

Lecture 6(20) :- Mixture of Binomial Dist \rightarrow HW3 PB (conts)

- Mixt of Poissons Dist
- Other Gen models
- Sampling techniques

Next
Wed 6/25

Demo HW3

Binomial Dist $P[X=k] = \binom{n}{k} p^k (1-p)^{n-k}$ $\rightarrow P[\text{tail}]$ # tails

$p = \text{coin bias} = \Pr[\text{Head}]$

$n = \# \text{ trials} = \# \text{ flips}$

$X = R.V = \# \text{ heads observed}$

prob to see k heads in n trials.

choice of k positions in n

heads

Binom th $\sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} = 1 \Rightarrow$ valid distribution over possible outcomes of $X=0, X=1, X=2, \dots, X=n$

$\Pr[\text{head}]$

Fit dist to data
 $x_i = \# \text{heads} \Rightarrow p \approx \frac{x_i}{n}$ chance of head

Multiple Coins $k=1, k=2, \dots, K$ $\rightarrow [K=k]$ b ($\#$ coins not heads)

Data x_1, x_2, \dots, x_M # of heads in n trials; M experiments each with a fixed unknown coin.

Mixture $P(x_i) = \sum_{k=1}^K w_k \cdot \text{Bin}(x_i | n, p_k)$

Membership $\pi_{ik} = \text{prob that coin } k \text{ generated } x_i = P(\text{coin } k | x_i)$

$$\frac{\Pr[x_i | \text{coin } k] \cdot \Pr[\text{coin } k]}{\text{normalized over all } k \text{ numerators}} = \frac{\text{Bin}(x_i | p_k, n) \cdot w_k}{\text{normalized}}$$

$$= \frac{w_k \cdot p_k^{x_i} (1-p_k)^{n-x_i}}{\text{normalized}}$$

for $i=1 : M=20$ experiments

- pick coin $k \in \{1, 2, \dots, K\}$
probs w_k

- flip coin $n=100$ times
- record $x_i = \# \text{heads}$.

$$\prod_{i=1}^M \left[\prod_{k=1}^K \Pr(\text{coin select}) \cdot P(x_i | \text{coin } k) \right]^{\pi_{ik}}$$

$\Pr(\text{coin } k, x_i \text{ heads})$

M Step

Log Likelihood = $\log \prod_{i=1}^M \prod_{k=1}^{T_i} [w_k \text{Bin}(x_i | p_k, n)]^{\pi_{ik}}$ filter π_{ik}
 $R.V \rightarrow 0$

$n = \# \text{Flips for each experiment } i = 1, 2, \dots, M$

$E[\ell]_{\text{LL}} = \sum_{i=1}^M \sum_{k=1}^{T_i} \langle \pi_{ik} \rangle [\log w_k + \log(x_i) + \log p_k + \log(1-p_k)]$

$\text{avg } \langle \pi_{ik} \rangle$ expected over π_{ik}

$\langle \pi_{ik} \rangle$ prob

$\log w_k$ ignore

$\log(x_i)$ ignore

$$\begin{aligned}\frac{\partial E[\ell]_{\text{LL}}}{\partial p_k} &= \frac{\partial}{\partial p_k} \cdot \sum_{i=1}^M \langle \pi_{ik} \rangle [x_i \log p_k + (n-x_i) \log(1-p_k)] \\ &= \sum_{i=1}^M \pi_{ik} x_i - \sum_{i=1}^M \frac{n-x_i}{1-p_k} \\ &= S_k \cdot \frac{1}{p_k} - n R_k \frac{1}{1-p_k} + S_k \frac{1}{1-p_k} \stackrel{\text{want}}{=} 0\end{aligned}$$

$S_k = \sum_i \pi_{ik} x_i$

$R_k = \sum_i \pi_{ik}$

$1 \cdot p_k$
 $1 \cdot (1-p_k)$

$$S_K(1 - P_K) - \boxed{n R_K \cdot P_K} + S_K \cdot P_K \stackrel{?}{=} 0$$

$$S_K = ? \quad n R_K \cdot P_K$$

$$P_K = \frac{S_K}{n \cdot R_K} = \frac{\sum_i \langle \pi_{ik} \rangle \cdot x_i}{\sum_i \langle \pi_{ik} \rangle} = \text{weighted AVG } \{x_i\} \text{ weights } \langle \pi_{ik} \rangle$$

for w_k : Lagrangian = $\sum_k \sum_i \pi_{ik} \log(w_k)$ subj to $\sum_k w_k = 1$

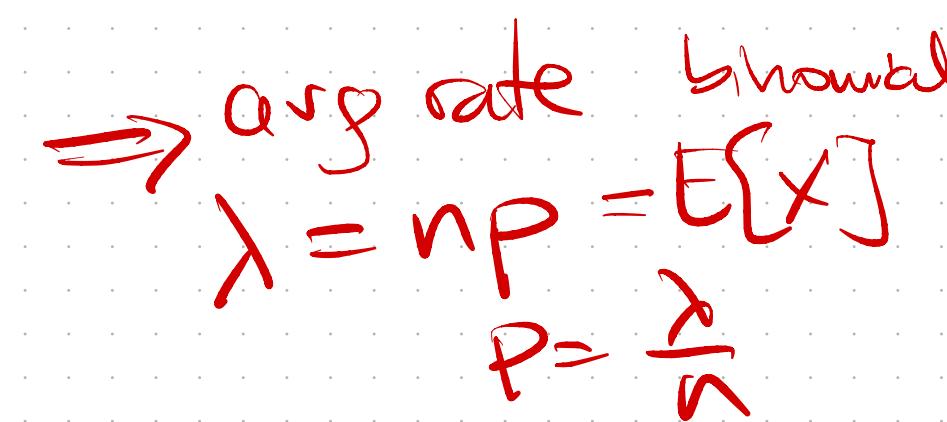
identical to mixt gauss

$$\Rightarrow w_k = \frac{1}{M} \sum_{i=1}^M \pi_{ik}$$

Poisson Distribution # events in fixed interval

- Events independent

- average occurrence rate = λ

Poisson dist = limit of Binomial dist $Bin(P, n)$  

$$\lambda = np = E[X]$$

$$P = \frac{\lambda}{n}$$

$$\begin{aligned}
 P[X=\text{# events}] &= \lim_{n \rightarrow \infty} \binom{n}{x} p^x (1-p)^{n-x} \\
 &= \lim_{n \rightarrow \infty} \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} \\
 &= \lim_{n \rightarrow \infty} \frac{n!}{x!(n-x)!} \cdot \frac{\lambda^x}{n^x} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-x} \cdot e^{-\lambda} \\
 &= \lim_{n \rightarrow \infty} \frac{n!}{x!(n-x)!} \cdot \frac{\lambda^x}{x!} \cdot \left[\left(1 - \frac{\lambda}{n}\right)^n\right]^{n-x} \cdot \left(\frac{n}{n}\right)^{n-x} \cdot e^{-\lambda}
 \end{aligned}$$

$$\begin{aligned}
 \left(1 + \frac{1}{n}\right)^n &\rightarrow e \\
 \left(1 - \frac{1}{n}\right)^n &\rightarrow \frac{1}{e}
 \end{aligned}$$

$$= \frac{\lambda^x}{x!} e^{-\lambda}$$

$$\text{Poisson } (x) \sim \frac{\lambda^x}{x!}$$

pr

$$\text{pr}(x) = \frac{\lambda^x / x!}{\text{Normalized over all } x \in \{0, 1, 2, \dots, \infty\}} = \frac{\lambda^x / x!}{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}} = \frac{\lambda^x / x!}{e^\lambda}$$

Series

Mixture of Poisson Distributions : K coins (flipped ∞ times.)

$$P(x_i) = \sum_{k=1}^K w_k \frac{\lambda_k^{x_i}}{x_i!} e^{-\lambda_k}$$

Prob of x_i heads
in fixed interval
with arg λ_k

$$\sum_{k=1}^K w_k = 1$$

fixed selection
prob.

E Step $\pi_{ik} = \text{membership prob. coin } k | x_i = \frac{pr(k \text{ coin}) \cdot P(x_i | \text{coin } k)}{\text{normalized}}$

$$= \frac{w_k \cdot \lambda_k^{x_i} / x_i! \cdot e^{-\lambda_k}}{\text{Normalized over coins } k=1 \dots K}$$

M step $w_k = \text{same as before}$

$$= \frac{1}{M} \cdot \sum_{i=1}^M \pi_{ik}$$

Prior prob of selecting
 k coin

M step for λ_k

$$P(x_i) = \sum_{k=1}^K w_k \frac{\lambda_k^{x_i}}{x_i!} e^{-\lambda_k}$$

$$E_{[\pi_{ik}]} \text{LogLik}_k = \sum_{i=1}^M \left[\sum_{k=1}^K \left(\pi_{ik} \right) \left[\log w_k + x_i \log \lambda_k - \lambda_k - \log(x_i!) \right] \right]$$

for λ_k : maximize $\sum_{i=1}^M \pi_{ik} [x_i \log \lambda_k - \lambda_k]$

$$\frac{\partial}{\partial \lambda_k} = \sum_{i=1}^M \left(\pi_{ik} x_i \cdot \frac{1}{\lambda_k} \right) - \sum_{i=1}^M \pi_{ik} = 0$$

↑ #events

$\lambda_k = \frac{\sum \pi_{ik} x_i}{\sum \pi_{ik}}$

weighted Avg (x_i) / hour

$= \text{AVG}(\# \text{events per hour})$

Gen. Models Overview

- Gauss Discriminant Analysis ✓
- Naïve Bayes ✓
- GMM (EM of mixt-gaussians) ✓
 - EM for Mixt-Binom
Mixt-Poisson

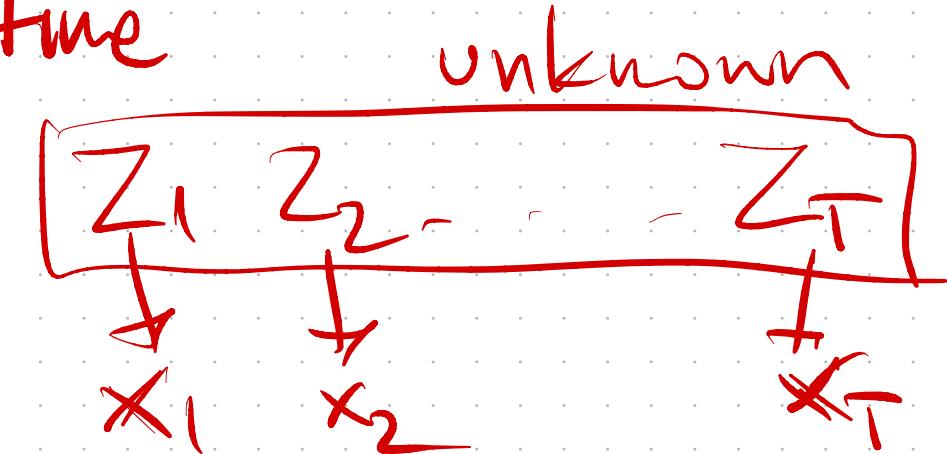
• Hidden Markov Models → next time

Markov Process

Sequence

Hidden states

Observation



Stationary State dist $[\pi_1 \ \pi_2 \ \dots \ \pi_k]$

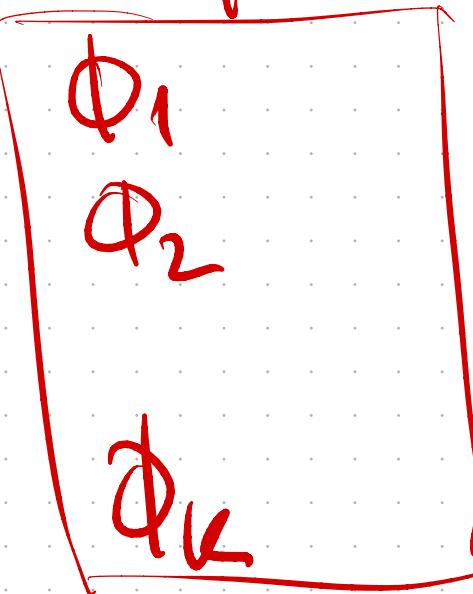
- transition probab $P_{ij} = \Pr [z_i \rightarrow z_j]$

- emission prob $z_i \xrightarrow{j} x_j$

• Latent Dirichlet Allocation "Topic Models" for text.

K topic dists.

doc
 $d = \text{dist over topics}$



topic
 $\phi = \text{dist over words}$

e SAMPLING (Unsup ML)
basic
(next)

• (***) VARIATIONAL ENCODER

Variational inference
alternative to EM