

Substitution

We want to define a notion of substitution so that if we substitute term t for variable x in formula φ , obtaining φ' , then φ' says about t what φ says about x .

Substitution is known to be error-prone.

Consider $\varphi = \exists z z + z \equiv x$.

$\langle \mathcal{N}, \beta \rangle \models \varphi$ iff ???.

Replacing x by y gives, $\varphi' = \exists z z + z \equiv y$, where $\langle \mathcal{N}, \beta \rangle \models \varphi$ iff $\beta.y$ is even.

What about replacing x by z ? This gives $\varphi' = \exists z z + z \equiv z$, but $\mathcal{N} \models \varphi$, so here we have a problem.

In order to get a φ' which expresses about z what φ expresses about x , we can first replace bound occurrences of z by a new variable u in φ , and then proceed as before.

Substitution for Terms

1. $x \frac{t_0 \dots t_r}{x_0 \dots x_r} = x$ if $x \neq x_0, \dots, x_r$
2. $x \frac{t_0 \dots t_r}{x_0 \dots x_r} = t_i$ if $x = x_i$
3. $c \frac{t_0 \dots t_r}{x_0 \dots x_r} = c$
4. $[ft'_1 \dots t'_n] \frac{t_0 \dots t_r}{x_0 \dots x_r} = f t'_1 \frac{t_0 \dots t_r}{x_0 \dots x_r} \dots t'_n \frac{t_0 \dots t_r}{x_0 \dots x_r}$

Substitution for Formulas

1. $[t'_1 \equiv t'_2] \frac{t_0 \dots t_r}{x_0 \dots x_r} = t'_1 \frac{t_0 \dots t_r}{x_0 \dots x_r} \equiv t'_2 \frac{t_0 \dots t_r}{x_0 \dots x_r}$
2. $[Rt'_1 \dots t'_n] \frac{t_0 \dots t_r}{x_0 \dots x_r} = Rt'_1 \frac{t_0 \dots t_r}{x_0 \dots x_r} \dots t'_n \frac{t_0 \dots t_r}{x_0 \dots x_r}$
3. $[\neg \varphi] \frac{t_0 \dots t_r}{x_0 \dots x_r} = \neg [\varphi] \frac{t_0 \dots t_r}{x_0 \dots x_r}$
4. $(\varphi \vee \psi) \frac{t_0 \dots t_r}{x_0 \dots x_r} = (\varphi \frac{t_0 \dots t_r}{x_0 \dots x_r} \vee \psi \frac{t_0 \dots t_r}{x_0 \dots x_r})$
5. Suppose x_{i_1}, \dots, x_{i_s} ($i_1 < \dots < i_s$) are exactly the variables x_i among the x_0, \dots, x_r such that $x_i \in \text{free}(\exists x \varphi)$ and $x_i \neq t_i$. Then, set

$$[\exists x \varphi] \frac{t_0 \dots t_r}{x_0 \dots x_r} = \exists u \left[\varphi \frac{t_{i_1} \dots t_{i_s} u}{x_{i_1} \dots x_{i_s} x} \right],$$

where u is x if x does not occur in $t_{i_1} \dots t_{i_s}$; otherwise u is the first variable in the list v_0, v_1, v_2, \dots which does not occur in $\varphi, t_{i_1} \dots t_{i_s}$.

Substitution Examples

1. $[Pv_0fv_1v_2]_{\frac{v_2v_0v_1}{v_1v_2v_3}} = ?$

2. $[\exists v_0Pv_0fv_1v_2]_{\frac{v_4fv_1v_1}{v_0v_2}} = ?$

3. $[\exists v_0Pv_0fv_1v_2]_{\frac{v_0v_2v_4}{v_1v_2v_0}} = ?$

Substitution Lemma

Let $\mathcal{J} = \langle \mathbf{U}, \beta \rangle$ with $a_0, \dots, a_r \in A$. Then:

$$\beta \frac{a_0 \dots a_r}{x_0 \dots x_r}(y) = \beta.y \text{ if } y \neq x_0, \dots, y \neq x_r$$

$$\beta \frac{a_0 \dots a_r}{x_0 \dots x_r}(y) = a_i \text{ if } y = x_i$$

$$\mathcal{J} \frac{a_0 \dots a_r}{x_0 \dots x_r} = \langle \mathbf{U}, \beta \frac{a_0 \dots a_r}{x_0 \dots x_r} \rangle$$

Here then is the main result about substitution.

Lemma 1 1. For every term t , $\mathcal{J}(t \frac{t_0 \dots t_r}{x_0 \dots x_r}) = \mathcal{J} \frac{\mathcal{J}(t_0) \dots \mathcal{J}(t_r)}{x_0 \dots x_r}(t)$

2. For every formula φ , $\mathcal{J} \models \varphi \frac{t_0 \dots t_r}{x_0 \dots x_r}$ iff $\mathcal{J} \frac{\mathcal{J}(t_0) \dots \mathcal{J}(t_r)}{x_0 \dots x_r} \models \varphi$

Proof By induction on terms and formulas. \square

Formalizations

Your book contains examples of simple formalizations; let's look at one.
How would you formalize an equivalence relation?

1. $\forall x Rxx$
2. $\forall x \forall y (Rxy) \rightarrow (Ryx)$
3. $\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz)$

The way I would write them is:

1. $\langle \forall x :: xRx \rangle$
2. $\langle \forall x, y :: xRy \Rightarrow yRx \rangle$
3. $\langle \forall x, y, z :: xRy \wedge yRz \Rightarrow xRz \rangle$

Define a new quantifier “there exists exactly one”, written $\exists^1 x \varphi$.

Uses of the Coincidence Lemma

Definition 1 *Let S and S' be symbol sets with $S \subseteq S'$; let $\mathbf{U} = \langle A, \mathbf{a} \rangle$ be an S -structure and $\mathbf{U}' = \langle A', \mathbf{a}' \rangle$ be an S' -structure. We call \mathbf{U} the S -reduct of \mathbf{U}' (and conversely \mathbf{U}' an expansion of \mathbf{U}) iff $A = A'$ and \mathbf{a} and \mathbf{a}' agree on S . We write $\mathbf{U} = \mathbf{U}'|_S$.*

For example, the ordered group $\mathbb{Z}^{<}$ of integers as an $S_{gr}^{<}$ -structure, written $\langle \mathbb{Z}, +, 0, < \rangle$, is an expansion of the group of integers as an S_{gr} -structure, written $\langle \mathbb{Z}, +, 0 \rangle$.

If $\mathbf{U} = \mathbf{U}'|_S$, then for every S -formula, φ , whose free variables are among v_0, \dots, v_{n-1} , and for all $a_0, \dots, a_{n-1} \in A$,

$$\mathbf{U} \models \varphi[a_0, \dots, a_{n-1}] \text{ iff } \mathbf{U}' \models \varphi[a_0, \dots, a_{n-1}]$$

Proof Apply the coincidence lemma.

Isomorphism

Definition 2 Let \mathbf{U} and \mathbf{B} be S -structures.

1. $\pi : A \rightarrow B$ is called an isomorphism of \mathbf{U} onto \mathbf{B} ($\pi : \mathbf{U} \cong \mathbf{B}$) iff
 - (a) π is a bijection of A onto B .
 - (b) For n -ary $R \in S$ and $a_1, \dots, a_n \in A$: $R^{\mathbf{U}} a_1 \dots a_n$ iff $R^{\mathbf{B}} \pi.a_1 \dots \pi.a_n$.
 - (c) For n -ary $f \in S$ and $a_1, \dots, a_n \in A$:

$$\pi(f^{\mathbf{U}}(a_1, \dots, a_n)) = f^{\mathbf{B}}(\pi.a_1, \dots, \pi.a_n).$$
 - (d) For $c \in S$, $\pi(c^{\mathbf{U}}) = c^{\mathbf{B}}$.
2. \mathbf{U} and \mathbf{B} are isomorphic iff there is an isomorphism $\pi : \mathbf{U} \cong \mathbf{B}$.

For example, the S_{gr} -structure $\langle \mathbb{N}, +, 0 \rangle$ (not a group) is isomorphic to the S_{gr} -structure $\langle E, +^E, 0 \rangle$, consisting of the even numbers and addition on them.

Give a suitable π .

Isomorphism

Lemma 2 *If \mathbf{U} and \mathcal{B} are isomorphic S -structures then for all S -sentences φ , $\mathbf{U} \models \varphi$ iff $\mathcal{B} \models \varphi$.*

Corollary 1 *If $\pi : \mathbf{U} \cong \mathcal{B}$ then for $\varphi \in L_n^S$ and $a_0, \dots, a_{n-1} \in A$, $\mathbf{U} \models \varphi[a_0, \dots, a_{n-1}]$ iff $\mathcal{B} \models \varphi[\pi.a_0, \dots, \pi.a_{n-1}]$*

Note that isomorphic structures cannot be distinguished in L_0^S .

Is it the case that S -structures which satisfy the same S -sentences are isomorphic?

This is a major question and we will see that the answer is ???

Substructures

Definition 3 Let \mathbf{U} and \mathcal{B} be S -structures. \mathbf{U} is called a substructure of \mathcal{B} (written $\mathbf{U} \subset \mathcal{B}$) if:

1. $A \subseteq B$
- 2.(a) For n -ary $R \in S$, $R^{\mathbf{U}} = R^{\mathcal{B}} \cap A^n$.
- (b) For n -ary $f \in S$, $f^{\mathbf{U}} = f^{\mathcal{B}}|_{A^n}$ (the restriction of $f^{\mathcal{B}}$ to A^n).
- (c) For $c \in S$, $c^{\mathbf{U}} = c^{\mathcal{B}}$.

For example $\langle \mathbb{Z}, +, 0 \rangle$ is a substructure of $\langle \mathbb{Q}, +, 0 \rangle$.

If $\mathbf{U} \subset \mathcal{B}$, then A is S -closed in \mathcal{B} . Conversely, every $X \subseteq B$ which is S -closed in \mathcal{B} is the domain of exactly one substructure of \mathcal{B} ($[X]^{\mathcal{B}}$).

For example, $\{2n \mid n \in \mathbb{N}\}$ is S_{gr} -closed, but $\{2n + 1 \mid n \in \mathbb{N}\}$ is not.

Substructures

For arbitrary X , we obtain $[X]^{\mathcal{B}}$ by starting with X and applying the functions in the symbol set of \mathcal{B} over and over until a fixed point is reached.

Lemma 3 *Let \mathbf{U} and \mathcal{B} be S -structures with $\mathbf{U} \subset \mathcal{B}$ and let $\beta : \{v_n \mid n \in \mathbb{N}\} \rightarrow A$ be an assignment in \mathbf{U} . Then the following holds for every S -term t :*

$$\langle \mathbf{U}, \beta \rangle(t) = \langle \mathcal{B}, \beta \rangle(t)$$

and for every quantifier-free S -formula φ :

$$\langle \mathbf{U}, \beta \rangle \models \varphi \text{ iff } \langle \mathcal{B}, \beta \rangle \models \varphi$$

What about quantified formulas?

Universal Formulas

Definition 4 *The set of universal formulas is the least set closed under the following rules.*

1. Every quantifier-free formula is a universal formula.
2. If φ, ψ are universal, so are $\varphi \wedge \psi$ and $\varphi \vee \psi$.
3. If φ is universal, so is $\forall x\varphi$.

Lemma 4 *Let \mathbf{U} and \mathcal{B} be S -structures with $\mathbf{U} \subset \mathcal{B}$ and $\varphi \in L_n^S$ be universal. Then the following holds for all $a_0, \dots, a_{n-1} \in A$:
If $\mathcal{B} \models \varphi[a_0, \dots, a_{n-1}]$ then $\mathbf{U} \models \varphi[a_0, \dots, a_{n-1}]$.*

Corollary 2 *If $\mathbf{U} \subset \mathcal{B}$ then the following holds for every universal sentence φ : If $\mathcal{B} \models \varphi$ then $\mathbf{U} \models \varphi$.*