Ordinal Notations

Theorem *(Cantor Normal Form)* For every ordinal $\alpha \neq 0$, there are unique $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ $(n \in \omega \backslash \{0\})$ s.t. $\alpha \geq \alpha_1$ and $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$.

Corollary For all $\alpha \in \epsilon_0$, there are unique $\alpha_1 > \alpha_2 > \cdots > \alpha_n > 0$ $(n \in \omega)$, $p \in \omega$, and $x_1, \ldots, x_n \in \omega \backslash \{0\}$, s.t. $\alpha > \alpha_1$ and
$\alpha = \omega^{\alpha_1} x_1 + \cdots + \omega^{\alpha_n} x_n + p$.

$CNF.\alpha = ((CNF.\alpha_1\ .\ x_1)\ (CNF.\alpha_2\ .\ x_2)\ \ldots\ (CNF.\alpha_n\ .\ x_n)\ .\ p)$

This representation is exponentially more succinct than the ACL2 representation.
Succinctness is critical for algorithms.

Helper Functions

atom(a) ;*false iff* a *is a list*

|a| ;*the length of* a

fe(a) ;*the first exponent of* a

atom(a)	:	0
true	:	**first**(**first**(a))

fc(a) ;*the first coefficient of* a

atom(a)	:	a
true	:	**rest**(**first**(a))

#a ;*the size of* a

atom(a)	:	1
true	:	#**fe**(a) + #**rest**(a)

Ordinal Ordering Function

cmp$_{\mathbf{o}}$`(a,b)` ;*ordering on ordinals*

atom`(a)` $\wedge$ **atom**`(b)`	:	**cmp**$_\omega$`(a,b)`
atom`(a)`	:	lt
atom`(b)`	:	gt
cmp$_{\mathbf{o}}$`(`**fe**`(a),`**fe**`(b))` $\neq eq$	:	**cmp**$_{\mathbf{o}}$`(`**fe**`(a),`**fe**`(b))`
cmp$_\omega$`(`**fc**`(a),`**fc**`(b))` $\neq eq$	:	**cmp**$_\omega$`(`**fc**`(a),`**fc**`(b))`
`true`	:	**cmp**$_{\mathbf{o}}$`(`**rest**`(a),`**rest**`(b))`

Key Insight $\omega^{\alpha_1}k_1 + \omega^{\alpha_2}k_2 + \ldots \omega^{\alpha_n}k_n + p \; < \; \omega^{\alpha_1}(k_1 + 1)$.

Complexity $O(\min(\#\mathtt{a}, \#\mathtt{b}))$.

Ordinal Predicate

cnfp(a) ;*ordinal recognizer*

atom(a)	:	a $\in \omega$
true	:	$\neg$**atom**(**first**(a))
		$\wedge$ **fc**(a) $\in \omega$
		$\wedge$ 0 $<_\omega$ **fc**(a)
		$\wedge$ **cnfp**(**fe**(a))
		$\wedge$ **cnfp**(**rest**(a))
		$\wedge$ **fe**(**rest**(a)) $<_o$ **fe**(a)

Complexity of cnfp

cnfp(a) ;*ordinal recognizer*

```
    atom(a)   :   a ∈ ω
    true      :   ¬atom(first(a))
                  ∧ fc(a) ∈ ω
                  ∧ 0 <_ω fc(a)
                  ∧ cnfp(fe(a))
                  ∧ cnfp(rest(a))
                  ∧ fe(rest(a)) <_o fe(a)
```

Complexity $O(\#\mathtt{a}(\log \#\mathtt{a}))$

Proof Complexity given by the (non-linear) recurrence relation

$$T(\mathtt{a}) = \begin{cases} c, & \text{if } \mathbf{atom}(\mathtt{a}) \\ T(\mathbf{fe}(\mathtt{a})) + T(\mathbf{rest}(\mathtt{a})) + \min(\#\mathbf{fe}(\mathtt{a}), \#\mathbf{rest}(\mathtt{a})) + c, & \text{otherwise} \end{cases}$$

Complexity of cnfp 2

$$T(\mathtt{a}) = \begin{cases} c, & \text{if } \mathbf{atom}(\mathtt{a}) \\ T(\mathbf{fe}(\mathtt{a})) + T(\mathbf{rest}(\mathtt{a})) + \min(\#\mathbf{fe}(\mathtt{a}), \#\mathbf{rest}(\mathtt{a})) + c, & \text{otherwise} \end{cases}$$

To Show $T(\mathtt{a}) \leq k(\#\mathtt{a})(\log \#\mathtt{a}) + t$,
where k, t are constants such that $t \geq c$ and $k \geq 3t$.

Base Case $T(\mathtt{a}) = c \leq t$
IS Let $x = \min(\#\mathbf{fe}(\mathtt{a}), \#\mathbf{rest}(\mathtt{a}))$ and $y = \max(\#\mathbf{fe}(\mathtt{a}), \#\mathbf{rest}(\mathtt{a}))$.
Note $x + y = \#\mathtt{a}$.

$T(\mathtt{a})$	$=$	{ Definition of T }	$T(\mathbf{fe}(\mathtt{a})) + T(\mathbf{rest}(\mathtt{a})) + x + c$
	$\leq$	{ IH }	$kx \log x + t + ky \log y + t + x + c$
	$\leq$	{ $kx \geq 2t + x$ as $k \geq 3t$ }	$k(x \log x + y \log y + x) + c$
	$\leq$	{ Lemma }	$k(x + y) \log(x + y) + c$
	$=$	{ $t \geq c$, $x + y = \#\mathtt{a}$ }	$k(\#\mathtt{a}) \log(\#\mathtt{a}) + t$ □

Complexity of cnfp 3

Lemma $x \leq y \quad \Rightarrow \quad x \log x + y \log y + x \;\leq\; (x+y)\log(x+y)$

$$
\begin{aligned}
x \log x + y \log y + x \quad &= \{\ \text{Log}\ \} \quad x \log x + y \log y + x \log 2 \\
&= \{\ \text{Log}\ \} \quad \log x^x + \log y^y + \log 2^x \\
&= \{\ \text{Log}\ \} \quad \log x^x y^y 2^x \\
&\leq \{\ \text{Lemma}\ \} \quad \log (x+y)^{x+y} \quad \square
\end{aligned}
$$

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\end{aligned}
$$

Lemma $x \leq y \quad \Rightarrow \quad x^x y^y 2^x \leq (x+y)^{x+y}$

$$
\begin{aligned}
(x+y)^{x+y} \quad &\geq \{\ \text{Binomial theorem}\ \} && x^x y^y \binom{x+y}{x} \\
&\geq \{\ y \geq x\ \} && x^x y^y \binom{2x}{x} \\
&\geq \{\ \text{Lemma}\ \} && x^x y^y 2^x \quad \square
\end{aligned}
$$

Complexity of cnfp 3

Lemma $x \leq y \quad \Rightarrow \quad x \log x + y \log y + x \ \leq \ (x+y)\log(x+y)$

$$
\begin{aligned}
x \log x + y \log y + x \quad & = \{ \text{ Log } \} \quad && x \log x + y \log y + x \log 2 \\
& = \{ \text{ Log } \} && \log x^x + \log y^y + \log 2^x \\
& = \{ \text{ Log } \} && \log x^x y^y 2^x \\
& \leq \{ \text{ Lemma } \} && \log (x+y)^{x+y} \quad \square
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Lemma $x \leq y \quad \Rightarrow \quad x^x y^y 2^x \leq (x+y)^{x+y}$

$$
\begin{aligned}
(x+y)^{x+y} \quad & \geq \{ \text{ Binomial theorem } \} \quad && x^x y^y \binom{x+y}{x} \\
& \geq \{ \ y \geq x \ \} && x^x y^y \binom{2x}{x} \\
& \geq \{ \text{ Lemma } \} && x^x y^y 2^x \quad \square
\end{aligned}
$$

Lemma $\binom{2x}{x} \geq 2^x$

$$\binom{2x}{x} = \frac{(2x)!}{x!x!} = \frac{(2x)(2x-1)\cdots(x+1)}{x(x-1)\cdots 1} \geq 2 \cdots 2 \geq 2^x \quad \square$$

Ordinal Addition

`a` $+_o$ `b` ;*ordinal addition*

atom(a) $\wedge$ **atom**(b)	:	a $+_\omega$ b
fe(a) $<_o$ **fe**(b)	:	b

Key Insight 1 $\alpha < \omega^\beta \Rightarrow \alpha + \omega^\beta = \omega^\beta$.

Examples

$(\omega 17 + 5) +_o (\omega^2 3 + 4) = \omega^2 3 + 4.$

Ordinal Addition

`a` $+_o$ `b` ;*ordinal addition*

atom(a) $\wedge$ **atom**(b)	:	a $+_\omega$ b
fe(a) $<_o$ **fe**(b)	:	b
fe(a) $=_o$ **fe**(b)	:	$\langle$**fe**(a),**fc**(a) $+_\omega$ **fc**(b)$\rangle$. **rest**(b)

Key Insight 1 $\alpha < \omega^\beta \Rightarrow \alpha + \omega^\beta = \omega^\beta$.

Key Insight 2 $\omega^\gamma x_1 + \omega^\gamma x_2 = \omega^\gamma (x_1 + x_2)$.

Examples

$(\omega 17 + 5) +_o (\omega^2 3 + 4) = \omega^2 3 + 4.$
$(\omega^2 5 + \omega 17 + 5) +_o (\omega^2 3 + 4) = \omega^2 8 + 4.$

Ordinal Addition

```
a +o b ;ordinal addition
    atom(a)  ∧  atom(b)   :  a +ω b
    fe(a) <o fe(b)        :  b
    fe(a) =o fe(b)        :  ⟨fe(a),fc(a) +ω fc(b)⟩ . rest(b)
    true                  :  ⟨fe(a),fc(a)⟩ . (rest(a) +o b)
```

Key Insight 1 $\alpha < \omega^\beta \Rightarrow \alpha + \omega^\beta = \omega^\beta$.

Key Insight 2 $\omega^\gamma x_1 + \omega^\gamma x_2 = \omega^\gamma (x_1 + x_2)$.

Examples

$(\omega 17 + 5) +_o (\omega^2 3 + 4) = \omega^2 3 + 4.$
$(\omega^2 5 + \omega 17 + 5) +_o (\omega^2 3 + 4) = \omega^2 8 + 4.$
$(\omega^3 + \omega^2 5 + \omega 17 + 5) +_o (\omega^2 3 + 4) = \omega^3 + \omega^2 8 + 4.$

Ordinal Addition

```
a +o b ;ordinal addition
    atom(a)  ∧  atom(b)   :  a +ω b
    fe(a) <o fe(b)        :  b
    fe(a) =o fe(b)        :  ⟨fe(a),fc(a) +ω fc(b)⟩ .  rest(b)
    true                  :  ⟨fe(a),fc(a)⟩ .  (rest(a) +o b)
```

Key Insight 1 $\alpha < \omega^\beta \Rightarrow \alpha + \omega^\beta = \omega^\beta$.

Key Insight 2 $\omega^\gamma x_1 + \omega^\gamma x_2 = \omega^\gamma(x_1 + x_2)$.

Examples

$(\omega 17 + 5) +_o (\omega^2 3 + 4) = \omega^2 3 + 4.$
$(\omega^2 5 + \omega 17 + 5) +_o (\omega^2 3 + 4) = \omega^2 8 + 4.$
$(\omega^3 + \omega^2 5 + \omega 17 + 5) +_o (\omega^2 3 + 4) = \omega^3 + \omega^2 8 + 4.$

Complexity $O(\min(\#\mathtt{a}, |\mathtt{a}|\#\mathbf{fe}(\mathtt{b})))$

Ordinal Multiplication

```
a *o b ;ordinal multiplication
    a = 0   ∨  b = 0        :  0
    atom(a)  ∧  atom(b)     :  a ·ω b
    atom(b)                 :  ⟨fe(a),fc(a) ·ω b⟩ .  rest(a)
```

Let $\alpha = \omega^{\alpha_1}x_1 + \omega^{\alpha_2}x_2 + \ldots \omega^{\alpha_n}x_n + p$, $\beta = \omega^{\beta_1}y_1 + \omega^{\beta_2}y_2 + \ldots \omega^{\beta_m}y_m + q$.

Key Insight 1

$$\begin{aligned}
\alpha \cdot q &= \omega^{\alpha_1}x_1 + \cdots + \omega^{\alpha_1}x_1 + \cdots + \omega^{\alpha_1}x_1 + \cdots + \omega^{\alpha_1}x_1 + \omega^{\alpha_2}x_2 + \cdots \omega^{\alpha_n}x_n + p \\
&= \omega^{\alpha_1}x_1 + \cdots + \omega^{\alpha_1}x_1 + \cdots + \omega^{\alpha_1}(x_1 \cdot 2) + \omega^{\alpha_2}x_2 + \cdots \omega^{\alpha_n}x_n + p \\
&= \omega^{\alpha_1}x_1 + \cdots + \omega^{\alpha_1}(x_1 \cdot 3) + \omega^{\alpha_2}x_2 + \cdots \omega^{\alpha_n}x_n + p \\
&= \omega^{\alpha_1}(x_1 \cdot q) + \omega^{\alpha_2}x_2 + \cdots \omega^{\alpha_n}x_n + p
\end{aligned}$$

Ordinal Multiplication

```
a *o b ;ordinal multiplication
    a = 0  ∨  b = 0          :  0
    atom(a)  ∧  atom(b)      :  a ·ω b
    atom(b)                  :  ⟨fe(a),fc(a) ·ω b⟩ .  rest(a)
    true                     :  ⟨fe(a) +o fe(b),fc(b)⟩ . (a *o rest(b))
```

Let $\alpha = \omega^{\alpha_1} x_1 + \omega^{\alpha_2} x_2 + \ldots \omega^{\alpha_n} x_n + p$, $\beta = \omega^{\beta_1} y_1 + \omega^{\beta_2} y_2 + \ldots \omega^{\beta_m} y_m + q$.

Key Insight 1 $\alpha \cdot q = \omega^{\alpha_1}(x_1 \cdot q) + \omega^{\alpha_2} x_2 + \cdots \omega^{\alpha_n} x_n + p$

Key Insight 2

$$\begin{aligned} \alpha \cdot \beta &= \alpha \cdot (\omega^{\beta_1} y_1 + \omega^{\beta_2} y_2 + \ldots \omega^{\beta_m} y_m + q) \\ &= \alpha \cdot \omega^{\beta_1} y_1 + \alpha \cdot \omega^{\beta_2} y_2 + \ldots \alpha \cdot \omega^{\beta_m} y_m + \alpha \cdot q \end{aligned}$$

So, we only need to deal with expressions of the form $\alpha \cdot \omega^{\beta_i} y_i$.

Ordinal Multiplication

```
a *o b ;ordinal multiplication
    a = 0   ∨   b = 0         :   0
    atom(a)   ∧   atom(b)     :   a ·ω b
    atom(b)                   :   ⟨fe(a),fc(a) ·ω b⟩ .  rest(a)
    true                      :   ⟨fe(a) +o fe(b),fc(b)⟩ . (a *o rest(b))
```

Let $\alpha = \omega^{\alpha_1}x_1 + \omega^{\alpha_2}x_2 + \ldots \omega^{\alpha_n}x_n + p$, $\beta = \omega^{\beta_1}y_1 + \omega^{\beta_2}y_2 + \ldots \omega^{\beta_m}y_m + q$.

Key Insight 1 $\alpha \cdot q = \omega^{\alpha_1}(x_1 \cdot q) + \omega^{\alpha_2}x_2 + \ldots \omega^{\alpha_n}x_n + p$

Key Insight 2 $\alpha \cdot \beta = \alpha \cdot \omega^{\beta_1}y_1 + \alpha \cdot \omega^{\beta_2}y_2 + \ldots \alpha \cdot \omega^{\beta_m}y_m + \alpha \cdot q$
So, we only need to deal with expressions of the form $\alpha \cdot \omega^{\beta_i}y_i$.

Key Insight 3
$\alpha \cdot \omega^{\beta_i}y_i \geq \omega^{\alpha_1}x_1 \cdot \omega^{\beta_i}y_i = \omega^{\alpha_1}(x_1 \cdot \omega^{\beta_i})y_i = \omega^{\alpha_1} \cdot \omega^{\beta_i}y_i = \omega^{\alpha_1+\beta_i}y_i$
$\alpha \cdot \omega^{\beta_i}y_i \leq \omega^{\alpha_1}(x_1 + 1) \cdot \omega^{\beta_i}y_i = \omega^{\alpha_1+\beta_i}y_i$
thus $\alpha \cdot \omega^{\beta_i}y_i = \omega^{\alpha_1+\beta_i}y_i$.

Ordinal Multiplication

a $*_o$ b ;*ordinal multiplication*

a = 0 ∨ b = 0	:	0
atom(a) ∧ **atom**(b)	:	a $\cdot_\omega$ b
atom(b)	:	⟨**fe**(a),**fc**(a) $\cdot_\omega$ b⟩ . **rest**(a)
true	:	⟨**fe**(a) $+_o$ **fe**(b),**fc**(b)⟩ . (a $*_o$ **rest**(b))

Let $\alpha = \omega^{\alpha_1}x_1 + \omega^{\alpha_2}x_2 + \ldots \omega^{\alpha_n}x_n + p$, $\beta = \omega^{\beta_1}y_1 + \omega^{\beta_2}y_2 + \ldots \omega^{\beta_m}y_m + q$.

Key Insight 1 $\alpha \cdot q = \omega^{\alpha_1}(x_1 \cdot q) + \omega^{\alpha_2}x_2 + \ldots \omega^{\alpha_n}x_n + p$

Key Insight 2 $\alpha \cdot \beta = \alpha \cdot \omega^{\beta_1}y_1 + \alpha \cdot \omega^{\beta_2}y_2 + \ldots \alpha \cdot \omega^{\beta_m}y_m + \alpha \cdot q$
So, we only need to deal with expressions of the form $\alpha \cdot \omega^{\beta_i}y_i$.

Key Insight 3 $\omega^{\alpha_1}x \cdot \omega^{\beta_i}y_i = \omega^{\alpha_1}(x \cdot \omega^{\beta_i})y_i = \omega^{\alpha_1} \cdot \omega^{\beta_i}y_i = \omega^{\alpha_1+\beta_i}y_i$.

Note This algorithm is inefficient.
We define an efficient version, $\cdot_o$, in the CADE paper.

Complexity Results

Algorithm	Complexity
$\mathbf{cmp_o}(a,b)$	$O(\min(\#a, \#b))$
$\mathbf{cnfp}(a)$	$O(\#a(\log \#a))$
$a +_o b$	$O(\min(\#a, \|a\| \cdot \#\mathbf{fe}(b)))$
$a -_o b$	$O(\min(\#a, \#b))$
$a \cdot_o b$	$O(\|\mathbf{fe}(a)\|\|b\| + \#\mathbf{fe}(a) + \#b)$
$\mathbf{exp_o}(a, b)$	$O(\mathbf{natpart}(b)[\|a\|\|b\| + \|\mathbf{fe}(a)\|\|a\| + \#a] + \#\mathbf{fe}(\mathbf{fe}(a))\|b\| + \#b)$