, v_5)$

by the induction hypothesis

3.3. QED

$$\frac{v_2 \lesssim \textit{add-trace}(b_0 b_1, v_4) \qquad v_3 \lesssim \textit{add-trace}(b_0 b_1, v_5)}{\langle v_2, v_3\rangle \lesssim \mathsf{trace_v}\,b_0 b_1\,\langle v_4, v_5\rangle}$$

4. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}\,b_0\,(\mathsf{mon}\,b_1\,(\mathsf{trace_v}\,\overline{b}_2\,\langle v_4, v_5\rangle))$

4.1. $v_2 \lesssim \mathsf{mon}^?\ b_0\ (\mathsf{mon}^?\ b_1\ (\mathit{add\text{-}trace}(\overline{b}_2, v_4)))$
and $v_3 \lesssim \mathsf{mon}^?\ b_0\ (\mathsf{mon}^?\ b_1\ (\mathit{add\text{-}trace}(\overline{b}_2, v_5)))$
by inversion $\lesssim$

4.2. $v_2 \lesssim \mathit{add\text{-}trace}(b_0 b_1, (\mathit{add\text{-}trace}(\overline{b}_2, v_4)))$
and $v_3 \lesssim \mathit{add\text{-}trace}(b_0 b_1, (\mathit{add\text{-}trace}(\overline{b}_2, v_5)))$
by the induction hypothesis

4.3. QED

$$\frac{v_2 \lesssim \mathit{add\text{-}trace}(b_0 b_1, (\mathit{add\text{-}trace}(\overline{b}_2, v_4))) \qquad v_3 \lesssim \mathit{add\text{-}trace}(b_0 b_1, (\mathit{add\text{-}trace}(\overline{b}_2, v_5)))}{\langle v_2, v_3 \rangle \lesssim \mathsf{trace}_{\mathsf{v}}\ b_0 b_1 \overline{b}_2\ \langle v_4, v_5 \rangle}$$

5. CASE $\langle v_2, v_3 \rangle \lesssim \mathsf{mon}\ b_0\ (\mathsf{mon}\ b_1\ (\mathsf{mon}\ b_2\ \langle v_4, v_5 \rangle))$

5.1. $v_2 \lesssim \mathsf{mon}^?\ b_0\ (\mathsf{mon}^?\ b_1\ (\mathsf{mon}^?\ b_2\ v_4))$
and $v_3 \lesssim \mathsf{mon}^?\ b_0\ (\mathsf{mon}^?\ b_1\ (\mathsf{mon}^?\ b_2\ v_5))$
by inversion $\lesssim$

5.2. $v_2 \lesssim \mathit{add\text{-}trace}(b_0 b_1, (\mathsf{mon}^?\ b_2\ v_4))$
and $v_3 \lesssim \mathit{add\text{-}trace}(b_0 b_1, (\mathsf{mon}^?\ b_2\ v_5))$
by the induction hypothesis

5.3. QED

$$\frac{v_2 \lesssim \mathit{add\text{-}trace}(b_0 b_1, (\mathsf{mon}^?\ b_2\ v_4)) \qquad v_3 \lesssim \mathit{add\text{-}trace}(b_0 b_1, (\mathsf{mon}^?\ b_2\ v_5))}{\langle v_2, v_3 \rangle \lesssim \mathsf{trace}_{\mathsf{v}}\ b_0 b_1\ (\mathsf{mon}\ b_2\ \langle v_4, v_5 \rangle)}$$

6. CASE $\langle v_2, v_3 \rangle \lesssim \mathsf{mon}\ b_0\ (\mathsf{mon}\ b_1\ (\mathsf{trace}_{\mathsf{v}}\ \overline{b}_2\ (\mathsf{mon}\ b_3\ \langle v_4, v_5 \rangle)))$

6.1. $v_2 \lesssim \mathsf{mon}^?\ b_0\ (\mathsf{mon}^?\ b_1\ (\mathit{add\text{-}trace}(\overline{b}_2, (\mathsf{mon}^?\ b_3\ v_4))))$
and $v_3 \lesssim \mathsf{mon}^?\ b_0\ (\mathsf{mon}^?\ b_1\ (\mathit{add\text{-}trace}(\overline{b}_2, (\mathsf{mon}^?\ b_3\ v_5))))$
by inversion $\lesssim$

6.2. $v_2 \lesssim \mathit{add\text{-}trace}(b_0 b_1, (\mathit{add\text{-}trace}(\overline{b}_2, (\mathsf{mon}^?\ b_3\ v_4))))$
and $v_3 \lesssim \mathit{add\text{-}trace}(b_0 b_1, (\mathit{add\text{-}trace}(\overline{b}_2, (\mathsf{mon}^?\ b_3\ v_5))))$
by the induction hypothesis

6.3. QED

$$\frac{v_2 \lesssim \mathit{add\text{-}trace}(b_0 b_1 \overline{b}_2, (\mathsf{mon}^?\ b_3\ v_4)) \qquad v_3 \lesssim \mathit{add\text{-}trace}(b_0 b_1 \overline{b}_2, (\mathsf{mon}^?\ b_3\ v_5))}{\langle v_2, v_3 \rangle \lesssim \mathsf{trace}_{\mathsf{v}}\ b_0 b_1 \overline{b}_2\ (\mathsf{mon}\ b_3\ \langle v_4, v_5 \rangle)}$$

7. CASE $\mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ v_2) \lesssim \mathsf{mon}\ b_0\ (\mathsf{mon}\ b_1\ \lambda x_3.\, e_3)$

7.1. $v_2 \lesssim \lambda x_3.\, e_3$
by inversion $\lesssim$

7.2. QED

$$\frac{v_2 \lesssim \lambda x_3.\, e_3}{\mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ v_2) \lesssim (\mathit{add\text{-}trace}(b_0 b_1, \lambda x_3.\, e_3))}$$

8. CASE $\mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ v_2) \lesssim \mathsf{mon}\ b_0\ (\mathsf{mon}\ b_n\ (\mathsf{trace}_{\mathsf{v}}\ \overline{b}_6\ \lambda x_3.\, e_3))$

8.1. $v_2 \lesssim (\mathsf{trace_v}\ \overline{b}_6\ \lambda x_3.\, e_3)$
by inversion $\lesssim$
8.2. QED

$$\frac{v_2 \lesssim \lambda x_3.\, e_3}{\mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ v_2) \lesssim (\mathit{add\text{-}trace}(b_0 b_1 \overline{b}_6, \lambda x_3.\, e_3))}$$

9. CASE $\mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ v_2) \lesssim \mathsf{mon}\ b_0\ (\mathsf{mon}\ b_1\ \lambda(x_3 : \tau_3).\, e_3)$
9.1. Contradiction:
$\cdot \vdash_{\mathsf{A}} v_1 : \mathcal{U}$

□

Lemma 8.17. *If* $\cdot \vdash_{\mathsf{N}} v_0 : \mathcal{U}$ *and* $\cdot \vdash_{\mathsf{A}} v_1 : \mathcal{U}$ *and* $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$ *and* $v_0 \lesssim v_1$ *then* $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_1)$

Proof. By induction on v_1 via case analysis of $v_0 \lesssim v_1$.

1. CASE $i_0 \lesssim i_0$
1.1. $\tau_0 \in \mathsf{Nat} \cup \mathsf{Int}$
by inversion *tag-match*
1.2. SCASE $\tau_0 = \mathsf{Nat}$
1.2.1. $i_0 \in \mathbb{N}$
by inversion *tag-match*
1.2.2. QED
$\mathit{tag\text{-}match}(\lfloor \mathsf{Nat} \rfloor, i_0)$
1.3. SCASE $\tau_0 = \mathsf{Int}$
1.3.1. QED
$\mathit{tag\text{-}match}(\lfloor \mathsf{Int} \rfloor, i_0)$
2. CASE $i_0 \lesssim \mathsf{trace_v}\ \overline{b}_0\ i_0$
similar to previous case
3. CASE $\langle v_2, v_3 \rangle \lesssim \langle v_4, v_5 \rangle$
similar to previous case
4. CASE $\langle v_2, v_3 \rangle \lesssim \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_1)\ \langle v_4, v_5 \rangle$
similar to previous case
5. CASE $\langle v_2, v_3 \rangle \lesssim \mathsf{trace_v}\ \overline{b}_0\ \langle v_4, v_5 \rangle$
similar to previous case
6. CASE $\langle v_2, v_3 \rangle \lesssim \mathsf{trace_v}\ \overline{b}_0\ (\mathsf{mon}\ (\ell_1 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_2)\ \langle v_4, v_5 \rangle)$
similar to previous case
7. CASE $\langle v_2, v_3 \rangle \lesssim \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_6 \times \tau_7 \blacktriangleleft \ell_1)\ (\mathsf{trace_v}\ \overline{b}_2\ \langle v_4, v_5 \rangle)$
similar to previous case
8. CASE $\langle v_2, v_3 \rangle \lesssim \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_6 \times \tau_6 \blacktriangleleft \ell_1)\ (\mathsf{mon}\ b_2\ \langle v_4, v_5 \rangle)$
similar to previous case
9. CASE $\langle v_2, v_3 \rangle \lesssim \mathsf{mon}\ \ell_0\ \tau_6 \times \tau_7 \ell_1 (\mathsf{trace_v}\ \overline{b}_2\ (\mathsf{mon}\ b_3\ \langle v_4, v_5 \rangle))$
similar to previous case
10. CASE $\langle v_2, v_3 \rangle \lesssim \mathsf{mon}^{+}\ \overline{b}_0\ (\mathsf{mon}\ b_1\ (\mathsf{mon}\ b_2\ \langle v_4, v_5 \rangle))$

similar to previous case

11. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}^{+}\, \overline{b}_0\, (\mathsf{mon}\ b_1\ (\mathsf{trace_v}\ \overline{b}_2\ (\mathsf{mon}\ b_3\ \langle v_4, v_5\rangle)))$
   11.1. $\tau_0 \in \tau\times\tau$
      by inversion *tag-match*
   11.2. QED
12. CASE $\lambda x_2.\, e_2 \lesssim \lambda x_2.\, e_3$
   12.1. $\tau_0 \in \tau \Rightarrow \tau$
      by inversion *tag-match*
   12.2. QED
13. CASE $\lambda(x_2 : \tau_2).\, e_2 \lesssim \lambda(x_2 : \tau_2).\, e_3$
   13.1. CONTRADICTION:
      $\cdot \vdash_{\mathsf{N}} v_0 : \mathcal{U}$
14. CASE $\mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_2 \Rightarrow \tau_3 \blacktriangleleft \ell_1)\,(\mathsf{mon}\ b_2\ v_2) \lesssim \mathsf{trace_v}\ \overline{b}_3\ \lambda x_3.\, e_3$
   14.1. $\tau_0 \in \tau \Rightarrow \tau$
      by inversion *tag-match*
   14.2. QED
15. CASE $\mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_2 \Rightarrow \tau_3 \blacktriangleleft \ell_1)\, v_2 \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_4 \Rightarrow \tau_5 \blacktriangleleft \ell_1)\, v_3$
   similar to previous case

□

LEMMA 8.18. *If v_0 and $\mathsf{mon}^{?}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\, v_1$ are reduced WF expressions and $\cdot \vdash_{\mathsf{N}} v_0 : \tau_0$and $\cdot \vdash_{\mathsf{A}} \mathsf{mon}^{?}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\, v_1 : \tau_1$ and $\tau_0 \leqslant: \tau_1$ and $v_0 \lesssim \mathsf{mon}^{?}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\, v_1$ then $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\, v_1 \rightarrow^{*}_{\mathsf{A}} v_2$ and $v_0 \lesssim v_2$.*

PROOF. By case analysis of τ_0 and $\lesssim$.

1. CASE $\tau_0 = \mathsf{Nat}$
   1.1. $v_0 \in \mathbb{N}$
      by inversion $\vdash_{\mathsf{N}}$
   1.2. SCASE $v_0 \lesssim v_0$
      1.2.1. QED
         $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\, v_1 \rhd_{\mathsf{A}} v_0 = \mathsf{mon}^{?}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\, v_1$
   1.3. SCASE $v_0 \lesssim \mathsf{trace_v}\ \overline{b}_0\ v_0$
      1.3.1. QED
         $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\, v_1 \rhd_{\mathsf{A}} \mathsf{trace_v}\ \overline{b}_0\ v_0 = \mathsf{mon}^{?}\,(\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\, v_1$
2. CASE $\tau_0 = \mathsf{Int}$
   similar to previous case
3. CASE $\tau_0 = \tau_1\times\tau_2$
   3.1. $v_0 \in \langle v, v\rangle$
      by inversion $\vdash_{\mathsf{N}}$
   3.2. SCASE $v_0 \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_3\times\tau_4 \blacktriangleleft \ell_1)\,\langle v_2, v_3\rangle$
      3.2.1. QED
         $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_3\times\tau_4 \blacktriangleleft \ell_1)\, v_1 \rhd_{\mathsf{A}} \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_3\times\tau_4 \blacktriangleleft \ell_1)\, v_1$
   3.3. SCASE $v_0 \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_3\times\tau_4 \blacktriangleleft \ell_1)\,(\mathsf{trace_v}\ \overline{b}_2\ \langle v_2, v_3\rangle)$

3.3.1. QED

$\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1)\ v_1 \rhd_{\mathsf{A}} \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1)\ v_1$

3.4. SCASE $v_0 \lesssim \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1)\ (\mathsf{mon}\ b_2\ \langle v_2, v_3 \rangle)$

3.4.1. QED

$\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1)\ v_1 \rhd_{\mathsf{A}} \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1)\ v_1$

3.5. SCASE $v_0 \lesssim \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1)\ (\mathsf{trace}_{\mathsf{v}}\ \overline{b}_2\ (\mathsf{mon}\ b_3\ \langle v_2, v_3 \rangle))$

3.5.1. CONTRADICTION:

$\cdot \vdash_{\mathsf{A}} v_1 : \mathcal{U}$

3.6. SCASE $v_0 \lesssim \mathsf{mon}^{+}\ \overline{b}_0\ (\mathsf{mon}\ b_1\ (\mathsf{mon}\ b_2\ \langle v_2, v_3 \rangle))$

3.6.1. CONTRADICTION:

by lemma 7.10

3.7. SCASE $v_0 \lesssim \mathsf{mon}^{+}\ \overline{b}_0\ (\mathsf{mon}\ b_1\ (\mathsf{trace}_{\mathsf{v}}\ \overline{b}_2\ (\mathsf{mon}\ b_3\ \langle v_2, v_3 \rangle)))$

3.7.1. CONTRADICTION:

by lemma 7.10

4. CASE $\tau_0 = \tau_1 \Rightarrow \tau_2$

4.1. $v_0 \in \lambda x.\, e \cup \mathsf{mon}\ b\ v$

by inversion $\vdash_{\mathsf{N}}$

4.2. SCASE $v_0 \lesssim \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_3 \Rightarrow \tau_4 \blacktriangleleft \ell_1)\ v_1$

4.2.1. QED

$\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_3 \Rightarrow \tau_4 \blacktriangleleft \ell_1)\ v_1 \rhd_{\mathsf{A}} \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_3 \Rightarrow \tau_4 \blacktriangleleft \ell_1)\ v_1$

□

LEMMA 8.19. *If* v_0 *and* $\mathsf{mon}^{?}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (\mathsf{mon}^{?}\ (\ell_2 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3)\ v_1)$ *are reduced WF expressions and* $\cdot \vdash_{\mathsf{N}} v_0 : \tau_0$ *and* $\cdot \vdash_{\mathsf{A}} \mathsf{mon}^{?}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (\mathsf{mon}^{?}\ (\ell_2 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3)\ v_1) : \tau_1$ *and* $v_0 \lesssim \mathsf{mon}^{?}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (\mathsf{mon}^{?}\ (\ell_2 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3)\ v_1)$ *and* $\tau_0 \leqslant: \tau_1$ *then* $\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (\mathsf{stat}\ (\ell_2 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3)\ v_1) \rightarrow^{*}_{\mathsf{A}} v_2$ *and* $v_0 \lesssim v_2$.

PROOF. By case analysis of τ_0 and $\lesssim$.

1. CASE $\tau_0 = \mathsf{Nat}$

1.1. $v_0 \in \mathbb{N}$

by inversion $\vdash_{\mathsf{N}}$

1.2. SCASE $v_0 \lesssim v_0$

1.2.1. QED

$\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (\mathsf{stat}\ (\ell_2 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3)\ v_1) \rightarrow^{*}_{\mathsf{A}} v_0 = \mathsf{mon}^{?}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (\mathsf{mon}^{?}\ (\ell_2 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3)\ v_1)$

1.3. SCASE $v_0 \lesssim \mathsf{trace}_{\mathsf{v}}\ \overline{b}_0\ v_0$

1.3.1. QED

$\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (\mathsf{stat}\ (\ell_2 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3)\ v_1) \rightarrow^{*}_{\mathsf{A}} v_0 = \mathsf{mon}^{?}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (\mathsf{mon}^{?}\ (\ell_2 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3)\ v_1)$

2. CASE $\tau_0 = \mathsf{Int}$

similar to previous case

3. CASE $\tau_0 = \tau_1 \times \tau_2$

3.1. $v_0 \in \langle v, v \rangle$

by inversion $\vdash_{\mathsf{N}}$

3.2. SCASE $v_0 \lesssim \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1)\ \langle v_2, v_3 \rangle$

3.2.1. QED

dyn $(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1)\ v_1 \rhd_{\mathsf{A}} \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1)\ v_1$

3.3. SCASE $v_0 \lesssim \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1)\ (\mathsf{trace_v}\ \overline{b}_2\ \langle v_2, v_3 \rangle)$

3.3.1. QED

dyn $(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1)\ v_1 \rhd_{\mathsf{A}} \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1)\ v_1$

3.4. SCASE $v_0 \lesssim \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1)\ (\mathsf{mon}\ (\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3)\ \langle v_2, v_3 \rangle)$

3.4.1. $\tau_5 \in \tau \times \tau$

by inversion $\vdash_{\mathsf{A}}$

3.4.2. QED

dyn $(\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1)\ (\mathsf{stat}\ (\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3)\ \langle v_2, v_3 \rangle) \rhd_{\mathsf{A}} \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_3 \times \tau_4 \blacktriangleleft \ell_1)\ (\mathsf{mon}\ (\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3)\ \langle v_2, v_3 \rangle)$

3.5. SCASE $v_0 \lesssim \mathsf{mon}^+\ \overline{b}_0\ (\mathsf{mon}\ b_1\ (\mathsf{mon}\ b_2\ \langle v_2, v_3 \rangle))$

3.5.1. CONTRADICTION:

by lemma 7.9

3.6. SCASE $v_0 \lesssim \mathsf{mon}^+\ \overline{b}_0\ (\mathsf{mon}\ b_1\ (\mathsf{trace_v}\ \overline{b}_2\ (\mathsf{mon}\ b_3\ \langle v_2, v_3 \rangle)))$

3.6.1. CONTRADICTION:

by lemma 7.9

4. CASE $\tau_0 = \tau_1 \Rightarrow \tau_2$

4.1. $v_0 \in \lambda x.\, e \cup \mathsf{mon}\ b\ v$

by inversion $\vdash_{\mathsf{N}}$

4.2. SCASE $v_0 \lesssim \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_3 \Rightarrow \tau_4 \blacktriangleleft \ell_1)\ (\mathsf{mon}\ (\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3)\ v_1)$

4.2.1. $\tau_5 \in \tau \Rightarrow \tau$

by inversion $\vdash_{\mathsf{A}}$

4.2.2. QED

$\mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_3 \Rightarrow \tau_4 \blacktriangleleft \ell_1)\ (\mathsf{mon}\ (\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3)\ v_1) \rhd_{\mathsf{A}} \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_3 \Rightarrow \tau_4 \blacktriangleleft \ell_1)\ (\mathsf{mon}\ (\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3)\ v_1)$

□

LEMMA 8.20. *If tag-match*$(\lfloor \tau_0 \rfloor, v_0)$ *and* $\tau_0 \leqslant: \tau_1$ *then tag-match*$(\lfloor \tau_1 \rfloor, v_0)$.

PROOF. By induction on v_0 via case analysis of *tag-match*$(\lfloor \tau_0 \rfloor, v_0)$.

1. CASE *tag-match*$(\lfloor \mathsf{Nat} \rfloor, i_0)$

1.1. QED

$\tau_1 \in \mathsf{Nat} \cup \mathsf{Int}$

2. CASE *tag-match*$(\lfloor \mathsf{Int} \rfloor, i_0)$

2.1. QED

$\tau_1 \in \mathsf{Int}$

3. CASE *tag-match*$(\lfloor \tau_2 \Rightarrow \tau_3 \rfloor, \lambda x_0.\, e_0)$

3.1. QED

$\tau_1 \in \tau \Rightarrow \tau$

4. CASE *tag-match*$(\lfloor \tau_3 \Rightarrow \tau_4 \rfloor, \lambda(x_0 : \tau_2).\, e_0)$

4.1. QED

$\tau_1 \in \tau \Rightarrow \tau$

5. CASE *tag-match*$(\lfloor \tau_2 \times \tau_3 \rfloor, \langle v_2, v_3 \rangle)$

5.1. QED

$\tau_1 \in \tau \times \tau$

6. CASE $\mathit{tag\text{-}match}(\lfloor \tau_4 \Rightarrow \tau_5 \rfloor, \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_2 \Rightarrow \tau_3 \blacktriangleleft \ell_1)\ v_1)$

6.1. QED

$\tau_1 \in \tau \Rightarrow \tau$

7. CASE $\mathit{tag\text{-}match}(\lfloor \tau_4 \times \tau_5 \rfloor, \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)\ v_1)$

7.1. QED

$\tau_1 \in \tau \times \tau$

8. CASE $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, \mathsf{trace}_{\mathsf{v}}\ \overline{b}_0\ v_1)$

8.1. $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_1)$

by inversion *tag-match*

8.2. $\mathit{tag\text{-}match}(\lfloor \tau_1 \rfloor, v_1)$

by the induction hypothesis

8.3. QED

8.4. QED

by definition *tag-match*

□

LEMMA 8.21. *If* $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$ *and* $\cdot \vdash_{\mathsf{A}} v_0 : \mathcal{U}$ *then* $\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0 \rhd_{\mathsf{A}} \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0$.

PROOF. By case analysis of $\mathit{tag\text{-}match}(\lfloor \tau_0 \rfloor, v_0)$.

1. CASE $\mathit{tag\text{-}match}(\lfloor \mathsf{Nat} \rfloor, n_0)$

1.1. QED

$\mathsf{dyn}\ (\ell_0 \blacktriangleleft \mathsf{Nat} \blacktriangleleft \ell_1)\ n_0 \rhd_{\mathsf{A}} n_0$

2. CASE $\mathit{tag\text{-}match}(\lfloor \mathsf{Int} \rfloor, i_0)$

2.1. QED

$\mathsf{dyn}\ (\ell_0 \blacktriangleleft \mathsf{Int} \blacktriangleleft \ell_1)\ i_0 \rhd_{\mathsf{A}} i_0$

3. CASE $\mathit{tag\text{-}match}(\lfloor \tau_1 \Rightarrow \tau_2 \rfloor, \lambda x_0.\, e_0)$

3.1. QED

$\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0 \rhd_{\mathsf{A}} \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0$

4. CASE $\mathit{tag\text{-}match}(\lfloor \tau_2 \Rightarrow \tau_3 \rfloor, \lambda (x_0 : \tau_1).\, e_0)$

4.1. CONTRADICTION:

$\cdot \vdash_{\mathsf{A}} v_0 : \mathcal{U}$

5. CASE $\mathit{tag\text{-}match}(\lfloor \tau_1 \times \tau_2 \rfloor, \langle v_1, v_2 \rangle)$

$\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0 \rhd_{\mathsf{A}} \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0$

6. CASE $\mathit{tag\text{-}match}(\lfloor \tau_3 \Rightarrow \tau_4 \rfloor, \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_1 \Rightarrow \tau_2 \blacktriangleleft \ell_1)\ v_1)$

$\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0 \rhd_{\mathsf{A}} \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0$

7. CASE $\mathit{tag\text{-}match}(\lfloor \tau_3 \times \tau_4 \rfloor, \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_1 \times \tau_2 \blacktriangleleft \ell_1)\ v_1)$

$\mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0 \rhd_{\mathsf{A}} \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0$

8. CASE $\mathit{tag\text{-}match}(\lfloor \tau_1 \rfloor, \mathsf{trace}_{\mathsf{v}}\ \overline{b}_0\ v_1)$

8.1. $\mathit{tag\text{-}match}(\lfloor \tau_1 \rfloor, v_1)$

by inversion *tag-match*

8.2. $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_1 \rhd_{\mathsf{A}} \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_1$
by the induction hypothesis
8.3. QED
$\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; \mathsf{trace_v}\, \overline{b}_0\; v_1 \rhd_{\mathsf{A}} \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; (\mathsf{trace_v}\, \overline{b}_0\; v_1)$

□

Lemma 8.22. *If v_0 and v_1 are reduced WF expressions and $unop\{\tau?\}\; v_0 \lesssim unop\{\tau?\}\; v_1$ and $\cdot \vdash_{\mathsf{N}} unop\{\tau?\}\; v_0 : \tau_0$ and $\cdot \vdash_{\mathsf{A}} unop\{\tau?\}\; v_1 : \tau_1$ and $\tau_0 \leqslant: \tau_1$ then $unop\{\tau?\}\; v_1 \rightarrow^*_{\mathsf{A}} v_2$ and $\delta_N(unop, v_0) \lesssim v_2$*

Proof. By induction on v_1 via case analysis of $\lesssim$.The cases for $unop = \mathsf{fst}\{\tau_2\}$ and $unop = \mathsf{snd}\{\tau_2\}$ are analogous; we show only the fst cases.

1. CASE $i_0 \lesssim i_0$
1.1. Contradiction:
$\cdot \vdash_{\mathsf{N}} unop\{\tau?\}\; v_0 : \tau_0$
2. CASE $i_0 \lesssim \mathsf{trace_v}\, \overline{b}_0\; i_0$
2.1. Contradiction:
$\cdot \vdash_{\mathsf{N}} unop\{\tau?\}\; v_0 : \tau_0$
3. CASE $\langle v_2, v_3\rangle \lesssim \langle v_4, v_5\rangle$
3.1. $v_2 \lesssim v_4$
by inversion $\lesssim$
3.2. $\delta_N(unop, v_0) = v_2$
by definition δ_N
3.3. $unop\{\tau?\}\; v_1 \rhd_{\mathsf{A}} v_4$
by definition $\rhd_{\mathsf{A}}$
3.4. QED
by 3.1
4. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_3{\times}\tau_4 \blacktriangleleft \ell_1)\; \langle v_4, v_5\rangle$
4.1. $v_2 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\; v_4$
by inversion $\lesssim$ and $\tau_3 \leqslant: \tau_2$
4.2. $\delta_N(unop, v_0) = v_2$
by definition δ_N
4.3. $unop\{\tau?\}\; v_1$
$\rightarrow_{\mathsf{A}} \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\; (unop\{\tau?\}\; \langle v_4, v_5\rangle)$
$\rightarrow_{\mathsf{A}} \mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\; v_4$
by definition $\rightarrow_{\mathsf{A}}$
4.4. QED
by lemma 8.18
5. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{trace_v}\, \overline{b}_0\; \langle v_4, v_5\rangle$
5.1. Contradiction:
$\cdot \vdash_{\mathsf{A}} v_1 : \tau_1$
6. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{trace_v}\, \overline{b}_0\; (\mathsf{mon}\,(\ell_1 \blacktriangleleft \tau_3{\times}\tau_4 \blacktriangleleft \ell_2)\; \langle v_4, v_5\rangle)$
6.1. Contradiction:

$\cdot \vdash_A v_1 : \tau_1$

7. CASE $\langle v_2, v_3\rangle \lesssim$ mon $(\ell_0 \blacktriangleleft \tau_3{\times}\tau_4 \blacktriangleleft \ell_1)$ $(\mathsf{trace}_v\ \overline{b}_2\ \langle v_4, v_5\rangle)$
 7.1. $v_2 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathit{add\text{-}trace}(\overline{b}_2, v_4))$
  by inversion $\lesssim$ and $\tau_3 \leqslant: \tau_2$
 7.2. $\delta_N(\mathit{unop}, v_0) = v_2$
  by definition δ_N
 7.3. $\mathit{unop}\{\tau?\}\ v_1$
  $\rightarrow_A$ dyn $(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathit{unop}\{\tau?\}\ (\mathsf{trace}_v\ \overline{b}_2\ \langle v_4, v_5\rangle))$
  $\rightarrow_A$ dyn $(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathit{add\text{-}trace}(\overline{b}_2, v_4))$
  by definition $\rightarrow_A$
 7.4. QED
  by lemma 8.18

8. CASE $\langle v_2, v_3\rangle \lesssim$ mon $(\ell_0 \blacktriangleleft \tau_3{\times}\tau_4 \blacktriangleleft \ell_1)$ (mon $(\ell_2 \blacktriangleleft \tau_5{\times}\tau_6 \blacktriangleleft \ell_3)\ \langle v_4, v_5\rangle$)
 8.1. $v_2 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathsf{mon}^?\ (\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3)\ v_4)$
  by inversion $\lesssim$ and $\tau_3 \leqslant: \tau_2$
 8.2. $\delta_N(\mathit{unop}, v_0) = v_2$
  by definition δ_N
 8.3. $\mathit{unop}\{\tau?\}\ v_1$
  $\rightarrow_A$ dyn $(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathit{unop}\{\tau?\}\ (\mathsf{mon}\ (\ell_2 \blacktriangleleft \tau_5{\times}\tau_6 \blacktriangleleft \ell_3)\ \langle v_4, v_5\rangle))$
  $\rightarrow_A$ dyn $(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathsf{stat}\ (\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3)\ (\mathit{unop}\{\tau?\}\ \langle v_4, v_5\rangle))$
  $\rightarrow_A$ dyn $(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathsf{stat}\ (\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3)\ v_4)$
  $\rightarrow_A$ dyn $(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathsf{stat}\ (\ell_2 \blacktriangleleft \tau_5 \blacktriangleleft \ell_3)\ v_4)$
  by definition $\rightarrow_A$
 8.4. QED
  by lemma 8.19

9. CASE $\langle v_2, v_3\rangle \lesssim$ mon $(\ell_0 \blacktriangleleft \tau_2{\times}\tau_3 \blacktriangleleft \ell_1)$ $(\mathsf{trace}_v\ \overline{b}_2\ (\mathsf{mon}\ (\ell_3 \blacktriangleleft \tau_4{\times}\tau_5 \blacktriangleleft \ell_4)\ \langle v_4, v_5\rangle))$
 9.1. $v_2 \lesssim \mathsf{mon}^?\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathit{add\text{-}trace}(\overline{b}_2, (\mathsf{mon}^?\ (\ell_3 \blacktriangleleft \tau_4 \blacktriangleleft \ell_4)\ v_4)))$
  by inversion $\lesssim$ and $\tau_3 \leqslant: \tau_2$
 9.2. $\delta_N(\mathit{unop}, v_0) = v_2$
  by definition δ_N
 9.3. $\mathit{unop}\{\tau?\}\ v_1$
  $\rightarrow_A$ dyn $(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathit{unop}\{\tau?\}\ (\mathsf{trace}_v\ \overline{b}_2\ (\mathsf{mon}\ (\ell_3 \blacktriangleleft \tau_4{\times}\tau_5 \blacktriangleleft \ell_4)\ \langle v_4, v_5\rangle)))$
  $\rightarrow_A$ dyn $(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathsf{trace}\ \overline{b}_2\ (\mathsf{stat}\ (\ell_3 \blacktriangleleft \tau_4 \blacktriangleleft \ell_4)\ (\mathit{unop}\{\tau?\}\ \langle v_4, v_5\rangle)))$
  $\rightarrow_A$ dyn $(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathsf{trace}\ \overline{b}_2\ (\mathsf{stat}\ (\ell_3 \blacktriangleleft \tau_4 \blacktriangleleft \ell_4)\ v_4))$
  by definition $\rightarrow_A$
 9.4. QED
  by lemma 8.19

10. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}^+\ \overline{b}_0\ (\mathsf{mon}\ b_1\ (\mathsf{mon}\ b_2\ \langle v_4, v_5\rangle))$
 10.1. CONTRADICTION:
  by lemma 7.9

11. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}^+\ \overline{b}_0\ (\mathsf{mon}\ b_1\ (\mathsf{trace}_v\ \overline{b}_2\ (\mathsf{mon}\ b_3\ \langle v_4, v_5\rangle)))$

11.1. Contradiction:
by lemma 7.9

12. case $\lambda x_2.\, e_2 \lesssim \lambda x_2.\, e_3$
12.1. Contradiction:
$\cdot \vdash_{\mathsf{N}} \mathit{unop}\{\tau?\}\, v_0 : \tau_0$

13. case $\lambda(x_2\!:\!\tau_2).\, e_2 \lesssim \lambda(x_2\!:\!\tau_2).\, e_3$
13.1. Contradiction:
$\cdot \vdash_{\mathsf{N}} \mathit{unop}\{\tau?\}\, v_0 : \tau_0$

14. case $\mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_2 \Rightarrow \tau_3 \blacktriangleleft \ell_1)\,(\mathsf{mon}\, b_2\, v_2) \lesssim \mathsf{trace}_{\mathsf{v}}\, \overline{b_3}\, \lambda x_3.\, e_3$
14.1. Contradiction:
$\cdot \vdash_{\mathsf{N}} \mathit{unop}\{\tau?\}\, v_0 : \tau_0$

15. case $\mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_2 \Rightarrow \tau_3 \blacktriangleleft \ell_1)\, v_2 \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_4 \Rightarrow \tau_5 \blacktriangleleft \ell_1)\, v_3$
15.1. Contradiction:
$\cdot \vdash_{\mathsf{N}} \mathit{unop}\{\tau?\}\, v_0 : \tau_0$

□

Lemma 8.23. *If v_0 and v_1 are reduced WF expressions and $\mathit{unop}\{\tau?\}\, v_0 \lesssim \mathit{unop}\{\tau?\}\, v_1$ and $\cdot \vdash_{\mathsf{N}} \mathit{unop}\{\tau?\}\, v_0 : \mathcal{U}$ and $\cdot \vdash_{\mathsf{A}} \mathit{unop}\{\tau?\}\, v_1 : \mathcal{U}$ then $\mathit{unop}\{\tau?\}\, v_0 \blacktriangleright_{\mathsf{N}} e_2$ and $\mathit{unop}\{\tau?\}\, v_1 \rightarrow^*_{\mathsf{A}} e_3$ and $e_2 \lesssim e_3$.*

Proof. By cases on $v_0 \lesssim v_1$.The cases for $\mathit{unop} = \mathsf{fst}\{\tau\}$ and $\mathit{unop} = \mathsf{snd}\{\tau\}$ are similar; we present only the fst cases.

1. case $i_0 \lesssim i_0$
1.1. qed
$\mathit{unop}\{\tau?\}\, v_0 \blacktriangleright_{\mathsf{N}} \mathsf{TagErr}\bullet$ and $\mathit{unop}\{\tau?\}\, v_1 \blacktriangleright_{\mathsf{A}} \mathsf{TagErr}\bullet$

2. case $i_0 \lesssim \mathsf{trace}_{\mathsf{v}}\, \overline{b_0}\, i_0$
2.1. qed
$\mathit{unop}\{\tau?\}\, v_0 \blacktriangleright_{\mathsf{N}} \mathsf{TagErr}\bullet$ and $\mathit{unop}\{\tau?\}\, v_1 \blacktriangleright_{\mathsf{A}} \mathsf{TagErr}\bullet$

3. case $\langle v_2, v_3\rangle \lesssim \langle v_4, v_5\rangle$
3.1. $v_2 \lesssim v_4$
by inversion $\lesssim$
3.2. qed
$\mathit{unop}\{\tau?\}\, v_0 \blacktriangleright_{\mathsf{N}} v_2$ and $\mathit{unop}\{\tau?\}\, v_1 \blacktriangleright_{\mathsf{A}} v_4$

4. case $\langle v_2, v_3\rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)\, \langle v_4, v_5\rangle$
4.1. $v_2 \lesssim \mathsf{mon}^?\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\, v_4$
by inversion $\lesssim$
4.2. $\mathit{unop}\{\tau?\}\, v_0 \blacktriangleright_{\mathsf{N}} v_2$
by definition $\blacktriangleright_{\mathsf{N}}$
4.3. $\mathit{unop}\{\tau?\}\, v_1$
$\rightarrow_{\mathsf{A}} \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\,(\mathit{unop}\{\tau?\}\, \langle v_4, v_5\rangle)$
$\rightarrow_{\mathsf{A}} \mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\, v_4$
by definition $\rightarrow_{\mathsf{A}}$
4.4. qed

by lemma 8.24

5. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{trace_v}\, \overline{b}_0\, \langle v_4, v_5\rangle$

5.1. $v_2 \lesssim \mathit{add\text{-}trace}(\overline{b}_0, v_4)$

by inversion $\lesssim$

5.2. $\mathit{unop}\{\tau?\}\, v_0 \blacktriangleright_{\mathsf{N}} v_2$

by definition $\blacktriangleright_{\mathsf{N}}$

5.3. $\mathit{unop}\{\tau?\}\, v_1 \blacktriangleright_{\mathsf{A}} \mathit{add\text{-}trace}(\overline{b}_0, v_4)$

by definition $\blacktriangleright_{\mathsf{A}}$

5.4. QED

by 5.1

6. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{trace_v}\, \overline{b}_0\, (\mathsf{mon}\, (\ell_1 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_2)\, \langle v_4, v_5\rangle)$

6.1. $v_2 \lesssim \mathit{add\text{-}trace}(\overline{b}_0, (\mathsf{mon}^?\, (\ell_1 \blacktriangleleft \tau_2 \blacktriangleleft \ell_2)\, v_4))$

by inversion $\lesssim$

6.2. $\mathit{unop}\{\tau?\}\, v_0 \blacktriangleright_{\mathsf{N}} v_2$

by definition $\blacktriangleright_{\mathsf{N}}$

6.3. $\mathit{unop}\{\tau?\}\, v_1$

$\rightarrow_{\mathsf{A}} \mathsf{trace}\, \overline{b}_0\, (\mathsf{stat}\, (\ell_1 \blacktriangleleft \tau_2 \blacktriangleleft \ell_2)\, (\mathit{unop}\{\tau?\}\, \langle v_4, v_5\rangle))$

$\rightarrow_{\mathsf{A}} \mathsf{trace}\, \overline{b}_0\, (\mathsf{stat}\, (\ell_1 \blacktriangleleft \tau_2 \blacktriangleleft \ell_2)\, v_4)$

$\rightarrow_{\mathsf{A}} \mathsf{trace}\, \overline{b}_0\, (\mathsf{mon}^?\, (\ell_1 \blacktriangleleft \tau_2 \blacktriangleleft \ell_2)\, v_4)$

$\rightarrow_{\mathsf{A}} \mathit{add\text{-}trace}(\overline{b}_0, (\mathsf{mon}^?\, (\ell_1 \blacktriangleleft \tau_2 \blacktriangleleft \ell_2)\, v_4))$

by definition $\rightarrow_{\mathsf{A}}$ and lemma 8.24

6.4. QED

7. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}\, (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)\, (\mathsf{trace_v}\, \overline{b}_2\, \langle v_4, v_5\rangle)$

7.1. CONTRADICTION:

$\cdot \vdash_{\mathsf{A}} v_1 : \mathcal{U}$

8. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}\, (\ell_0 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_1)\, (\mathsf{mon}\, (\ell_2 \blacktriangleleft \tau_4 \times \tau_5 \blacktriangleleft \ell_3)\, \langle v_4, v_5\rangle)$

8.1. CONTRADICTION:

by lemma 7.10

9. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}\, \ell_0\, \tau_2 \times \tau_3 \ell_1 (\mathsf{trace_v}\, \overline{b}_2\, (\mathsf{mon}\, b_3\, \langle v_4, v_5\rangle))$

9.1. CONTRADICTION:

by lemma 7.10

10. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}^+\, \overline{b}_0\, (\mathsf{mon}\, b_1\, (\mathsf{mon}\, b_2\, \langle v_4, v_5\rangle))$

10.1. CONTRADICTION:

by lemma 7.10

11. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}^+\, \overline{b}_0\, (\mathsf{mon}\, b_1\, (\mathsf{trace_v}\, \overline{b}_2\, (\mathsf{mon}\, b_3\, \langle v_4, v_5\rangle)))$

11.1. CONTRADICTION:

by lemma 7.10

12. CASE $\lambda x_2.\, e_2 \lesssim \lambda x_2.\, e_3$

12.1. QED

$\mathit{unop}\{\tau?\}\, v_0 \blacktriangleright_{\mathsf{N}} \mathsf{TagErr}\bullet$ and $\mathit{unop}\{\tau?\}\, v_1 \blacktriangleright_{\mathsf{A}} \mathsf{TagErr}\bullet$

13. CASE $\lambda(x_2 : \tau_2).\, e_2 \lesssim \lambda(x_2 : \tau_2).\, e_3$

13.1. CONTRADICTION:

$\cdot \vdash_{\mathsf{N}} v_0 : \mathcal{U}$

14. CASE $\mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_2 \Rightarrow \tau_3 \blacktriangleleft \ell_1)\,(\mathsf{mon}\; b_2\; v_2) \lesssim \mathsf{trace}_{\mathsf{v}}\; \overline{b_3}\; \lambda x_3.\, e_3$

14.1. QED

$unop\{\tau?\}\; v_0 \blacktriangleright_{\mathsf{N}} \mathsf{TagErr}\bullet$ and $unop\{\tau?\}\; v_1 \blacktriangleright_{\mathsf{A}} \mathsf{TagErr}\bullet$

15. CASE $\mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_2 \Rightarrow \tau_3 \blacktriangleleft \ell_1)\; v_2 \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_4 \Rightarrow \tau_5 \blacktriangleleft \ell_1)\; v_3$

15.1. QED

$unop\{\tau?\}\; v_0 \blacktriangleright_{\mathsf{N}} \mathsf{TagErr}\bullet$ and $unop\{\tau?\}\; v_1 \blacktriangleright_{\mathsf{A}} \mathsf{TagErr}\bullet$

□

LEMMA 8.24. *If v_0 and v_1 are reduced WF expressions and $\cdot \vdash_{\mathsf{N}} v_0 : \mathcal{U}$ and $\cdot \vdash_{\mathsf{A}} v_1 : \tau_0$ and $v_0 \lesssim \mathsf{mon}^{?}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_1$ then $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_1 \rightarrow_{\mathsf{A}}^{*} v_2$ and $v_0 \lesssim v_2$.*

PROOF. By case analysis of $\lesssim$.

1. CASE $i_0 \lesssim i_0$

1.1. QED

$\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_1 \blacktriangleright_{\mathsf{A}} v_1$

2. CASE $i_0 \lesssim \mathsf{trace}_{\mathsf{v}}\; \overline{b_2}\; i_0$

2.1. QED

$\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_1 \blacktriangleright_{\mathsf{A}} v_0$

3. CASE $\langle v_3, v_4\rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; \langle v_5, v_6\rangle$

3.1. QED

$\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_1 \blacktriangleright_{\mathsf{A}} \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_1$

4. CASE $\langle v_3, v_4\rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; (\mathsf{trace}_{\mathsf{v}}\; \overline{b_2}\; \langle v_5, v_6\rangle)$

4.1. CONTRADICTION:

$\cdot \vdash_{\mathsf{A}} v_1 : \tau_0$

5. CASE $\langle v_3, v_4\rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; (\mathsf{mon}\; b_2\; \langle v_5, v_6\rangle)$

5.1. $v_3 \lesssim \mathsf{mon}^{?}\,(\ell_0 \blacktriangleleft fst(\tau_0) \blacktriangleleft \ell_1)\; (\mathsf{mon\; fst}\,(b_2)\; v_5)$

and $v_4 \lesssim \mathsf{mon}^{?}\,(\ell_0 \blacktriangleleft snd(\tau_0) \blacktriangleleft \ell_1)\; (\mathsf{mon\; snd}\,(b_2)\; v_6)$

by inversion $\lesssim$

5.2. $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; (\mathsf{mon}\; b_2\; \langle v_5, v_6\rangle) \blacktriangleright_{\mathsf{A}} \mathsf{trace}_{\mathsf{v}}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) b_2\; \langle v_5, v_6\rangle$

by definition $\blacktriangleright_{\mathsf{A}}$

6. CASE $\langle v_3, v_4\rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; (\mathsf{trace}_{\mathsf{v}}\; \overline{b_2}\; (\mathsf{mon}\; b_3\; \langle v_4, v_5\rangle))$

6.1. CONTRADICTION:

$\cdot \vdash_{\mathsf{A}} v_1 : \tau_0$

7. CASE $\langle v_3, v_4\rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; (\mathsf{mon}\; b_2\; (\mathsf{trace}_{\mathsf{v}}^{?}\; \overline{b_3}\; (\mathsf{mon}\; b_4\; \langle v_4, v_5\rangle)))$

7.1. $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; v_1 \blacktriangleright_{\mathsf{A}} \mathsf{trace}_{\mathsf{v}}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) b_2 \overline{b_3}\; (\mathsf{mon}\; b_4\; \langle v_4, v_5\rangle)$

by definition $\blacktriangleright_{\mathsf{A}}$

7.2. QED

by lemma 8.25

8. CASE $\langle v_3, v_4\rangle \lesssim \mathsf{mon}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; (\mathsf{mon}\; b_2\; (\mathsf{mon}\; b_3\; (\mathsf{trace}_{\mathsf{v}}^{?}\; \overline{b_4}\; (\mathsf{mon}\; b_5\; \langle v_4, v_5\rangle))))$

8.1. CONTRADICTION:

by lemma 7.10

9. CASE mon b_2 $(\lambda x_2.\, e_2) \lesssim$ mon $(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ $(\lambda x_2.\, e_3)$

9.1. CONTRADICTION:

$\cdot \vdash_A v_1 : \tau_0$

10. CASE mon b_2 $(\lambda(x_2 : \tau_2).\, e_2) \lesssim$ mon $(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ $(\lambda(x_2 : \tau_2).\, e_3)$

10.1. QED

stat $(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ $v_1 \blacktriangleright_A$ mon $(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ v_1

11. CASE mon b_2 (mon b_3 $(\lambda x_2.\, e_2)) \lesssim$ mon $(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ (mon b_4 (trace$^?_v$ $\overline{b}_5$ $(\lambda x_2.\, e_3)$))

11.1. stat $(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ $v_1 \blacktriangleright_N$ *add-trace*$((\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) b_4 \overline{b}_5, (\lambda x_2.\, e_3))$

by definition $\blacktriangleright_N$

11.2. QED

by lemma 8.25

12. CASE mon b_2 (mon b_3 (mon b_4 $(\lambda x_2.\, e_2))) \lesssim$ mon $(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ (mon b_5 (trace$^?_v$ $\overline{b}_6$ (mon b_7 $(\lambda(x_2 : \tau_2).\, e_3)$)))

12.1. stat $(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ $v_1 \blacktriangleright_N$ *add-trace*$((\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1) b_4 b_5 \overline{b}_6$, (mon b_7 $(\lambda(x_2 : \tau_2).\, e_3)))$

by definition $\blacktriangleright_N$

12.2. QED

by lemma 8.25

13. CASE mon b_2 (mon b_3 (mon b_4 (mon b_5 $(\lambda x_2.\, e_2)))) \lesssim$ mon $(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)$ (mon b_5 (mon b_6 (mon b_7 v_2)))

13.1. CONTRADICTION:

by lemma 7.10

□

LEMMA 8.25. *If v_0 and v_1 are reduced WF expressions and $\cdot \vdash_N v_0 : \mathcal{U}$ and $\cdot \vdash_A v_1 : \mathcal{U}$ and $v_0 \lesssim$ mon$^?$ b_0 (mon$^?$ b_1 v_1) then $v_0 \lesssim$ add-trace$(b_0 b_1, v_1)$.*

PROOF. By induction on v_0 via case analysis of $\lesssim$.

1. CASE $i_0 \lesssim i_0$

1.1. QED

$i_0 \lesssim$ trace$_v$ $b_0 b_1$ i_0

2. CASE $i_0 \lesssim$ trace$_v$ $\overline{b}_2$ i_0

2.1. QED

$i_0 \lesssim$ trace$_v$ $b_0 b_1 \overline{b}_2$ i_0

3. CASE $\langle v_2, v_3 \rangle \lesssim$ mon b_0 (mon b_1 $\langle v_4, v_5 \rangle$)

3.1. $v_2 \lesssim$ mon$^?$ fst(b_0) (mon$^?$ fst(b_1) v_4)
and $v_3 \lesssim$ mon$^?$ snd(b_0) (mon$^?$ snd(b_1) v_5)

by inversion $\lesssim$

3.2. $v_2 \lesssim$ *add-trace*(fst(b_0)fst(b_1), v_4)
and $v_3 \lesssim$ *add-trace*(snd(b_0)snd(b_1), v_5)

by the induction hypothesis

3.3. QED

4. CASE $\langle v_2, v_3 \rangle \lesssim$ mon b_0 (mon b_1 (trace$^?_v$ $\overline{b}_2$ (mon b_3 $\langle v_4, v_5 \rangle$)))

4.1. $v_2 \lesssim \mathsf{mon}^{?}\ \mathsf{fst}\ (b_0)\ (\mathsf{mon}^{?}\ \mathsf{fst}\ (b_1)\ (\mathit{add\text{-}trace}\,(\mathsf{fst}\ (\overline{b}_2), (\mathsf{mon}\ b_3\ v_4))))$
and $v_3 \lesssim \mathsf{mon}^{?}\ \mathsf{snd}\ (b_0)\ (\mathsf{mon}^{?}\ \mathsf{snd}\ (b_1)\ (\mathit{add\text{-}trace}\,(\mathsf{snd}\ (\overline{b}_2), (\mathsf{mon}\ b_3\ v_5))))$
by inversion $\lesssim$

4.2. $v_2 \lesssim \mathit{add\text{-}trace}\,(\mathsf{fst}\ (b_0)\mathsf{fst}\ (b_1), (\mathit{add\text{-}trace}\,(\mathsf{fst}\ (\overline{b}_2), (\mathsf{mon}\ b_3\ v_4))))$
and $v_3 \lesssim \mathit{add\text{-}trace}\,(\mathsf{snd}\ (b_0)\mathsf{snd}\ (b_1), (\mathit{add\text{-}trace}\,(\mathsf{snd}\ (\overline{b}_2), (\mathsf{mon}\ b_3\ v_5))))$
by the induction hypothesis

4.3. QED

5. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}\ b_0\ (\mathsf{mon}\ b_1\ (\mathsf{mon}\ b_2\ (\mathsf{trace}^{?}_{\mathsf{v}}\ \overline{b}_3\ (\mathsf{mon}\ b_4\ \langle v_4, v_5\rangle)))))$

5.1. CONTRADICTION:
by lemma 7.10

6. CASE $\mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ (\lambda x_2.\, e_2)) \lesssim \mathsf{mon}\ b_0\ (\mathsf{mon}\ b_1\ (\lambda x_2.\, e_3))$

6.1. $\lambda x_2.\, e_2 \lesssim \lambda x_2.\, e_3$
by inversion $\lesssim$

6.2. QED
by definition $\lesssim$

7. CASE $\mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ (\mathsf{mon}\ b_4\ (\lambda(x_2\!:\!\tau_2).\, e_2))) \lesssim \mathsf{mon}\ b_0\ (\mathsf{mon}\ b_1\ (\mathsf{mon}\ b_5\ (\lambda x_2.\, e_3)))$

7.1. $\lambda(x_2\!:\!\tau_2).\, e_2 \lesssim \lambda(x_2\!:\!\tau_2).\, e_3$
by inversion $\lesssim$

7.2. QED
by definition $\lesssim$

8. CASE $\mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ (\mathsf{mon}\ b_4\ (\mathsf{mon}\ b_5\ v_2))) \lesssim \mathsf{mon}\ b_0\ (\mathsf{mon}\ b_1\ (\mathsf{mon}\ b_6\ (\mathsf{mon}\ b_7\ v_3)))$

8.1. CONTRADICTION:
by lemma 7.10

□

LEMMA 8.26. *If* $\mathsf{app}\{\mathcal{U}\}\ v_0\ v_1$ *and* $\mathsf{app}\{\mathcal{U}\}\ (\mathsf{trace}_{\mathsf{v}}\ b_0\overline{b}_1\ v_2)\ v_3$ *are reduced WF expressions and* $\mathsf{app}\{\mathcal{U}\}\ v_0\ v_1 \lesssim \mathsf{app}\{\mathcal{U}\}\ (\mathsf{trace}_{\mathsf{v}}\ b_0\overline{b}_1\ v_2)\ v_3$ *then one of the following holds:*

- $\mathsf{app}\{\mathcal{U}\}\ v_0\ v_1 \rightarrow^{*}_{\mathsf{N}} E_0[\mathsf{app}\{\tau?\}\ v_4\ (\mathsf{BndryErr}\,(b_3, v_5))]$
  and $E_0 \lesssim \mathsf{trace}\ b_0 \cdots b_i\ [\,]$
  and $v_4 \lesssim \mathit{add\text{-}trace}\,(b_i \cdots b_n, v_2)$
- $\mathsf{app}\{\mathcal{U}\}\ v_0\ v_1 \rightarrow^{*}_{\mathsf{N}} E_0[\mathsf{app}\{\tau?\}\ v_4\ v_5]$
  and $E_0 \lesssim \mathsf{trace}\ b_0\overline{b}_1\ [\,]$
  and $v_4 \lesssim v_2$
  and $v_5 \lesssim \mathit{add\text{-}trace}\,(\mathit{flip}\,(b_0\overline{b}_1), v_3)$

PROOF. By induction on the length of $\overline{b}_1$.

1. $\overline{b}_1 = \cdot$

1.1. CONTRADICTION:
by $\lesssim$

2. $\overline{b}_1 = b_1$

2.1. $v_0 = \mathsf{mon}\ b_2\ (\mathsf{mon}\ b_3\ v_4)$ and $b_2 \leqslant: b_0$ and $b_3 \leqslant: b_1$

by inversion $\lesssim$

2.2. $\mathsf{app}\{\mathcal{U}\}\ v_0\ v_1 \blacktriangleright_{\mathsf{N}} \mathsf{stat}\ \mathsf{cod}\,(b_2)\ (\mathsf{app}\{\tau_1\}\ (\mathsf{mon}\ b_3\ v_4)\ (\mathsf{dyn}\ \mathsf{dom}\,(b_2)\ v_1))$

by definition $\blacktriangleright_{\mathsf{N}}$

2.3. ASSUME $\mathsf{dyn}\ \mathsf{dom}\,(b_2) v_1 = v_5$

otherwise, end with a boundary error

2.4. $\mathsf{app}\{\tau_1\}\ (\mathsf{mon}\ b_3\ v_4)\ v_5 \rhd_{\mathsf{N}} \mathsf{dyn}\ \mathsf{cod}\,(b_3)\ (\mathsf{app}\{\mathcal{U}\}\ v_4\ (\mathsf{stat}\ \mathsf{dom}\,(b_3)\ v_5))$ and $\mathsf{stat}\ \mathsf{dom}\,(b_3)\ v_5 \blacktriangleright_{\mathsf{N}} v_6$

by definition $\rhd_{\mathsf{N}}$ and $\blacktriangleright_{\mathsf{N}}$

2.5. $v_6 \lesssim \mathit{add\text{-}trace}(\mathit{flip}(\mathsf{dom}\,(b_0)\mathsf{dom}\,(b_1)), v_3)$

by lemma 8.27

2.6. QED

$\mathsf{stat}\ \mathsf{cod}\,(b_2)\ (\mathsf{dyn}\ \mathsf{cod}\,(b_3)\ (\mathsf{app}\{\mathcal{U}\}\ v_4\ v_6)) \lesssim \mathsf{trace}_{\mathsf{v}}\ b_0 b_1\ (\mathsf{app}\{\mathcal{U}\}\ v_2\ \mathsf{trace}_{\mathsf{v}}\ \mathit{flip}(b_0 b_1)\ v_3)$

3. $\overline{b}_1 = b_1 b_2 \overline{b}_3$

3.1. $v_0 = \mathsf{mon}\ b_3\ (\mathsf{mon}\ b_4\ v_4)$ and $b_3 \leqslant: b_1$ and $b_4 \leqslant: b_2$

by inversion $\lesssim$

3.2. $\mathsf{app}\{\mathcal{U}\}\ v_0\ v_1 \to_{\mathsf{N}}^{*} \mathsf{stat}\ \mathsf{cod}\,(b_3)\ (\mathsf{dyn}\ \mathsf{dom}\,(b_4)\ (\mathsf{app}\{\mathcal{U}\}\ v_4\ v_6))$

and $v_4 \lesssim \mathsf{trace}_{\mathsf{v}}\ \overline{b}_3\ v_2$ and $v_6 \lesssim \mathsf{trace}_{\mathsf{v}}\ \mathit{flip}(b_0 b_1)\ v_3$

by similar reasoning as the previous case

3.3. $\mathsf{app}\{\mathcal{U}\}\ v_4\ v_6 \to_{\mathsf{N}}^{*} E_1[\mathsf{app}\{\mathcal{U}\}\ v_7\ v_8]$

by the induction hypothesis, assuming no boundary errors

3.4. QED

$\mathsf{stat}\ b_3\ (\mathsf{dyn}\ b_4\ (E_1[\mathsf{app}\{\mathcal{U}\}\ v_7\ v_8])) \lesssim \mathit{add\text{-}trace}(b_0 b_1 b_2 \overline{b}_3, \mathsf{app}\{\mathcal{U}\}\ v_2\ (\mathsf{trace}_{\mathsf{v}}\ \mathit{flip}(b_0 b_1 \overline{b}_2)\ v_3)$

□

LEMMA 8.27. *If v_0 and v_1 are reduced WF expressions and $\cdot \vdash_{\mathsf{N}} v_0 : \mathcal{U}$ and $\cdot \vdash_{\mathsf{A}} v_1 : \mathcal{U}$ and $v_0 \lesssim v_1$ and $b_0 \leqslant: b_2$ and $b_1 \leqslant: b_3$ then* $\mathsf{stat}\ b_0\ (\mathsf{dyn}\ b_1\ v_0) \to_{\mathsf{N}}^{*} e_2$ *and* $e_2 \lesssim \mathit{add\text{-}trace}(b_2 b_3, v_1)$.

PROOF. By induction on v_0 via cases on $v_0 \lesssim v_1$.

We assume below that every subexpression of the form $\mathsf{dyn}\ b\ v$ steps to a value, because otherwise e_2 is a boundary error and the result is immediate.

1. CASE $i_0 \lesssim i_0$

1.1. QED

$\mathsf{stat}\ b_0\ (\mathsf{dyn}\ b_1\ i_0) \to_{\mathsf{N}}^{*} i_0 \lesssim \mathsf{trace}_{\mathsf{v}}\ b_2 b_3\ i_0$

2. CASE $i_0 \lesssim \mathsf{trace}_{\mathsf{v}}\ \overline{b}_0\ \mathsf{trace}_{\mathsf{v}}\ \overline{b}_4\ i_0$

2.1. QED

$\mathsf{stat}\ b_0\ (\mathsf{dyn}\ b_1\ i_0) \to_{\mathsf{N}}^{*} i_0 \lesssim \mathsf{trace}_{\mathsf{v}}\ b_2 b_3 \overline{b}_4\ i_0$

3. CASE $\langle v_2, v_3\rangle \lesssim \langle v_4, v_5\rangle$

3.1. $v_2 \lesssim v_4$ and $v_3 \lesssim v_5$

by inversion $\lesssim$

3.2. $\mathsf{stat}\ b_0\ (\mathsf{dyn}\ b_1\ \langle v_2, v_3\rangle) \to_{\mathsf{N}} \mathsf{stat}\ b_0\ \langle \mathsf{dyn}\ \mathsf{fst}\,(b_1)\ v_2, \mathsf{dyn}\ \mathsf{snd}\,(b_1)\ v_3\rangle$

by definition $\to_{\mathsf{N}}$

3.3. $\mathsf{stat}\ b_0\ \langle \mathsf{dyn}\ \mathsf{fst}\,(b_1)\ v_2, \mathsf{dyn}\ \mathsf{snd}\,(b_1)\ v_3\rangle \to_{\mathsf{N}}^{*} \langle v_6, v_7\rangle$

if and only if $\langle \mathsf{stat}\ \mathsf{fst}\,(b_0)\ (\mathsf{dyn}\ \mathsf{fst}\,(b_1)\ v_2), \mathsf{stat}\ \mathsf{snd}\,(b_0)\ (\mathsf{dyn}\ \mathsf{snd}\,(b_1)\ v_3)\rangle \to_{\mathsf{N}}^{*} \langle v_6, v_7\rangle$

by lemma 8.28

3.4. $v_6 \lesssim \mathit{add\text{-}trace}(\mathsf{fst}\,(b_2)\mathsf{fst}\,(b_3), v_4)$
and $v_7 \lesssim \mathit{add\text{-}trace}(\mathsf{snd}\,(b_2)\mathsf{snd}\,(b_3), v_4)$
by the induction hypothesis

3.5. QED
by definition $\lesssim$

4. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}\ b_4\ \langle v_4, v_5\rangle$

4.1. $v_2 \lesssim \mathsf{mon}^?\ \mathsf{fst}\,(b_4)\ v_4$ and $v_3 \lesssim \mathsf{mon}^?\ \mathsf{snd}\,(b_4)\ v_5$
by inversion $\lesssim$

4.2. $\mathsf{stat}\ b_0\ (\mathsf{dyn}\ b_1\ \langle v_2, v_3\rangle) \rightarrow^*_{\mathsf{N}} \langle v_6, v_7\rangle$
and $v_6 \lesssim \mathit{add\text{-}trace}(\mathsf{fst}\,(b_2)\mathsf{fst}\,(b_3), \mathsf{mon}^?\ \mathsf{fst}\,(b_4)\ v_4)$
and $v_7 \lesssim \mathit{add\text{-}trace}(\mathsf{snd}\,(b_2)\mathsf{snd}\,(b_3), \mathsf{mon}^?\ \mathsf{snd}\,(b_4)\ v_5)$
by definition $\rightarrow_{\mathsf{N}}$ and lemma 8.28 and the induction hypothesis

4.3. QED
by definition $\lesssim$

5. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{trace}_{\mathsf{v}}\ \overline{b}_4\ \langle v_4, v_5\rangle$

5.1. $v_2 \lesssim \mathit{add\text{-}trace}(\mathsf{fst}\,(\overline{b}_4), v_4)$ and $v_3 \lesssim \mathit{add\text{-}trace}(\mathsf{snd}\,(\overline{b}_4), v_5)$
by inversion $\lesssim$

5.2. $\mathsf{stat}\ b_0\ (\mathsf{dyn}\ b_1\ \langle v_2, v_3\rangle) \rightarrow^*_{\mathsf{N}} \langle v_6, v_7\rangle$
and $v_6 \lesssim \mathit{add\text{-}trace}(\mathsf{fst}\,(b_2)\mathsf{fst}\,(b_3), \mathit{add\text{-}trace}(\mathsf{fst}\,(\overline{b}_4), v_4))$
and $v_7 \lesssim \mathit{add\text{-}trace}(\mathsf{snd}\,(b_2)\mathsf{snd}\,(b_3), \mathit{add\text{-}trace}(\mathsf{snd}\,(\overline{b}_4), v_5))$
by definition $\rightarrow_{\mathsf{N}}$ and lemma 8.28 and the induction hypothesis

5.3. QED
by definition $\lesssim$

6. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{trace}_{\mathsf{v}}\ \overline{b}_4\ (\mathsf{mon}\ b_5\ \langle v_4, v_5\rangle)$

6.1. $v_2 \lesssim \mathit{add\text{-}trace}(\mathsf{fst}\,(\overline{b}_4), (\mathsf{mon}^?\ \mathsf{fst}\,(b_5)\ v_4))$ and $v_3 \lesssim \mathit{add\text{-}trace}(\mathsf{snd}\,(\overline{b}_4), (\mathsf{mon}^?\ \mathsf{snd}\,(b_5)\ v_5))$
by inversion $\lesssim$

6.2. $\mathsf{stat}\ b_0\ (\mathsf{dyn}\ b_1\ \langle v_2, v_3\rangle) \rightarrow^*_{\mathsf{N}} \langle v_6, v_7\rangle$
and $v_6 \lesssim \mathit{add\text{-}trace}(\mathsf{fst}\,(b_2)\mathsf{fst}\,(b_3), \mathit{add\text{-}trace}(\mathsf{fst}\,(\overline{b}_4), (\mathsf{mon}^?\ \mathsf{fst}\,(b_5)\ v_4)))$
and $v_7 \lesssim \mathit{add\text{-}trace}(\mathsf{snd}\,(b_2)\mathsf{snd}\,(b_3), \mathit{add\text{-}trace}(\mathsf{snd}\,(\overline{b}_4), (\mathsf{mon}^?\ \mathsf{snd}\,(b_5)\ v_5)))$
by definition $\rightarrow_{\mathsf{N}}$ and lemma 8.28 and the induction hypothesis

6.3. QED
by definition $\lesssim$

7. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}\ b_4\ (\mathsf{trace}_{\mathsf{v}}\ \overline{b}_5\ \langle v_4, v_5\rangle)$

7.1. CONTRADICTION:
$\cdot \vdash_{\mathsf{A}} v_1 : \mathcal{U}$

8. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}\ b_4\ (\mathsf{mon}\ b_5\ \langle v_4, v_5\rangle)$

8.1. CONTRADICTION:
by lemma 7.10

9. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}\ b_4\ (\mathsf{trace}_{\mathsf{v}}\ \overline{b}_5\ (\mathsf{mon}\ b_6\ \langle v_4, v_5\rangle))$

9.1. CONTRADICTION:

by lemma 7.10

10. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}^{+}\, \overline{b}_4\, (\mathsf{mon}\; b_5\, (\mathsf{mon}\; b_6\, \langle v_4, v_5\rangle))$

xxx

11. CASE $\langle v_2, v_3\rangle \lesssim \mathsf{mon}^{+}\, \overline{b}_4\, (\mathsf{mon}\; b_5\, (\mathsf{trace_v}\; b_6\, (\mathsf{mon}\; b_7\, \langle v_4, v_5\rangle)))$

11.1. CONTRADICTION:

by lemma 7.10

12. CASE $\lambda x_2.\, e_2 \lesssim \lambda x_2.\, e_3$

12.1. QED

$\mathsf{stat}\; b_0\, (\mathsf{dyn}\; b_1\; v_0) \rightarrow^{*}_{\mathsf{N}} \mathsf{mon}\; b_0\, (\mathsf{mon}\; b_1\; v_0) \lesssim \mathsf{trace_v}\; b_2 b_3\, v_1$

13. CASE $\lambda(x_2 : \tau_2).\, e_2 \lesssim \lambda(x_2 : \tau_2).\, e_3$

13.1. CONTRADICTION:

$\cdot \vdash_{\mathsf{A}} v_1 : \mathcal{U}$

14. CASE $\mathsf{mon}\; b_4\, (\mathsf{mon}\; b_5\; v_3) \lesssim \mathsf{trace_v}\; b_6 b_7\, v_4$

and $v_4 \in \lambda x.\, e \cup \mathsf{mon}\; b\; (\lambda(x : \tau).\, e)$

14.1. QED

$\mathsf{stat}\; b_0\, (\mathsf{dyn}\; b_1\; v_0) \rightarrow^{*}_{\mathsf{N}} \mathsf{mon}\; b_0\, (\mathsf{mon}\; b_1\; v_0) \lesssim \mathsf{trace_v}\; b_2 b_3 b_6 b_7\, v_4$

15. CASE $\mathsf{mon}\; b_4\, (\mathsf{mon}\; b_5\; v_3) \lesssim \mathsf{trace_v}\; b_6 b_7 \overline{b}_8\, v_4$

and $v_4 \in \lambda x.\, e \cup \mathsf{mon}\; b\; (\lambda(x : \tau).\, e)$

15.1. QED

$\mathsf{stat}\; b_0\, (\mathsf{dyn}\; b_1\; v_0) \rightarrow^{*}_{\mathsf{N}} \mathsf{mon}\; b_0\, (\mathsf{mon}\; b_1\; v_0) \lesssim \mathsf{trace_v}\; b_2 b_3 b_6 b_7 \overline{b}_8\, v_4$

16. CASE $\mathsf{mon}\; b_4\; v_3 \lesssim \mathsf{mon}\; b_5\; v_4$

16.1. $v_4 \in \lambda(x : \tau).\, e$

by inversion $\vdash_{\mathsf{A}}$ and $\lesssim$ and lemma 7.10

16.2. $\mathsf{stat}\; b_0\, (\mathsf{dyn}\; b_1\; v_0) \rightarrow^{*}_{\mathsf{N}} \mathsf{mon}\; b_0\, (\mathsf{mon}\; b_1\; v_0)$

by definition $\rightarrow_{\mathsf{N}}$

16.3. QED

by definition $\lesssim$

□

LEMMA 8.28. $\mathsf{stat}\; b_0\, \langle \mathsf{dyn}\; b_1\; v_0, \mathsf{dyn}\; b_2\; v_1\rangle \rightarrow^{*}_{\mathsf{N}} \langle v_2, v_3\rangle$ *if and only if* $\langle \mathsf{stat}\; \mathsf{fst}\, (b_0)\, (\mathsf{dyn}\; b_1\; v_0), \mathsf{stat}\; \mathsf{snd}\, (b_0)\, (\mathsf{dyn}\; b_2\; v_1)\rangle \rightarrow^{*}_{\mathsf{N}} \langle v_2, v_3\rangle$

PROOF. By definition of $\rhd_{\mathsf{N}}$ and $\blacktriangleright_{\mathsf{N}}$.

□

LEMMA 8.29. *If* $binop\{\tau?\}\; v_0\; v_1 \lesssim binop\{\tau?\}\; v_2\; v_3$ *then* $binop\{\tau?\}\; v_0\; v_1 \rightarrow_{\mathsf{N}} e_4$ *if and only if* $binop\{\tau?\}\; v_2\; v_3 \rightarrow_{\mathsf{A}} e_5$

PROOF. The binary operations are only defined for integers and $\lesssim$ relates integers iff they are equal.

1. CASE $binop\{\tau?\}\; v_0\; v_1 \blacktriangleright_{\mathsf{N}} \mathsf{TagErr}\, \bullet$

1.1. SCASE $v_0 \notin i$

1.1.1. $v_2 \notin i$

by inversion $\lesssim$

1.1.2. QED
$binop\{\tau?\}\ svalue_2\ v_3 \blacktriangleright_{\mathsf{A}} \mathsf{TagErr}\bullet$
1.2. SCASE $v_1 \notin i$
1.2.1. $v_3 \notin i$
by inversion $\lesssim$
1.2.2. QED
$binop\{\tau?\}\ v_2\ v_3 \blacktriangleright_{\mathsf{A}} \mathsf{TagErr}\bullet$
2. CASE $binop\{\tau?\}\ v_2\ v_3 \blacktriangleright_{\mathsf{A}} \mathsf{TagErr}\bullet$
2.1. SCASE $\mathit{rem\text{-}trace}(v_2) \notin i$
2.1.1. $v_0 \notin i$
by inversion $\lesssim$
2.1.2. QED
$binop\{\tau?\}\ svalue_0\ v_1 \blacktriangleright_{\mathsf{N}} \mathsf{TagErr}\bullet$
2.2. SCASE $\mathit{rem\text{-}trace}(v_3) \notin i$
2.2.1. $v_1 \notin i$
by inversion $\lesssim$
2.2.2. QED
$binop\{\tau?\}\ v_0\ v_1 \blacktriangleright_{\mathsf{N}} \mathsf{TagErr}\bullet$
3. CASE $binop = \mathsf{sum}$ and $v_0 = i_0$ and $v_1 = i_1$
3.1. $binop\{\tau?\}\ v_0\ v_1 \rightarrow_{\mathsf{N}} v_0 + v_1$
by definition $\triangleright_{\mathsf{N}}$, $\blacktriangleright_{\mathsf{N}}$, and δ_N
3.2. $\mathit{rem\text{-}trace}(v_2) = v_0$ and $\mathit{rem\text{-}trace}(v_3) = v_1$
by inversion $\lesssim$
3.3. QED
$binop\{\tau?\}\ v_2\ v_3 \rightarrow_{\mathsf{A}} v_0 + v_1$
4. CASE $binop = \mathsf{sum}$ and $\mathit{rem\text{-}trace}(v_2) = i_0$ and $\mathit{rem\text{-}trace}(v_3) = i_1$
4.1. $binop\{\tau?\}\ v_2\ v_3 \rightarrow_{\mathsf{A}} i_0 + i_1$
by definition $\triangleright_{\mathsf{A}}$, $\blacktriangleright_{\mathsf{A}}$, and δ_A
4.2. $v_0 = i_0$ and $v_1 = i_1$
by inversion $\lesssim$
4.3. QED
$binop\{\tau?\}\ v_0\ v_1 \rightarrow_{\mathsf{N}} v_2 + v_3$
5. CASE $binop = \mathsf{quotient}$ and $v_0 = i_0$ and $v_1 = i_1$ and $i_1 = 0$
5.1. $binop\{\tau?\}\ v_0\ v_1 \rightarrow_{\mathsf{N}} \mathsf{TagErr}\bullet$
by definition $\triangleright_{\mathsf{N}}$, $\blacktriangleright_{\mathsf{N}}$, and δ_N
5.2. $\mathit{rem\text{-}trace}(v_2) = i_0$ and $\mathit{rem\text{-}trace}(v_3) = 0$
by inversion $\lesssim$
5.3. QED
$binop\{\tau?\}\ v_2\ v_3 \rightarrow_{\mathsf{A}} \mathsf{TagErr}\bullet$
6. CASE $binop = \mathsf{quotient}$ and $\mathit{rem\text{-}trace}(v_2) = i_2$ and $\mathit{rem\text{-}trace}(v_3) = i_3$ and $i_3 = 0$
6.1. $binop\{\tau?\}\ v_2\ v_3 \rightarrow_{\mathsf{A}} \mathsf{TagErr}\bullet$

by definition $\triangleright_{\mathsf{A}}$, $\blacktriangleright_{\mathsf{A}}$, and δ_A

6.2. $v_0 = i_2$ and $v_1 = 0$

by inversion $\lesssim$

6.3. QED

$binop\{\tau?\}\, v_0\, v_1 \rightarrow_{\mathsf{N}} \mathsf{TagErr}\bullet$

7. CASE $binop$ = quotient and $v_0 = i_0$ and $v_1 = i_1$ and $i_1 \neq 0$

7.1. $binop\{\tau?\}\, v_0\, v_1 \rightarrow_{\mathsf{N}} \lfloor i_0/i_1 \rfloor$

by definition $\triangleright_{\mathsf{N}}$, $\blacktriangleright_{\mathsf{N}}$, and δ_N

7.2. $rem\text{-}trace(v_2) = i_0$ and $rem\text{-}trace(v_3) = i_1$

by inversion $\lesssim$

7.3. QED

$binop\{\tau?\}\, v_2\, v_3 \rightarrow_{\mathsf{A}} \lfloor i_0/i_1 \rfloor$

8. CASE $binop$ = quotient and $rem\text{-}trace(v_2) = i_2$ and $rem\text{-}trace(v_3) = i_3$ and $i_3 \neq 0$

8.1. $binop\{\tau?\}\, v_2\, v_3 \rightarrow_{\mathsf{A}} \lfloor i_2/i_3 \rfloor$

by definition $\triangleright_{\mathsf{A}}$, $\blacktriangleright_{\mathsf{A}}$, and δ_A

8.2. $v_0 = i_0$ and $v_1 = i_1$

by inversion $\lesssim$

8.3. QED

$binop\{\tau?\}\, v_0\, v_1 \rightarrow_{\mathsf{N}} \lfloor i_0/i_1 \rfloor$

□

LEMMA 8.30. *If* $e_0 \lesssim e_1$ *and* $v_0 \lesssim v_1$ *then* $e_0[x_0 \leftarrow v_0] \lesssim e_1[x_0 \leftarrow v_1]$

PROOF. By induction on the structure of both expressions (or rather, the pair $\langle e_0, e_1 \rangle$) via case analysis of $e_0 \lesssim e_1$.

□

LEMMA 8.31. *If* $e_0 \lesssim e_1$ *and* $E_0 \lesssim E_1$ *then* $E_0[e_0] \lesssim E_1[e_1]$

PROOF. By induction on the structure of E_0.

□

9 A/T SIMULATION

This simulation uses the Transient notion of reduction from the paper and a modified Amnesic notion of reduction that inserts check expressions that are guaranteed (by type soundness) to be no-ops.

$\boxed{e \sim e; \mathcal{H}; \mathcal{B}}$

$$\frac{v_0 \sim v_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{trace}\,\overline{b}_0\, v_0 \sim v_1; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{v_0 \sim \mathcal{H}_0(\mathsf{p}_0); \mathcal{H}_0; \mathcal{B}_0}{v_0 \sim \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{}{i_0 \sim i_0; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{}{\mathsf{trace}_{\mathsf{v}}\,\overline{b}_0\, i_0 \sim i_0; \mathcal{H}_0; \mathcal{B}_0}$$

$$\frac{v_0 \sim v_2; \mathcal{H}_0; \mathcal{B}_0 \qquad v_1 \sim v_3; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{trace}^{?}_{\mathsf{v}}\,\overline{b}_0\, \langle v_0, v_1\rangle \sim \langle v_2, v_3\rangle; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{v_0 \sim v_2; \mathcal{H}_0; \mathcal{B}_0 \qquad v_1 \sim v_3; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{trace}^{?}_{\mathsf{v}}\,\overline{b}_0\, (\mathsf{mon}\; b_0\; (\mathsf{trace}^{?}_{\mathsf{v}}\,\overline{b}_1\, \langle v_0, v_1\rangle)) \sim \langle v_2, v_3\rangle; \mathcal{H}_0; \mathcal{B}_0}$$

$$\frac{v_0 \sim v_2; \mathcal{H}_0; \mathcal{B}_0 \qquad v_1 \sim v_3; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{mon}\; b_0\; (\mathsf{trace}^{?}_{\mathsf{v}}\,\overline{b}_0\, (\mathsf{mon}\; b_1\; \langle v_0, v_1\rangle)) \sim \langle v_2, v_3\rangle; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{trace}^{?}_{\mathsf{v}}\,\overline{b}_0\, (\lambda x_0.\, e_0) \sim \lambda x_0.\, e_1; \mathcal{H}_0; \mathcal{B}_0}$$

$$\frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{mon}\; b_0\; (\mathsf{trace}^{?}_{\mathsf{v}}\,\overline{b}_0\, (\lambda x_0.\, e_0)) \sim \lambda x_0.\, e_1; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{trace}^{?}_{\mathsf{v}}\,\overline{b}_0\, (\lambda(x_0 : \tau_0).\, e_0) \sim \lambda(x_0 : \tau_0).\, e_1; \mathcal{H}_0; \mathcal{B}_0}$$

$$\frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{trace}^{?}_{\mathsf{v}}\,\overline{b}_0\, (\mathsf{mon}\; b_0\; (\lambda(x_0 : \tau_0).\, e_0)) \sim \lambda(x_0 : \tau_0).\, e_1; \mathcal{H}_0; \mathcal{B}_0}$$

$$\frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{mon}\; b_0\; (\mathsf{trace}^{?}_{\mathsf{v}}\,\overline{b}_0\, (\mathsf{mon}\; b_1\; (\lambda(x_0 : \tau_0).\, e_0))) \sim \lambda(x_0 : \tau_0).\, e_1; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{}{x_0 \sim x_0; \mathcal{H}_0; \mathcal{B}_0}$$

$$\frac{e_0 \sim e_2; \mathcal{H}_0; \mathcal{B}_0 \qquad e_1 \sim e_3; \mathcal{H}_0; \mathcal{B}_0}{\langle e_0, e_1\rangle \sim \langle e_2, e_3\rangle; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{e_0 \sim e_2; \mathcal{H}_0; \mathcal{B}_0 \qquad e_1 \sim e_3; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{app}\{\tau?\}\; e_0\; e_1 \sim \mathsf{app}\{\tau?\}\; e_2\; e_3; \mathcal{H}_0; \mathcal{B}_0}$$

$$\frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\mathit{unop}\{\tau?\}\; e_0 \sim \mathit{unop}\{\tau?\}\; e_1; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{e_0 \sim e_2; \mathcal{H}_0; \mathcal{B}_0 \qquad e_1 \sim e_3; \mathcal{H}_0; \mathcal{B}_0}{\mathit{binop}\{\tau?\}\; e_0\; e_1 \sim \mathit{binop}\{\tau?\}\; e_2\; e_3; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{dyn}\; b_0\; e_0 \sim \mathsf{dyn}\; b_0\; e_1; \mathcal{H}_0; \mathcal{B}_0}$$

$$\frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{stat}\; b_0\; e_0 \sim \mathsf{stat}\; b_0\; e_1; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{dyn}\; (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; e_0 \sim \mathsf{check}\; \tau_0\; e_1\; \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0}$$

$$\frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{stat}\; (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; e_0 \sim \mathsf{check}\; \mathcal{U}\; e_1\; \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{\tau_1 \leqslant: \tau_0 \qquad e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{dyn}\; (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\; (\mathsf{stat}\; (\ell_2 \blacktriangleleft \tau_1 \blacktriangleleft \ell_3)\; e_0) \sim \mathsf{check}\; \tau_0\; e_1\; \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0}$$

$$\frac{e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{check}\; \tau?\; e_0\; \bullet \sim \mathsf{check}\; \tau?\; e_1\; \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{}{\mathsf{TagErr}\circ \sim \mathsf{TagErr}\circ; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{}{\mathsf{TagErr}\bullet \sim \mathsf{TagErr}\bullet; \mathcal{H}_0; \mathcal{B}_0}$$

$$\frac{}{\mathsf{DivErr} \sim \mathsf{DivErr}; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{v_0 \sim v_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{BndryErr}\,(b_0, v_0) \sim \mathsf{BndryErr}\,(b_0, v_1); \mathcal{H}_0; \mathcal{B}_0}$$

$\boxed{E \sim E; \mathcal{H}; \mathcal{B}}$

$$\frac{E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{trace}\,\overline{b}_0\, E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{}{[\,] \sim [\,]; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{E_0 \sim E_2; \mathcal{H}_0; \mathcal{B}_0 \qquad e_1 \sim e_3; \mathcal{H}_0; \mathcal{B}_0}{\langle E_0, e_1 \rangle \sim \langle E_2, e_3 \rangle; \mathcal{H}_0; \mathcal{B}_0}$$

$$\frac{v_0 \sim v_2; \mathcal{H}_0; \mathcal{B}_0 \qquad E_1 \sim E_3; \mathcal{H}_0; \mathcal{B}_0}{\langle v_0, E_1 \rangle \sim \langle v_2, E_3 \rangle; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{E_0 \sim E_2; \mathcal{H}_0; \mathcal{B}_0 \qquad e_1 \sim e_3; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{app}\{\tau?\}\, E_0\, e_1 \sim \mathsf{app}\{\tau?\}\, E_2\, e_3; \mathcal{H}_0; \mathcal{B}_0}$$

$$\frac{v_0 \sim v_2; \mathcal{H}_0; \mathcal{B}_0 \qquad E_1 \sim E_3; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{app}\{\tau?\}\, v_0\, E_1 \sim \mathsf{app}\{\tau?\}\, v_2\, E_3; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0}{\mathit{unop}\{\tau?\}\, E_0 \sim \mathit{unop}\{\tau?\}\, E_1; \mathcal{H}_0; \mathcal{B}_0}$$

$$\frac{E_0 \sim E_2; \mathcal{H}_0; \mathcal{B}_0 \qquad e_1 \sim e_3; \mathcal{H}_0; \mathcal{B}_0}{\mathit{binop}\{\tau?\}\, E_0\, e_1 \sim \mathit{binop}\{\tau?\}\, E_2\, e_3; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{v_0 \sim v_2; \mathcal{H}_0; \mathcal{B}_0 \qquad E_1 \sim E_3; \mathcal{H}_0; \mathcal{B}_0}{\mathit{binop}\{\tau?\}\, v_0\, E_1 \sim \mathit{binop}\{\tau?\}\, v_2\, E_3; \mathcal{H}_0; \mathcal{B}_0}$$

$$\frac{E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{dyn}\, b_0\, E_0 \sim \mathsf{dyn}\, b_0\, E_1; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{stat}\, b_0\, E_0 \sim \mathsf{stat}\, b_0\, E_1; \mathcal{H}_0; \mathcal{B}_0}$$

$$\frac{E_0 \notin \mathsf{stat}\, b\, E \qquad E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, E_0 \sim \mathsf{check}\, \tau_0\, E_1\, \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{stat}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, E_0 \sim \mathsf{check}\, \mathcal{U}\, E_1\, \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0}$$

$$\frac{\tau_1 \leqslant: \tau_0 \qquad E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, (\mathsf{stat}\, (\ell_2 \blacktriangleleft \tau_1 \blacktriangleleft \ell_3)\, E_0) \sim \mathsf{check}\, \tau_0\, E_1\, \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0} \qquad \frac{E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0}{\mathsf{check}\, \tau?\, E_0 \bullet \sim \mathsf{check}\, \tau?\, E_1\, \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0}$$

Corollary 9.1. $\mathsf{eval}_A(\Uparrow s_0) \in \mathsf{Err}$ *if and only if* $\mathsf{eval}_T(\Uparrow s_0) \in \mathsf{Err}$

Proof. By lemma 9.2

□

Lemma 9.2. *If* $\mathsf{eval}_A(\Uparrow s_0) = r_0$ *and* $\mathsf{eval}_T(\Uparrow s_0) = r_1; \mathcal{H}_0; \mathcal{B}_0$ *then one of the following holds:*

- $r_0 = r_1 = \Omega$
- $r_0 \neq \Omega$ *and* $r_1 \neq \Omega$ *and* $r_0 \sim r_1; \mathcal{H}_0; \mathcal{B}_0$

Proof. By lemma 9.3 and lemma 9.4.

□

Lemma 9.3. $\uparrow s_0 \sim \uparrow s_0; \emptyset; \emptyset$ *for all* $\uparrow s_0$

Proof. By induction on the structure of s_0.

1. case $s_0 = x_0$
   1.1. qed
   $x_0 \sim x_0; \mathcal{H}_0; \mathcal{B}_0$
2. case $s_0 = i_0$
   2.1. qed
   $i_0 \sim i_0; \mathcal{H}_0; \mathcal{B}_0$
3. case $s_0 = \lambda x_0.\, s_0$
   3.1. qed
   by the induction hypothesisand the definition of $\sim$
4. case $s_0 = \lambda(x_0 : \tau_0).\, s_1$
   4.1. qed
   by the induction hypothesisand the definition of $\sim$
5. case $s_0 = \langle s_1, s_2 \rangle$
   5.1. qed
   by the induction hypothesis
6. case $s_0 = \mathsf{app}\{\tau_0\}\, s_1\, s_2$
   6.1. qed
   by the induction hypothesis
7. case $s_0 = \mathit{unop}\{\tau?\}\, s_1$
   7.1. qed
   by the induction hypothesis
8. case $s_0 = \mathit{binop}\{\tau?\}\, s_1\, s_2$
   8.1. qed
   by the induction hypothesis
9. case $s_0 = \mathsf{dyn}\ \tau_0\ s_1$
   9.1. qed
   by the induction hypothesis
10. case $s_0 = \mathsf{stat}\ \tau_0\ s_1$

10.1. QED

by the induction hypothesis

□

Lemma 9.4. *If* $e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0$ *then:*

- *if* $e_0 \rightarrow_{\mathsf{A}} e_2$ *then* $e_2 \rightarrow^*_{\mathsf{A}} e_3$ *and* $e_1; \mathcal{H}_0; \mathcal{B}_0 \rightarrow^*_{\mathsf{T}} e_4; \mathcal{H}_1; \mathcal{B}_1$ *and* $e_3 \sim e_4; \mathcal{H}_1; \mathcal{B}_1$
- *if* $e_1; \mathcal{H}_0; \mathcal{B}_0 \rightarrow_{\mathsf{T}} e_3; \mathcal{H}_1; \mathcal{B}_1$ *then* $e_0 \rightarrow^*_{\mathsf{A}} e_2$ *and* $e_2 \sim e_3; \mathcal{H}_1; \mathcal{B}_1$

Proof. By lemma 9.5 and lemma 9.6 and lemma 9.7 and lemma 9.8 and lemma 9.9.

□

Lemma 9.5. *If* $E_0[e_0] \sim e_1; \mathcal{H}_0; \mathcal{B}_0$ *then* $e_1 = E_1[e_2]$ *and* $E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0$ *and* $e_0 \sim e_2; \mathcal{H}_0; \mathcal{B}_0$

Proof. By induction on the structure of E_0.

1. CASE $E_0 = [\,]$
   1.1. QED
   $E_1 = [\,]$
2. CASE $E_0 = \langle E_2, e_2 \rangle$
   2.1. QED
   by the induction hypothesis
3. CASE $E_0 = \langle v_0, E_2 \rangle$
   3.1. QED
   by the induction hypothesis
4. CASE $E_0 = \mathsf{app}\{\tau?\}\, E_2\, e_2$
   4.1. QED
   by the induction hypothesis
5. CASE $E_0 = \mathsf{app}\{\tau?\}\, v_0\, E_2$
   5.1. QED
   by the induction hypothesis
6. CASE $E_0 = \mathit{unop}\{\tau?\}\, E_2$
   6.1. QED
   by the induction hypothesis
7. CASE $E_0 = \mathit{binop}\{\tau?\}\, E_2\, e_2$
   7.1. QED
   by the induction hypothesis
8. CASE $E_0 = \mathit{binop}\{\tau?\}\, E_2\, e_2$
   8.1. QED
   by the induction hypothesis
9. CASE $E_0 = \mathsf{dyn}\; b_0\; E_2$ and $E_2 \neq \mathsf{check}\; \tau?\; E_3 \bullet$
   9.1. SCASE $E_1 = \mathsf{dyn}\; b_0\; E_3$
      9.1.1. QED
      by the induction hypothesis

9.2. SCASE $E_1 = \mathsf{check}\ \tau_0\ E_3\ \mathsf{p}_0$

9.2.1. QED

by the induction hypothesis

10. CASE $E_0 = \mathsf{dyn}\ b_0\ (\mathsf{stat}\ b_1\ E_2)$

10.0.1. QED

by the induction hypothesis

11. CASE $E_0 = \mathsf{stat}\ b_0\ E_2$

11.1. SCASE $E_1 = \mathsf{stat}\ b_0\ E_3$

11.1.1. QED

by the induction hypothesis

11.2. SCASE $E_1 = \mathsf{check}\ \mathcal{U}\ E_3\ \mathsf{p}_0$

11.2.1. QED

by the induction hypothesis

12. CASE $E_0 = \mathsf{check}\ \tau?\ E_2\ \bullet$

12.1. QED

by the induction hypothesis

□

LEMMA 9.6. *If* $e_0 \sim E_1[e_1]; \mathcal{H}_0; \mathcal{B}_0$ *then* $e_0 = E_0[e_2]$ *and* $E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0$ *and* $e_2 \sim e_1; \mathcal{H}_0; \mathcal{B}_0$

PROOF. By induction on the structure of E_1 and E_0; the latter is because E_0 may be a suffix context ($\mathsf{trace}\ \overline{b}\ E$).

1. CASE $E_1 = [\,]$

1.1. SCASE $E_0 = [\,]$

1.1.1. QED

1.2. SCASE $E_0 = \mathsf{trace}\ \overline{b}_0\ E_2$

1.2.1. QED

by the induction hypothesis

2. CASE $E_1 = \langle E_2, e_3\rangle$

2.1. QED

by the induction hypothesis

3. CASE $E_1 = \langle v_0, E_2\rangle$

3.1. QED

by the induction hypothesis

4. CASE $E_1 = \mathsf{app}\{\tau?\}\ E_2\ e_3$

4.1. QED

by the induction hypothesis

5. CASE $E_1 = \mathsf{app}\{\tau?\}\ v_0\ E_2$

5.1. QED

by the induction hypothesis

6. CASE $E_1 = \mathit{unop}\{\tau?\}\ E_2$

6.1. QED

by the induction hypothesis

7. CASE $E_1 = binop\{\tau?\}\ E_2\ e_3$
   7.1. QED
   by the induction hypothesis
8. CASE $E_1 = binop\{\tau?\}\ v_0\ E_2$
   8.1. QED
   by the induction hypothesis
9. CASE $E_1 = \mathsf{dyn}\ b_0\ E_2$
   9.1. QED
   by the induction hypothesis
10. CASE $E_1 = \mathsf{stat}\ b_0\ E_2$
   10.1. QED
   by the induction hypothesis
11. CASE $E_1 = \mathsf{check}\ \tau_0\ E_2\ \mathsf{p}_0$
   11.1. SCASE $E_0 = \mathsf{dyn}\ b_0\ E_3$ and $E_3 \notin \mathsf{stat}\ b\ E$
   by the induction hypothesis
   11.2. SCASE $E_0 = \mathsf{dyn}\ b_0\ (\mathsf{stat}\ b_1\ E_3)$
   by the induction hypothesis
   11.3. SCASE $E_0 = \mathsf{check}\ \tau_0\ E_3\ \bullet$
   by the induction hypothesis
12. CASE $E_1 = \mathsf{check}\ \mathcal{U}\ E_2\ \mathsf{p}_0$
   12.1. SCASE $E_0 = \mathsf{stat}\ b_0\ E_3$
   by the induction hypothesis
   12.2. SCASE $E_0 = \mathsf{check}\ \mathcal{U}\ E_3\ \bullet$
   by the induction hypothesis

□

LEMMA 9.7. *If* $E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0$ *and* $e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0$ *then* $E_0[e_0] \sim E_1[e_1]; \mathcal{H}_0; \mathcal{B}_0$

PROOF. By case analysis of $E_0 \sim E_1; \mathcal{H}_0; \mathcal{B}_0$ and induction on the structure of E_0 and E_1.

1. CASE $\mathsf{trace}\ \overline{b}_0\ E_2 \sim E_1; \mathcal{H}_0; \mathcal{B}_0$
   1.1. QED
   by the induction hypothesis
2. CASE $[\,] \sim [\,]; \mathcal{H}_0; \mathcal{B}_0$
   2.1. QED
   by $e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0$
3. CASE $\langle E_2, e_2 \rangle \sim \langle E_3, e_3 \rangle; \mathcal{H}_0; \mathcal{B}_0$
   3.1. QED
   by the induction hypothesis
4. CASE $\langle v_0, E_2 \rangle \sim \langle v_1, E_3 \rangle; \mathcal{H}_0; \mathcal{B}_0$
   4.1. QED
   by the induction hypothesis
5. CASE $\mathsf{app}\{\tau?\}\ E_2\ e_2 \sim \mathsf{app}\{\tau?\}\ E_3\ e_3; \mathcal{H}_0; \mathcal{B}_0$

5.1. QED

by the induction hypothesis

6. CASE $\mathsf{app}\{\tau?\}\ v_0\ E_2 \sim \mathsf{app}\{\tau?\}\ v_1\ E_3; \mathcal{H}_0; \mathcal{B}_0$

6.1. QED

by the induction hypothesis

7. CASE $unop\{\tau?\}\ E_2 \sim unop\{\tau?\}\ E_3; \mathcal{H}_0; \mathcal{B}_0$

7.1. QED

by the induction hypothesis

8. CASE $binop\{\tau?\}\ E_2\ e_2 \sim binop\{\tau?\}\ E_3\ e_3; \mathcal{H}_0; \mathcal{B}_0$

8.1. QED

by the induction hypothesis

9. CASE $binop\{\tau?\}\ v_0\ E_2 \sim binop\{\tau?\}\ v_1\ E_3; \mathcal{H}_0; \mathcal{B}_0$

9.1. QED

by the induction hypothesis

10. CASE $\mathsf{dyn}\ b_0\ E_2 \sim \mathsf{dyn}\ b_0\ E_3; \mathcal{H}_0; \mathcal{B}_0$

10.1. QED

by the induction hypothesis

11. CASE $\mathsf{stat}\ b_0\ E_2 \sim \mathsf{stat}\ b_0\ E_3; \mathcal{H}_0; \mathcal{B}_0$

11.1. QED

by the induction hypothesis

12. CASE $\mathsf{dyn}\ b_0\ E_2 \sim \mathsf{check}\ \tau_0\ E_3\ \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0$

12.1. QED

by the induction hypothesis

13. CASE $\mathsf{stat}\ b_0\ E_2 \sim \mathsf{check}\ \mathcal{U}\ E_3\ \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0$

13.1. QED

by the induction hypothesis

14. CASE $\mathsf{dyn}\ b_0\ (\mathsf{stat}\ b_1\ E_2) \sim \mathsf{check}\ \tau_0\ E_3\ \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0$

14.1. QED

by the induction hypothesis

15. CASE $\mathsf{check}\ \tau?\ E_2\ \bullet \sim \mathsf{check}\ \tau?\ E_3\ \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0$

15.1. QED

by the induction hypothesis

□

LEMMA 9.8. *If* $e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0$ *and* $e_0\ (\rhd_{\mathsf{A}} \cup \blacktriangleright_{\mathsf{A}})\ e_2$ *then* $e_2 \rightarrow^*_{\mathsf{A}} e_3$ *and* $e_1; \mathcal{H}_0; \mathcal{B}_0 \rightarrow^*_{\mathsf{T}} e_4; \mathcal{H}_1; \mathcal{B}_1$ *and* $e_3 \sim e_4; \mathcal{H}_1; \mathcal{B}_1$

PROOF. By case analysis of $(\rhd_{\mathsf{A}} \cup \blacktriangleright_{\mathsf{A}})$ and inversion on the $\sim$ relation. In short, the shape of the Amnesic expression e_0 determines (and matches) the shape of the Transient expression.

Expressions on the Transient side may be pre-values. This proof assumes that all pre-values have already been allocated to the heap by stutter steps (lemma 9.10).

1. CASE $unop\{\tau?\}\ \langle v_0, v_1\rangle \rhd_{\mathsf{A}} \delta(unop, \langle v_0, v_1\rangle)$

1.1. QED

by lemma 9.14

2. CASE $\mathit{binop}\{\tau?\}\, v_0\, v_1 \rhd_{\mathsf{A}} \delta(\mathit{binop}, v_0, v_1)$
   2.1. QED
   by lemma 9.15

3. CASE $\mathsf{fst}\{\tau_0\}\, (\mathsf{mon}\, (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\, v_0) \rhd_{\mathsf{A}} \mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, (\mathsf{fst}\{\mathcal{U}\}\, v_0)$
   3.1. CASE $v_0 = \langle v_1, v_2 \rangle$
      3.1.1. $\mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, (\mathsf{fst}\{\mathcal{U}\}\, v_0) \rightarrow_{\mathsf{A}} \mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, \delta(\mathsf{fst}\{\mathcal{U}\}, v_0)$
      3.1.2. $e_1 = \mathit{unop}\{\tau?\}\, v_2$ and $e_1; \mathcal{H}_0; \mathcal{B}_0 \rightarrow^*_{\mathsf{T}} \mathsf{check}\, \tau_0\, \delta(\mathsf{fst}\{\tau_0\}, v_2)\, v_2; \mathcal{H}_0; \mathcal{B}_0$
      3.1.3. QED
      by definition $\sim$
   3.2. CASE $v_0 = \mathsf{mon}\, (\ell_2 \blacktriangleleft \tau_2 \blacktriangleleft \ell_3)\, \langle v_1, v_2 \rangle$
      3.2.1. $\mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, (\mathsf{fst}\{\mathcal{U}\}\, v_0) \rightarrow_{\mathsf{A}} \mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, (\mathsf{stat}\, (\ell_2 \blacktriangleleft \mathit{fst}(\tau_2) \blacktriangleleft \ell_3)\, \delta(\mathsf{fst}\{\mathit{fst}(\tau_2) v_0\}))$
      3.2.2. $e_1 = \mathit{unop}\{\tau?\}\, v_2$ and $e_1; \mathcal{H}_0; \mathcal{B}_0 \rightarrow^*_{\mathsf{T}} \mathsf{check}\, \tau_0\, \delta(\mathsf{fst}\{\tau_0\}, v_2)\, v_2; \mathcal{H}_0; \mathcal{B}_0$
      3.2.3. QED
      by definition $\sim$

4. CASE $\mathsf{snd}\{\tau_0\}\, (\mathsf{mon}\, b_0\, v_0) \rhd_{\mathsf{A}} \ldots$
   4.1. QED
   similar to fst case

5. CASE $\mathsf{app}\{\tau_0\}\, (\lambda(x_0 : \tau_0).\, e_2)\, v_1 \rhd_{\mathsf{A}} \mathsf{check}\, \tau_0\, e_2[x_0 \leftarrow v_0] \bullet$
   5.1. QED
   by lemma 9.17

6. CASE $\mathsf{app}\{\tau_0\}\, (\mathsf{mon}\, (\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2) \blacktriangleleft \ell_1)\, v_0)\, v_1 \rhd_{\mathsf{A}} \mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, (\mathsf{app}\{\mathcal{U}\}\, v_0\, (\mathsf{stat}\, (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\, v_1))$
   6.1. $\mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, (\mathsf{app}\{\mathcal{U}\}\, v_0\, (\mathsf{stat}\, (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\, v_1)) \rightarrow_{\mathsf{A}} \mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, (\mathsf{app}\{\mathcal{U}\}\, v_0\, v_2)$
   6.2. QED
   by lemma 9.12/lemma 9.13 for stat and the definition of $\sim$ for the whole

7. CASE $\mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, v_0 \rhd_{\mathsf{A}} \mathsf{mon}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, v_0$
   7.1. QED
   by lemma 9.16 and lemma 9.13

8. CASE $\mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, (\mathsf{trace}^?_{\mathsf{v}}\, \overline{b}_0\, i_0) \rhd_{\mathsf{A}} i_0$
   8.1. QED
   by lemma 9.16

9. CASE $\mathsf{dyn}\, (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\, v_0 \rhd_{\mathsf{A}} \mathsf{BndryErr}\, (\overline{b}_0, v_0)$
   9.1. QED
   by lemma 9.16

10. CASE $\mathsf{check}\, \tau_0\, v_0 \bullet \rhd_{\mathsf{A}} v_0$
    10.1. QED
    by inversion $\sim$

11. CASE $\mathsf{check}\, \tau_0\, v_0 \bullet \rhd_{\mathsf{A}} \mathsf{BndryErr}\, (\overline{b}_0, v_0)$
    11.1. CONTRADICTION:
    by THEOREM 7.2

12. CASE $\mathsf{trace}\, \overline{b}_0\, v_0 \rhd_{\mathsf{A}} \mathit{add\text{-}trace}(\overline{b}_0, v_0)$

12.1. QED

by lemma 9.12

13. CASE $\mathit{unop}\{\tau?\}\ v_0 \blacktriangleright_{\mathsf{A}} \mathsf{TagErr}\bullet$

13.1. QED

by lemma 9.14

14. CASE $\mathit{unop}\{\tau?\}\ v_0 \blacktriangleright_{\mathsf{A}} \mathsf{check}\,\mathcal{U}\,\delta(\mathit{unop}, v_0)\bullet$

14.1. QED

by lemma 9.14

15. CASE $\mathit{binop}\{\tau?\}\ v_0\ v_1 \blacktriangleright_{\mathsf{A}} \mathsf{TagErr}\bullet$

15.1. QED

by lemma 9.15

16. CASE $\mathit{binop}\{\tau?\}\ v_0\ v_1 \blacktriangleright_{\mathsf{A}} \delta(\mathit{binop}, v_0, v_1)$

16.1. QED

by lemma 9.15

17. CASE $\mathsf{fst}\{\mathcal{U}\}\ (\mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_0\ (\mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_0 \times \tau_1) \blacktriangleleft \ell_1)\ v_0)) \blacktriangleright_{\mathsf{A}} \mathsf{trace}\,\overline{b}_0\ (\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ \mathsf{fst}\{\tau_0\}\ v_0)$

17.1. $\mathsf{trace}\,\overline{b}_0\ (\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ \mathsf{fst}\{\tau_0\}\ v_0) \rightarrow_{\mathsf{A}} \mathsf{trace}\,\overline{b}_0\ (\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ \delta(\mathsf{fst}\{\mathit{fst}(\tau_0)\}, v_0))$

17.2. $e_1 = \mathsf{fst}\{\mathcal{U}\}\ v_1$ and $e_1; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{check}\,\mathcal{U}\,\delta(\mathsf{fst}\{\mathcal{U}\}, v_1)\ v_1; \mathcal{H}_0; \mathcal{B}_0$

17.3. QED

by definition $\sim$

18. CASE $\mathsf{snd}\{\mathcal{U}\}\ \ldots \blacktriangleright_{\mathsf{A}} \ldots$

18.1. QED

similar to previous case

19. CASE $\mathsf{app}\{\mathcal{U}\}\ (\mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_0\ (\lambda x_0.\, e_0))\ v_0 \blacktriangleright_{\mathsf{A}} \mathsf{trace}\,\overline{b}_0\ \mathsf{check}\,\mathcal{U}\ (e_0[x_0 \leftarrow v_0])\bullet$

19.1. QED

by lemma 9.17

20. CASE $\mathsf{app}\{\mathcal{U}\}\ (\mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_0\ (\mathsf{mon}\ (\ell_0 \blacktriangleleft (\tau_1 \Rightarrow \tau_2) \blacktriangleleft \ell_1)\ v_0))\ v_1 \blacktriangleright_{\mathsf{A}}$
$\mathsf{trace}\,\overline{b}_0\ (\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\tau_2\}\ v_0\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ v_1)))$

20.1. CASE $\mathsf{trace}\,\overline{b}_0\ (\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\tau_2\}\ v_0\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ v_1))) \rightarrow_{\mathsf{A}}$
$\mathsf{trace}\,\overline{b}_0\ (\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\tau_2\}\ v_0\ v_2))$

20.1.1. QED

by lemma 9.16

20.2. CASE $\mathsf{trace}\,\overline{b}_0\ (\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\tau_2\}\ v_0\ (\mathsf{dyn}\ (\ell_1 \blacktriangleleft \tau_1 \blacktriangleleft \ell_0)\ v_1))) \rightarrow_{\mathsf{A}}$
$\mathsf{trace}\,\overline{b}_0\ (\mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_2 \blacktriangleleft \ell_1)\ (\mathsf{app}\{\tau_2\}\ v_0\ \mathsf{BndryErr}\,(\overline{b}_0, v_1)))$

20.2.1. QED

by lemma 9.16

21. CASE $\mathsf{stat}\ b_0\ v_0 \blacktriangleright_{\mathsf{A}} \mathsf{mon}\ b_0\ v_0$

21.1. QED

by lemma 9.16 and lemma 9.13

22. CASE $\mathsf{stat}\ b_0\ (\mathsf{mon}\ b_1\ (\mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_0\ v_0)) \blacktriangleright_{\mathsf{A}} \mathsf{trace}\,(b_0 b_1 \overline{b}_0)\ v_0$

22.1. QED

by lemma 9.16

23. CASE $\mathsf{stat}\ b_0\ i_0 \blacktriangleright_{\mathsf{A}} i_0$
    23.1. QED
        by lemma 9.16
24. CASE $\mathsf{check}\,\mathcal{U}\ v_0\ \mathsf{p}_0 \blacktriangleright_{\mathsf{A}} v_0$
    24.1. QED
        by inversion $\sim$

□

Lemma 9.9. *If* $e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0$ *and* $e_1; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} e_3; \mathcal{H}_1; \mathcal{B}_1$ *then* $e_0 \to^*_{\mathsf{A}} e_2$ *and* $e_2 \sim e_3; \mathcal{H}_1; \mathcal{B}_1$

Proof. By case analysis of $\rhd_{\mathsf{T}}$.

Any Amnesic expression can have the form $(\mathsf{trace}\,\overline{b}\,v)$; the proof assumes these have been reduced by stutter steps (lemma 9.11).

1. CASE $\mathsf{w}_0; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{p}_0; \mathsf{p}_0 \mapsto \mathsf{w}_0, \mathcal{H}_0; \mathsf{p}_0 \mapsto \emptyset, \mathcal{B}_0$
    1.1. QED
        by inversion $\sim$ and lemma 9.18
2. CASE $\mathit{unop}\{\tau?\}\ \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{check}\,\tau?\,\delta(\mathit{unop}, v_0)\ \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0$
    2.1. SCASE $e_0 \in \mathit{unop}\{\tau?\}\ \mathsf{trace}^?_{\mathsf{v}}\ \overline{b}\ \langle v, v\rangle$
        2.1.1. QED
            by lemma 9.14
    2.2. SCASE $e_0 = \mathit{unop}\{\tau?\}\ \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (\mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_0\ \langle v_2, v_3\rangle)$
        and $\tau? = \tau_0$
        2.2.1. $e_0 \to^*_{\mathsf{A}} \mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ \delta(\mathit{unop}, \mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_0\ \langle v_2, v_3\rangle)$
        2.2.2. QED
    2.3. SCASE $e_0 = \mathit{unop}\{\tau?\}\ \mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ (\mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_0\ (\mathsf{mon}\ (\ell_2 \blacktriangleleft \tau_2 \times \tau_3 \blacktriangleleft \ell_3)\ \langle v_2, v_3\rangle))$
        and $\tau? = \tau_0$
        2.3.1. $e_0 \to^*_{\mathsf{A}} \mathsf{dyn}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ (\mathsf{stat}\ (\ell_2 \blacktriangleleft \tau_1 \blacktriangleleft \ell_3)\ \delta(\mathit{unop}, \mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_0\ \langle v_2, v_3\rangle))$
        2.3.2. QED
    2.4. SCASE $e_0 = \mathit{unop}\{\tau?\}\ \mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_0\ (\mathsf{mon}\ (\ell_0 \blacktriangleleft \tau_1 \blacktriangleleft \ell_1)\ \langle v_2, v_3\rangle)$
        and $\tau? = \mathcal{U}$
        2.4.1. $e_0 \to^*_{\mathsf{A}} \mathsf{stat}\ (\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ \delta(\mathit{unop}, \mathsf{trace}^?_{\mathsf{v}}\ \overline{b}_0\ \langle v_2, v_3\rangle)$
        2.4.2. QED
3. CASE $\mathit{unop}\{\tau?\}\ v_0; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{TagErr}\,\bullet; \mathcal{H}_0; \mathcal{B}_0$
    3.1. QED
        by inversion $\sim$
4. CASE $\mathit{binop}\{\tau?\}\ i_0\ i_1; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \delta(\mathit{binop}, v_0, v_1); \mathcal{H}_0; \mathcal{B}_0$
    4.1. QED
        by lemma 9.15
5. CASE $\mathit{binop}\{\tau?\}\ i_0\ i_1; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{TagErr}\,\bullet; \mathcal{H}_0; \mathcal{B}_0$
    5.1. QED
        by lemma 9.15
6. CASE $\mathsf{app}\{\tau?\}\ \mathsf{p}_0\ v_0; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{check}\,\tau?\,e_2[x_0 \leftarrow v_0]\ \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0[v_0 \cup \mathit{rev}(\mathcal{B}_0(\mathsf{p}_0))]$

6.1. QED

by lemma 9.17

7. CASE $\mathsf{app}\{\tau?\}\ \mathsf{p}_0\ v_0; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{BndryErr}\,(\overline{b}_0, v_0); \mathcal{H}_0; \mathcal{B}_0$

7.1. QED

by lemma 9.16

8. CASE $\mathsf{app}\{\tau_0\}\ \mathsf{p}_0\ v_0; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{check}\ \tau_0\ e_2[x_0 \leftarrow v_0]\ \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0[v_0 \cup rev(\mathcal{B}_0(\mathsf{p}_0))]$

8.1. QED

by lemma 9.17

9. CASE $\mathsf{app}\{\mathcal{U}\}\ \mathsf{p}_0\ v_0; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} e_2[x_0 \leftarrow v_0]; \mathcal{H}_0; \mathcal{B}_0$

9.1. QED

similar to previous case

10. CASE $\mathsf{app}\{\mathcal{U}\}\ v_0\ v_1; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{TagErr}\ \bullet; \mathcal{H}_0; \mathcal{B}_0$

10.1. QED

by lemma 9.16

11. CASE $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} v_0; \mathcal{H}_0; \mathcal{B}_0[v_0 \cup \{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\}]$

11.1. QED

by lemma 9.16

12. CASE $\mathsf{dyn}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{BndryErr}\,(\overline{b}_0, v_0); \mathcal{H}_0; \mathcal{B}_0$

12.1. QED

by lemma 9.16

13. CASE $\mathsf{stat}\,(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\ v_0; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} v_0; \mathcal{H}_0; \mathcal{B}_0[v_0 \cup \{(\ell_0 \blacktriangleleft \tau_0 \blacktriangleleft \ell_1)\}]$

13.1. QED

by inversion $\sim$ and lemma 9.12 and lemma 9.13

14. CASE $\mathsf{check}\ \mathcal{U}\ v_0\ \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} v_0; \mathcal{H}_0; \mathcal{B}_0$

14.1. QED

by inversion $\sim$

15. CASE $\mathsf{check}\ \tau_0\ v_0\ \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} v_0; \mathcal{H}_0; \mathcal{B}_0[v_0 \cup \mathcal{B}_0(\mathsf{p}_0)]$

15.1. QED

by lemma 9.16

16. CASE $\mathsf{check}\ \tau_0\ v_0\ \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0 \rhd_{\mathsf{T}} \mathsf{BndryErr}\,(\overline{b}_0, v_0); \mathcal{H}_0; \mathcal{B}_0$

16.1. QED

by lemma 9.16

□

LEMMA 9.10. *If* $v_0 \sim e_0; \mathcal{H}_0; \mathcal{B}_0$ *then* $e_0; \mathcal{H}_0; \mathcal{B}_0 \rightarrow^*_{\mathsf{T}} v_1; \mathcal{H}_1; \mathcal{B}_1$ *and* $v_0 \sim v_1; \mathcal{H}_1; \mathcal{B}_1$

PROOF. 1. CASE $e_0 \in v$

1.1. QED

2. CASE $e_0 \in \mathsf{w}$

2.1. $e_0; \mathcal{H}_0; \mathcal{B}_0 \rightarrow^*_{\mathsf{T}} \mathsf{p}_0; \mathsf{p}_0 \mapsto e_0, \mathcal{H}_0; \mathsf{p}_0 \mapsto \emptyset, \mathcal{B}_0$

2.2. QED

by $v_0 \sim e_0; \mathcal{H}_0; \mathcal{B}_0$ and lemma 9.18

□

Lemma 9.11. *If* $e_0 \sim v_0; \mathcal{H}_0; \mathcal{B}_0$ *then* $e_0 \rightarrow^*_{\mathsf{A}} v_1$ *and* $v_1 \sim v_0; \mathcal{H}_0; \mathcal{B}_0$

Proof. 1. CASE $e_0 \in v$

1.1. QED

2. CASE $e_0 = \mathsf{trace}\, \overline{b}_0\, v_1$

2.1. $e_0 \rightarrow^*_{\mathsf{A}}$ *add-trace* $(\overline{b}_0, v_1)$

2.2. QED

by lemma 9.12

□

Lemma 9.12. *If* $v_0 \sim v_1; \mathcal{H}_0; \mathcal{B}_0$ *and* $\Gamma_0 \vdash_{\mathsf{A}} v_0 : \mathcal{U}$ *then* *add-trace* $(\overline{b}_0, v_0) \sim v_1; \mathcal{H}_0; \mathcal{B}_0$

Proof. By case analysis of $v_0 \sim v_1; \mathcal{H}_0; \mathcal{B}_0$

1. CASE *add-trace* $(\overline{b}_0, v_0) = v_0$

1.1. QED

2. CASE *add-trace* $(\overline{b}_0, v_0) = \mathsf{trace}_{\mathsf{v}}\, \overline{b}_1\, v_2$

2.1. SCASE $v_0 = \mathsf{trace}^?_{\mathsf{v}}\, \overline{b}_2\, i_0$

2.1.1. QED

2.2. SCASE $v_0 = \mathsf{trace}^?_{\mathsf{v}}\, \overline{b}_2\, \langle v_2, v_3 \rangle$

2.2.1. QED

2.3. SCASE $v_0 = \mathsf{trace}^?_{\mathsf{v}}\, \overline{b}_2\, (\mathsf{mon}\, b_0\, \langle v_2, v_3 \rangle)$

2.3.1. QED

2.4. SCASE $v_0 = \mathsf{trace}^?_{\mathsf{v}}\, \overline{b}_2\, (\mathsf{mon}\, b_0\, (\mathsf{trace}^?_{\mathsf{v}}\, \overline{b}_3\, (\mathsf{mon}\, b_1\, \langle v_2, v_3 \rangle)))$

2.4.1. Contradiction:

by lemma 7.10

2.5. SCASE $v_0 = \mathsf{trace}^?_{\mathsf{v}}\, \overline{b}_2\, (\lambda x_0.\, e_0)$

2.5.1. QED

2.6. SCASE $v_0 = \mathsf{trace}^?_{\mathsf{v}}\, \overline{b}_2\, (\mathsf{mon}\, b_0\, (\lambda x_0.\, e_0))$

2.6.1. Contradiction:

by $\Gamma_0 \not\vdash_{\mathsf{A}} v_0 : \mathcal{U}$

2.7. SCASE $v_0 = \mathsf{trace}^?_{\mathsf{v}}\, \overline{b}_2\, (\lambda(x_0 : \tau_0).\, e_0)$

2.7.1. Contradiction:

by $\Gamma_0 \not\vdash_{\mathsf{A}} v_0 : \mathcal{U}$

2.8. SCASE $v_0 = \mathsf{trace}^?_{\mathsf{v}}\, \overline{b}_2\, (\mathsf{mon}\, b_0\, (\lambda(x_0 : \tau_0).\, e_0))$

2.8.1. QED

2.9. SCASE $v_0 = \mathsf{trace}^?_{\mathsf{v}}\, \overline{b}_2\, (\mathsf{mon}\, b_0\, (\mathsf{trace}^?_{\mathsf{v}}\, \overline{b}_3\, (\mathsf{mon}\, b_1\, (\lambda(x_0 : \tau_0).\, e_0))))$

2.9.1. Contradiction:

by lemma 7.10

□

Lemma 9.13. *If* $v_0 \sim v_1; \mathcal{H}_0; \mathcal{B}_0$ *and* $v_0 \in (\lambda(x : \tau).\, e) \cup \langle v, v \rangle$ *then* $\mathsf{mon}\, b_0\, v_0 \sim v_1; \mathcal{H}_0; \mathcal{B}_0$

Proof. Immediate from the definition of $\sim$.

□

Lemma 9.14. *If* $unop\{\tau?\}\, v_0 \sim unop\{\tau?\}\, v_1; \mathcal{H}_0; \mathcal{B}_0$ *and* $v_0 \notin \mathsf{mon}\ b\ v$ *then* $\delta(unop, v_0)$ *is defined iff* $\delta(unop, v_1)$ *is defined. Furthermore, if both are defined, then* $\delta(unop, v_0) \sim \delta(unop, v_1); \mathcal{H}_0; \mathcal{B}_0$

Proof. 1. $v_0 \in \mathsf{trace}^?_{\mathsf{v}}\, \overline{b}\, \langle v, v\rangle$ iff $v_1 \in \langle v, v\rangle$
by definition $\sim$
2. qed
by definition δ

□

Lemma 9.15. *If* $binop\{\tau?\}\, v_0\, v_1 \sim binop\{\tau?\}\, v_2\, v_3; \mathcal{H}_0; \mathcal{B}_0$ *then* $\delta(binop, v_0, v_1)$ *is defined iff* $\delta(binop, v_2, v_3)$ *is defined. Furthermore, if both are defined, then* $\delta(binop, v_0, v_1) \sim \delta(binop, v_2, v_3); \mathcal{H}_0; \mathcal{B}_0$

Proof. 1. $v_0 \in i$ iff $v_2 \in i$
and $v_1 \in i$ iff $v_3 \in i$
by definition $\sim$
2. qed
by definition δ

□

Lemma 9.16. *If* $v_0 \sim v_1; \mathcal{H}_0; \mathcal{B}_0$ *then* $\textit{tag-match}(\lfloor \tau_0 \rfloor, v_0)$ *iff* $\textit{tag-match}(\lfloor \tau_1 \rfloor, v_1)$

Proof. By case analysis of $v_0 \sim v_1;$; and *tag-match*

1. case $\mathsf{trace}^?_{\mathsf{v}}\, \overline{b_0}\, i_0 \sim i_0; \mathcal{H}_0; \mathcal{B}_0$
   1.1. qed
   by the definition of *tag-match*
2. case $\mathsf{trace}^?_{\mathsf{v}}\, \overline{b_0}\, \langle v_2, v_3\rangle \sim \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0$
   and $\mathcal{H}_0(\mathsf{p}_0) = \langle v_4, v_5\rangle$
   2.1. qed
   by *tag-match*
3. case $\mathsf{trace}^?_{\mathsf{v}}\, \overline{b_0}\, (\mathsf{mon}\ b_0\ \langle v_2, v_3\rangle) \sim \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0$
   and $\mathcal{H}_0(\mathsf{p}_0) = \langle v_4, v_5\rangle$
   3.1. qed
   by *tag-match*
4. case $\mathsf{mon}\ b_0\ (\mathsf{trace}^?_{\mathsf{v}}\, \overline{b_0}\, (\mathsf{mon}\ b_1\ \langle v_2, v_3\rangle)) \sim \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0$
   and $\mathcal{H}_0(\mathsf{p}_0) = \langle v_4, v_5\rangle$
   4.1. qed
   by *tag-match*
5. case $\mathsf{trace}^?_{\mathsf{v}}\, \overline{b_0}\, (\lambda x_0.\, e_0) \sim \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0$
   and $\mathcal{H}_0(\mathsf{p}_0) = \lambda x_0.\, e_1$
   5.1. qed

by *tag-match*

6. CASE $\mathsf{trace}^?_\mathsf{v}\,\overline{b}_0\,(\mathsf{mon}\;b_0\;(\lambda x_0.\,e_0)) \sim \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0$ and $\mathcal{H}_0(\mathsf{p}_0) = \lambda x_0.\,e_1$
   6.1. QED
   by *tag-match*
7. CASE $\mathsf{trace}^?_\mathsf{v}\,\overline{b}_0\,(\lambda(x_0 : \tau_0).\,e_0) \sim \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0$ and $\mathcal{H}_0(\mathsf{p}_0) = \lambda(x_0 : \tau_0).\,e_1$
   7.1. QED
   by *tag-match*
8. CASE $\mathsf{trace}^?_\mathsf{v}\,\overline{b}_0\,(\mathsf{mon}\;b_0\;(\lambda(x_0 : \tau_0).\,e_0)) \sim \mathsf{p}_0; \mathcal{H}_0; \mathcal{B}_0$ and $\mathcal{H}_0(\mathsf{p}_0) = \lambda(x_0 : \tau_0).\,e_1$
   8.1. QED
   by *tag-match*

□

LEMMA 9.17. *If* $e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0$ *and* $v_0 \sim v_1; \mathcal{H}_0; \mathcal{B}_0$ *then* $e_0[x_0 \leftarrow v_0] \sim e_1[x_0 \leftarrow v_1]; \mathcal{H}_0; \mathcal{B}_0$

PROOF. By case analysis of $e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0$ and induction on the structure of e_0 and e_1

□

LEMMA 9.18. *If* $e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0$ *and* $\mathcal{H}_1$ *adds bindings to* $\mathcal{H}_0$ *and* $\mathcal{B}_1$ *adds bindings and blame information to* $\mathcal{B}_0$ *then* $e_0 \sim e_1; \mathcal{H}_1; \mathcal{B}_1$

PROOF. By induction on the derivation of $e_0 \sim e_1; \mathcal{H}_0; \mathcal{B}_0$

□