Consistency

The syntactic counterpart of satisfiability is consistency.

Definition 1 *Φ is consistent, written Con Φ, iff there is no formula φ such that $\Phi \vdash \varphi$ and $\Phi \vdash \neg\varphi$.*

*Φ is inconsistent, written Inc Φ iff Φ is not consistent (*i.e.*, there is a formula φ such that $\Phi \vdash \varphi$ and $\Phi \vdash \neg\varphi$).*

Lemma 1 *Inc Φ iff for all φ: $\Phi \vdash \varphi$.*

Lemma 2 *Con Φ iff there is a φ such that not $\Phi \vdash \varphi$.*

Lemma 3 *For all Φ, Con Φ iff Con Φ_0 for all finite subsets Φ_0 of Φ.*

Lemma 4 *Sat Φ implies Con Φ.*

Consistency

Lemma 5 *For all Φ and φ the following holds:*

1. $\Phi \vdash \varphi$ *iff* $Inc\ \Phi \cup \{\neg\varphi\}$.
2. $\Phi \vdash \neg\varphi$ *iff* $Inc\ \Phi \cup \{\varphi\}$.
3. *If* $Con\ \Phi$, *then* $Con\ \Phi \cup \{\varphi\}$ *or* $Con\ \Phi \cup \{\neg\varphi\}$.

We have assumed a fixed symbol set S. When we need to consider several symbol sets simultaneously, we will use $\Phi \vdash_S \varphi$ to indicate that that there is a derivation with underlying symbol set S. Similarly $\mathsf{Con}_S\ \Phi$ denotes $\mathsf{Con}\ \Phi$ with underlying symbol set S.

Lemma 6 *For all* $i \in \omega$, S_i *is a symbol set and* $S_i \subseteq S_{i+1}$. *Similarly for all* $i \in \omega$, Φ_i *is a set of* S_i*-formulas such that* $Con_{S_i}\ \Phi_i$ *and* $\Phi_i \subseteq \Phi_{i+1}$.

Let $S = \cup_{i\in\omega} S_i$ *and* $\Phi = \cup_{i\in\omega}\Phi_i$. *Then* $Con_S\ \Phi$.

Completeness Theorem

To show: For all Φ and φ: If $\Phi \models \varphi$ then $\Phi \vdash \varphi$. We will instead show: Every consistent set of formulas is satisfiable.

Proof

$$\text{not } \Phi \vdash \varphi \quad \text{implies} \quad \text{not } \Phi \models \varphi$$

$\equiv$ { Lemma 5 }

$$\text{Con } \Phi \cup \{\neg\varphi\} \quad \text{implies} \quad \text{Sat } \Phi \cup \{\neg\varphi\}$$

$\Leftarrow$ { Instance of }

$$\text{Con } \Psi \quad \text{implies} \quad \text{Sat } \Psi \,\square$$

The Idea of Henkin's Theorem

If Φ is consistent, then we use the syntactical info that this provides to find a model $\mathcal{J} = \langle \mathbf{U}, \beta \rangle$ of Φ. If A is T^S and $\beta(v_i) = v_i$, $f^{\mathbf{U}}(t) = ft$, ..., then for variable x we have $\mathcal{J}(fx) = f^{\mathbf{U}}(\beta.x) = fx$, so $\mathcal{J}(fv_0) \neq \mathcal{J}(fv_1)$, but what if $fv_0 \equiv fv_1 \in \Phi$? To overcome this, we define an equivalence relation on terms.

First, we define an equivalence relation on T^S: $t_1 \sim t_2$ iff $\Phi \vdash t_1 \equiv t_2$.

Lemma 7

1. *$\sim$ is an equivalence relation.*

2. *If $t_1 \sim t'_1, \ldots, t_n \sim t'_n$ then for n-ary $f \in S$: $ft_1 \ldots t_n \sim ft'_1 \ldots t'_n$ and for n-ary $R \in S$: $\Phi \vdash Rt_1 \ldots t_n$ iff $\Phi \vdash Rt'_1 \ldots t'_n$.*

Let $\bar{t} = \{t' \in T^S : t \sim t'\}$, *i.e.*, $\bar{t}$ is the equivalence class of t.

Term Structure

Let T^Φ be the set of equivalence classes: $T^\Phi = \{\overline{t} : t \in T^S\}$. Note that T^Φ is not empty. We now define the term structure over T^Φ, $\mathcal{T}^\Phi$ as follows.

1. $c^{\mathcal{T}^\Phi} = \overline{c}$
2. $f^{\mathcal{T}^\Phi}(\overline{t_1}, \ldots, \overline{t_n}) = \overline{ft_1 \ldots t_n}$
3. $R^{\mathcal{T}^\Phi}\overline{t_1} \ldots \overline{t_n}$ iff $\Phi \vdash Rt_1 \ldots t_n$

Note that by Lemma 7, the definitions of $f^{\mathcal{T}^\Phi}$ and $R^{\mathcal{T}^\Phi}$ make sense.

Term Interpretation

We define the *term interpretation* associated with Φ to be $\mathcal{J}^\Phi = \langle \mathcal{T}^\Phi, \beta^\Phi \rangle$, where $\beta^\Phi(x) = \overline{x}$.

Lemma 8

1. *For all* t, $\mathcal{J}^\Phi(t) = \bar{t}$.
2. *For every atomic formula* φ, $\mathcal{J}^\Phi \models \varphi$ *iff* $\Phi \vdash \varphi$.
3. *For every formula* φ *and pairwise disjoint variables* $x_1, \ldots, x_n$

   (a) $\mathcal{J}^\varphi \models \exists x_1 \ldots \exists x_n \varphi$ *iff there are* $t_1, \ldots, t_n \in T^S$ *s.t.* $\mathcal{J}^\Phi \models \varphi\frac{t_1 \ldots t_n}{x_1 \ldots x_n}$.

   (b) $\mathcal{J}^\varphi \models \forall x_1 \ldots \forall x_n \varphi$ *iff for all* $t_1, \ldots, t_n \in T^S$ *we have* $\mathcal{J}^\Phi \models \varphi\frac{t_1 \ldots t_n}{x_1 \ldots x_n}$.

By the previous lemma $\mathcal{J}^\Phi$ is a model of the atomic formulas in Φ, but we do not know that it is a model of all formulas in Φ. In fact, it isn't. Why?

Closure Conditions

Definition 2

Φ is *negation complete* iff for every formula φ, $\Phi \vdash \varphi$ or $\Phi \vdash \neg\varphi$.

Φ *contains witnesses* iff for every formula of the form $\exists x \varphi$, there is a term t such that $\Phi \vdash (\exists x \varphi \rightarrow \varphi\frac{t}{x})$.

Lemma 9 *If Φ is consistent, negation complete, and contains witnesses, then for all φ and ψ.*

1. *$\Phi \vdash \neg\varphi$ iff not $\Phi \vdash \varphi$*
2. *$\Phi \vdash (\varphi \vee \psi)$ iff $\Phi \vdash \varphi$ or $\Phi \vdash \psi$*
3. *$\Phi \vdash \exists x \varphi$ iff there is a term t s.t. $\Phi \vdash \varphi\frac{t}{x}$*

Henkin's Theorem

Theorem 1 *(Henkin's Theorem) If Φ is consistent, negation complete, and contains witnesses, then for all φ, $\mathcal{J}^{\Phi} \models \varphi$ iff $\Phi \vdash \varphi$.*

What we can do now is to show that and consistent set of formulas can be extended to one that is consistent, negation complete, and contains witnesses. Then, from Henkin's theorem we get the completeness theorem.

Theorem 2 *(a) $\Phi \models \varphi$ iff there is a finite $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \models \varphi$.*
(b) Sat Φ iff for all finite $\Phi_0 \subseteq \Phi$, Sat Φ_0.

In addition, given that the term interpretation is a model of a set of formulas and that the size of the term interpretation is bound by the size of T^S, we have the Löwenheim-Skolem theorem.

Theorem 3 *Every satisfiable and at most countable set of formulas is satisfiable over a domain which is at most countable.*