Set Theory & Ordinals

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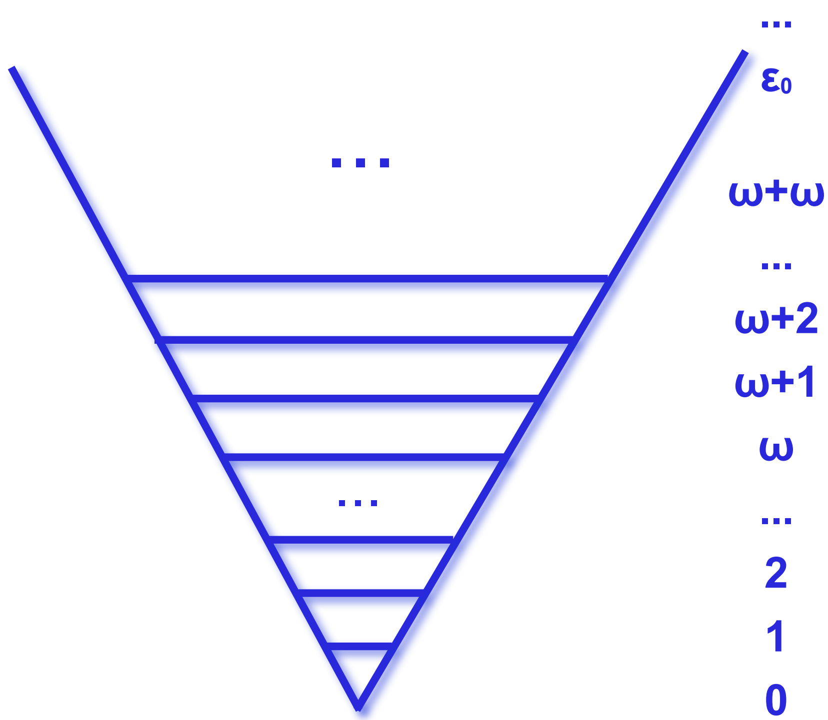
Standard Model of Set Theory



$$V_0 = \{\}$$

$$V_{\alpha+1} = \wp(V_\alpha)$$

$$V_\alpha = \bigcup_{\beta<\alpha} V_\beta$$

$$V = \bigcup_{\alpha\in On} V_\alpha$$

Axiomatic Set Theory

- From Kunen's Set Theory book
- Axiom 0. Set Existence: $\langle \exists x :: x=x \rangle$
- Axiom 1. Extensionality: $\langle \forall x,y :: \langle \forall z :: z \in x \equiv z \in y \rangle \Rightarrow x=y \rangle$
- What about an axiom that allows $\{x : P(x)\}$?
- What would that mean?
- $\langle \exists y :: \langle \forall x :: x \in y \equiv P(x) \rangle \rangle$
- But, this is problematic
- Why?
- Russell's paradox: Let $P(x)$ be $x \notin x$
- Idea: restrict the sets we can define in this way so that they are subsets of existing sets

Comprehension

- Axiom 3. Comprehension Scheme: For each formula φ without y free, the universal closure of the following is an axiom: $\langle \exists y :: \langle \forall x :: x \in y \equiv x \in z \wedge \varphi \rangle \rangle$
- We write this $\{ x \in z : \varphi \}$
- Note: this scheme yields an *infinite* number of axioms
- Why is y not free in φ?
- Consider $\langle \exists y :: \langle \forall x :: x \in y \equiv (x \in z \wedge x \notin y) \rangle \rangle$
- Definition: 0 is the unique set y s.t. $\langle \forall x :: x \notin y \rangle$
- Why is this a definition?
- Comprehension: $\{ x \in z : x \neq x \}$
  - By Axiom 0, some set z exists, so an empty set exists
  - Extensionality yields uniqueness

Pairing

- We just showed 0 exists.
- From Axioms 0,1, and 3 can we prove other sets exist?
- No. Domain = {0}, $\in$ ={} is a model of axioms 0, 1, and 3
- We can't even refute $\langle \forall x :: x=0 \rangle$
- Axioms 4-8 posit the existence of sets
- Axiom 4. Pairing: $\langle \forall x,y :: \langle \exists z :: x \in z \wedge y \in z \rangle \rangle$
- Can define {x, y} by Pairing, Comprehension, Extensionality
- What about ordered sets?
- $\langle x,y \rangle = \{\{x\}, \{x, y\}\}$
- How would you prove that this is a reasonable definition?

Union

- We also want to write $A = \cup F$ (every member of F is a $\subseteq$ A)
- Axiom 5. Union: $\langle \forall F :: \langle \exists A :: \langle \forall x,Y :: (Y \in F \wedge x \in Y) \Rightarrow x \in A \rangle\rangle\rangle$
- So $\cup F = \{x : \langle \exists Y \in F :: x \in Y \rangle\}$ is well defined
- Why can't we define $\cup F$ with comprehension?
- How would you define $\cap F$?
- Define $A \cup B = \cup\{A,B\}$, $A \cap B = \cap\{A,B\}$, $A \backslash B = \{x \in A : x \notin B\}$

Replacement

- Axiom 6. Replacement Scheme. For each formula φ without Y free, the universal closure of the following is an axiom: $\langle \forall x \in A :: \langle \exists! z :: \varphi(x,z) \rangle\rangle \Rightarrow \langle \exists Y :: \langle \forall x \in A :: \langle \exists z \in Y :: \varphi(x,z) \rangle\rangle\rangle$
- Infinite collection of axioms: one for each φ
- Can define A × B. How?
- We can define relations and functions as in the handout
- Relations are sets whose elements are ordered pairs
- A woset is a pair $\langle X, < \rangle$: $<$ is a well-founded relation on X that is transitive, irreflexive, and for which trichotomy holds
- Axiom 9. Choice. $\langle \forall A :: \langle \exists R :: R \text{ well-orders } A \rangle\rangle$
- There are many equivalent formulations of 9

What are the Ordinals?

- Let $\langle X, <\rangle$ be a woset
- Define $X_a = \{x \in X \mid x < a\}$
- An *ordinal* is a woset $\langle X, <\rangle$, such that $\langle \forall a \in X :: a = X_a\rangle$
- Theorem: if $\langle X, <\rangle$ is an ordinal, then $<$ is $\in$ (is $\subset$)
- Theorem: every woset is order-isomorphic to a unique ordinal
- Def: Ord(X, $<$) is the ordinal corresponding to woset $\langle X, <\rangle$
- Existence of infinite ordinals does not follow, yet
- Axiom 7. Infinity: $\langle \exists x :: 0 \in x \wedge \langle \forall y \in x :: y+ \in x\rangle\rangle$ $(y+ = y \cup \{y\})$
- ω is set set of naturals

Transfinite Induction

- ON is the *class* of ordinals
- $0, 1, 2, \ldots, \omega, \omega+1, \omega+2, \ldots, \omega+\omega = \omega\cdot 2, \omega\cdot 2+1, \ldots, \omega\cdot 3, \ldots, \omega\cdot\omega = \omega^2, \ldots, \omega^3, \ldots, \omega^\omega, \ldots, = \varepsilon_0, \ldots$
- Three types of ordinals: 0, successor, limit
- Transfinite induction on ON
  - If $C \subseteq ON$ and $C \neq 0$ then C has a least element
  - This is really a theorem schema
  - Proof:
    - Fix $\alpha \in C$
    - If α is not the least element of C, let β be the least element of $\alpha \cap C$
    - Then β is the least element of C

Transfinite Recursion

- Transfinite recursions on ON
  - If F: V → V, then there is a unique G: ON → V such that G.a = F(G|a)
  - We can define recursive (class) functions if they only depend on smaller values

Ordinal Addition

- $\alpha+\beta = Ord(A, <_A)$, where
  - $A = (\{0\} \times \alpha) \cup (\{1\} \times \beta)$
  - $<_A$ is the lexicographic ordering on A
- Examples
  - $1+\omega \approx \langle 0,0\rangle,\langle 1,0\rangle,\langle 1,1\rangle,\langle 1,2\rangle, \ldots \approx \omega$
  - $\omega+1 \approx \langle 0,0\rangle,\langle 0,1\rangle,\langle 0,2\rangle,\ldots, \langle 1,0\rangle \approx \omega+1$
- Properties of addition:
  - $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$ (associativity)
  - $(\beta < \gamma) \Rightarrow \alpha+\beta < \alpha+\gamma$ (strict right monotonicity)
  - $(\beta < \gamma) \Rightarrow \beta+\alpha \leqslant \gamma+\alpha$ (weak left monotonicity)
  - $\alpha < \omega^\beta \Rightarrow \alpha + \omega^\beta = \omega^\beta$ (additive principal property)
  - $\alpha, \beta < \omega^\gamma \Rightarrow \alpha + \beta < \omega^\gamma$ (closure of additive principal ordinals)

Ordinal Multiplication

- $\alpha \cdot \beta = Ord(A, <_A)$, where
  - $A = \beta \times \alpha$
  - $<_A$ is the lexicographic ordering on A
- Examples
  - $2 \cdot \omega \approx \langle 0,0\rangle,\langle 0,1\rangle,\langle 1,0\rangle,\langle 1,1\rangle,\langle 2,0\rangle,\langle 2,1\rangle, \ldots \quad \approx \omega$
  - $\omega \cdot 2 \approx \langle 0,0\rangle,\langle 0,1\rangle,\langle 0,2\rangle,\ldots,\langle 1,0\rangle,\langle 1,1\rangle,\langle 1,2\rangle \ldots \approx \omega \cdot 2$
- Properties of multiplication:
  - $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$ (associativity)
  - $\alpha \cdot 0 = 0$, $\alpha \cdot 1 = \alpha$
  - $\alpha(\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$ (left distributivity (but not right))
  - $(0 < \alpha \wedge \beta < \gamma) \Rightarrow \beta \cdot \alpha < \gamma \cdot \alpha$ (strict right monotonicity)
  - $\beta < \gamma \Rightarrow \beta \cdot \alpha \leq \gamma \cdot \alpha$ (weak left monotonicity)
  - If β is a limit, $\alpha \cdot \beta = \cup\{\alpha \cdot \gamma : \gamma < \beta\}$

Ordinal Exponentiation

- $\alpha^0 = 1$, $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$, and for limit ordinals, $\alpha^\beta = \bigcup_{\gamma<\beta} \alpha^\gamma$
- Examples
  - $2^\omega = \bigcup_{n<\omega} 2^n = \omega$
  - $2^{\omega+1} = 2^\omega \cdot 2 = \omega \cdot 2$ (*not* $2^{\omega+1} = 2 \cdot 2^\omega = 2 \cdot \omega = \omega$)
- Properties of exponentiation:
  - $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$
  - $(\alpha^\beta)^\gamma = \alpha^{(\beta \cdot \gamma)}$
  - $\alpha < \omega^\beta \Rightarrow \alpha + \omega^\beta = \omega^\beta$ (additive principal property)
  - $\alpha, \beta < \omega^\gamma \Rightarrow \alpha + \beta < \omega^\gamma$ (closure of additive principal ordinals)
  - $(1<\alpha \wedge \beta<\gamma) \Rightarrow \alpha^\beta < \alpha^\gamma$ (strict right monotonicity)
  - $\beta<\gamma \Rightarrow \beta^\alpha \leq \gamma^\alpha$ (weak left monotonicity)