Multicut Survey

Zhifeng Sun

September 17, 2008

Graph cut problems are widely studied in the area of approximation algorithms. The most basic cut problem is the s-t minimum cut problem, for which Ford and Fulkerson gave an exact algorithm and illustrated max-flow min-cut relationship. This min-max theorem has led researchers to seek its generalization to the case of multicommodity flow. In this setting, each commodity has its own source and sink, and the object is to maximaze the sum of the flows subject to capacity and flow conservation requirements. The notion of a multicut generalizes that of a cut, and is defined as a set of edges whose removal disconnects each source from its corresponding sink.

Clearly, maximum multicommodity flow is bounded by minimum multicut; however, the equality doesn't hold often times for $k \geq 3$, where k is the number of source and sink pairs. And minimum multicut is proved to be NP-hard when $k \geq 3$. In this situation, the best one can hope for is an approximation algorithm for minimum multicut problems. This suvey collects approximation and complexity results for multicut problems and its variants.

1 Minimum multicut problem

Problem 1. Let $G = (V, E)$ be an undirected graph with nonnegative capacity c_e for each edge $e \in E$. Let $\{(s_1, t_1), \ldots, (s_k, t_k)\}$ be a specified set of pairs of vertices, where each pair is distinct, but vertices in different pairs are not required to be distinct. A *multicut* is a set of edges whose removal separates each of the pairs. The problem is to find a minimum capacity multicut in G.

Minimum multicut problem is NP-hard and MAX-SNP-hard even for $k = 3$ by [3], which means it's unlikely to have polynomial time approximation scheme. Moreover, it is NP-hard even if graphs are restricted to trees of height 1 and unit capacity edges. In [1], Chawla et al show an arbitrary large constant factor hardness, assuming the Unique Games Conjecture. A stronger version of this conjecture leads to a hardness result of $\Omega(\log \log n)$.

Garg et al give a 2-approximation algorithm in [7] when the underlining graph is restricted to trees, and a $O(\log k)$-approximation for general graphs in [6], which is the best know approximation ratio so far.

Minimum multicut can be generalized to vertex multicut and preserve the approximation ratio $O(\log n)$, which was stated in [4] (the journal version of [7]) without proof.

2 k-Multicut problem

Problem 2. Let $G = (V, E)$ be an undirected graph with nonnegative capacity c_e for each edge $e \in E$. Let $\{(s_1, t_1), \dots, (s_k, t_k)\}$ be a specified set of pairs of vertices, where each pair is distinct, but vertices in different pairs are not required to be distinct. A *k-multicut* is a set of edges whose removal separates *at least* k pairs. The problem is to find a minimum capacity k-multicut in G.

In [8], Glovin et al show that the k-multicut problem on trees can be approximated within a factor of $\frac{8}{3}+\epsilon$, for any fixed $\epsilon > 0$, and within $O(\log^2 n \log\log n)$ on general graphs by Racke decomposition, where n is the number of vertices in the graph. Recently, Räcke imporved the congetion gap to $O(\log n)$ in [11], which in turn imporved the approximation ratio for k-multicut in general graph to $O(\log n)$.

For any fixed $\epsilon > 0$, they also obtain a polynomial time algorithm for k-multicut on trees which returns a solution of cost at most $(2+\epsilon)\dot{O}PT$, that separates at least $(1-\epsilon)\dot{k}$ pairs, where OPT is the cost of the optimal solution separaating k pairs.

By applying the same techniques in [8], they also give a simple 2-approximation algorithm for the multicut problem on trees, which matches the best known algorithm [7].

Levin and Segev obtained the same result independently using similar techniques [10].

3 Multiway Cut

Problem 3. Given a set of terminals $S = \{s_1, s_2, \dots, s_k\} \subseteq V$, a *multiway cut* is a set of edges whose removal disconnects the terminals from each other. The multiway cut problem asks for the minimum weight such set.

The problem of finding a minimum weight multiway cut is MAX SNP-hard for any fixed $k \geq 3$; even if all edge weights are equal to 1, multiway cut is still NP-hard for all fixed $k \geq 3$; when k is not fixed, multiway cut is NP-hard even if graph is restricted to planar graphs and all edge weigths are equal to 1, shown in [3]. Ovserve that the case $k = 2$ is precisely the minimum $s - t$ cut problem.

In [3], there is a simple algorithm that gives $(2-2/k)$-approximation. Later, [2] improved the approximation ratio to $3/2$. The current best guarantee known for the multiway cut problem is 1.3438, due to [9].

Multiway cut can be generalized to node multiway cut, which is stated as follows.

Problem 4. Given a connected, undirected graph $G = (V, E)$ with an assignment of costs to vertices, and a set of terminals $S = \{s_1, s_2, \ldots, s_k\} \subseteq V$ that form an independent set in G, a *node multiway cut* is a subset of $V \setminus S$ whose removal disconnects the terminals from each other. The node multiway cut problem asks for the minimum cost such subset.

In [5], Garg et al give a $(2 - 2/k)$-approximation.

4 k-Cut

Problem 5. A set of edges whose removal leaves k connected components is called k-cut. The minimum k-cut problem asks for a minimum weight k-cut.

The minimum k-cut problem is polynomial time solvable for fixed k; however, it is NP-hard if k is specified as part of the input.

In [12], there is an algorithm that gives $(2 - 2/k)$-approximation.

References

[1] Shuchi Chawla, Robert Krauthgamer, Ravi Kumar, Yuval Rabani, and D. Sivakumar. On the hardness of approximating multicut and sparsest-cut. *Comput. Complex.*, 15(2):94–114, 2006.

[2] Gruia Călinescu, Howard Karloff, and Yuval Rabani. An improved approximation algorithm for multiway cut. *J. Comput. Syst. Sci.*, 60(3):564–574, 2000.

[3] E. Dahlhaus, D. S. Johnson, C. H. Papadimitriou, P. D. Seymour, and M. Yannakakis. The complexity of multiterminal cuts. *SIAM Journal on Computing*, 23:864–894, 1994.

[4] Naveen Garg, Vijay V. Vazirani, and Mihalis Yannakakis. Primal-dual approximation algorithms for integral flow and multicut in trees, with applications to matching and set cover. In *ICALP '93: Proceedings of the 20th International Colloquium on Automata, Languages and Programming*, pages 64–75, London, UK, 1993. Springer-Verlag.

[5] Naveen Garg, Vijay V. Vazirani, and Mihalis Yannakakis. Multiway cuts in directed and node weighted graphs. In *ICALP '94: Proceedings of the 21st International Colloquium on Automata, Languages and Programming*, pages 487–498, 1994.

[6] Naveen Garg, Vijay V. Vazirani, and Mihalis Yannakakis. Approximate max-flow min-(multi)cut theorems and their applications. *SIAM J. Comput.*, 25(2):235–251, 1996.

[7] Naveen Garg, Vijay V. Vazirani, and Mihalis Yannakakis. Primal-dual approximation algorithms for integral flow and multicut in trees. *Algorithmica*, 18(1):3–20, 1997.

[8] Daniel Golovin, Viswanath Nagarajan, and Mohit Singh. Approximating the k-multicut problem. In *SODA '06: Proceedings of the seventeenth annual ACM-SIAM symposium on Discrete algorithm*, pages 621–630, 2006.

[9] David R. Karger, Philip Klein, Cliff Stein, Mikkel Thorup, and Neal E. Young. Rounding algorithms for a geometric embedding of minimum multiway cut. *Proc. 29th STOC*, pages 668–678, 1999.

[10] Asaf Levin and Danny Segev. Partial multicuts in trees. *Theor. Comput. Sci.*, 369(1):384–395, 2006.

[11] Harald Räcke. Optimal hierarchical decompositions for congestion minimization in networks. In *STOC '08: Proceedings of the 40th annual ACM symposium on Theory of computing*, pages 255–264, 2008.

[12] H. Saran and V.V. Vazirani. Finding k-cuts within twice the optimal. *SIAM Journal on Computing*, 24:101–108, 1995.