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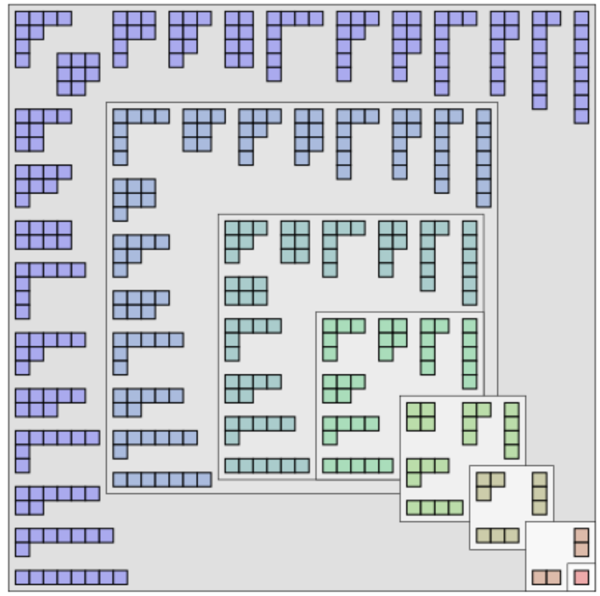
Partition (number theory)

In number theory and combinatorics, a **partition** of a positive integer n, also called an **integer partition**, is a way of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered the same partition. (If order matters, the sum becomes a composition.) For example, 4 can be partitioned in five distinct ways:

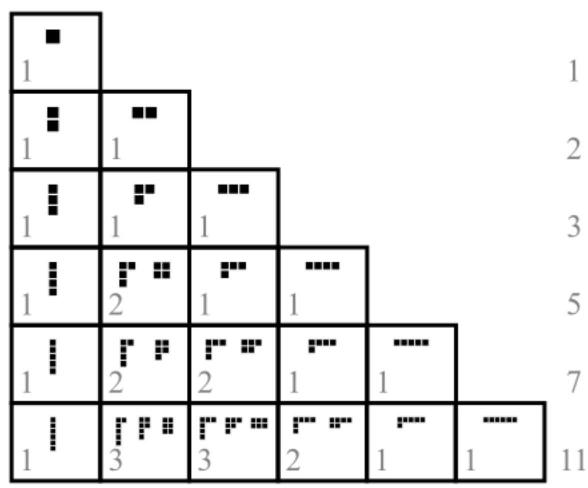
4
3 + 1
2 + 2
2 + 1 + 1
1 + 1 + 1 + 1



Young diagrams associated to the partitions of the positive integers 1 through 8. They are arranged so that images under the reflection about the main diagonal of the square are conjugate partitions.

The order-dependent composition 1 + 3 is the same partition as 3 + 1, while the two distinct compositions 1 + 2 + 1 and 1 + 1 + 2 represent the same partition 2 + 1 + 1.

A summand in a partition is also called a **part**. The number of partitions of n is given by the partition function $p(n)$. So $p(4) = 5$. The notation $\lambda \vdash n$ means that λ is a partition of n.



Partitions of n with biggest addend k

Partitions can be graphically visualized with Young diagrams or Ferrers diagrams. They occur in a number of branches of mathematics and physics, including the study of symmetric polynomials and of the symmetric group and in group representation theory in general.

Contents

Examples

The seven partitions of 5 are:

- 5
- 4 + 1
- 3 + 2
- 3 + 1 + 1
- 2 + 2 + 1
- 2 + 1 + 1 + 1
- 1 + 1 + 1 + 1 + 1

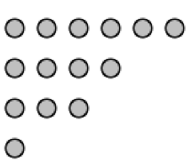
In some sources partitions are treated as the sequence of summands, rather than as an expression with plus signs. For example, the partition 2 + 2 + 1 might instead be written as the tuple (2, 2, 1) or in the even more compact form $(2^2, 1)$ where the superscript indicates the number of repetitions of a term.

Representations of partitions

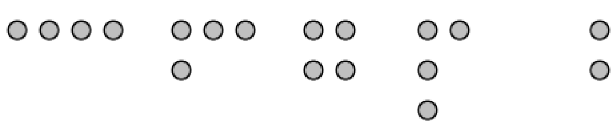
There are two common diagrammatic methods to represent partitions: as Ferrers diagrams, named after Norman Macleod Ferrers, and as Young diagrams, named after the British mathematician Alfred Young. Both have several possible conventions; here, we use *English notation*, with diagrams aligned in the upper-left corner.

Ferrers diagram

The partition 6 + 4 + 3 + 1 of the positive number 14 can be represented by the following diagram:



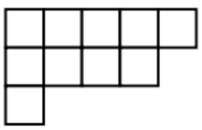
The 14 circles are lined up in 4 rows, each having the size of a part of the partition. The diagrams for the 5 partitions of the number 4 are listed below:



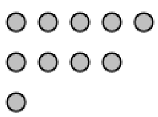
4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1

Young diagram

An alternative visual representation of an integer partition is its *Young diagram* (often also called a Ferrers diagram). Rather than representing a partition with dots, as in the Ferrers diagram, the Young diagram uses boxes or squares. Thus, the Young diagram for the partition 5 + 4 + 1 is



while the Ferrers diagram for the same partition is



While this seemingly trivial variation doesn't appear worthy of separate mention, Young diagrams turn out to be extremely useful in the study of symmetric functions and group representation theory: filling the boxes of Young diagrams with numbers (or sometimes more complicated objects) obeying various rules leads to a family of objects called Young tableaux, and these tableaux have combinatorial and representation-theoretic significance.[1] As a type of shape made by adjacent squares joined together, Young diagrams are a special kind of polyomino.[2]

Partition function

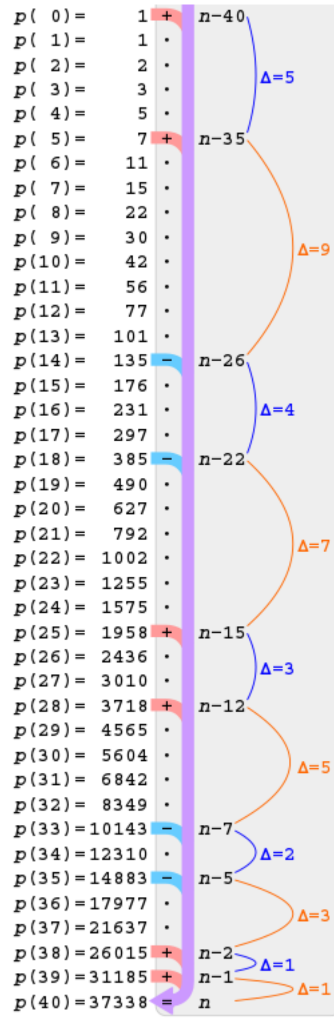
The partition function $p(n)$ represents the number of possible partitions of a non-negative integer n. For instance, $p(4) = 5$ because the integer 4 has the five partitions $1+1+1+1$, $1+1+2$, $1+3$, $2+2$, and 4. The values of this function for $n = 0, 1, 2, \ldots$ are:

1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, 2436, 3010, 3718, 4565, 5604, ... (sequence A000041 in the OEIS).

The generating function of p is

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{j=1}^{\infty} \sum_{i=0}^{\infty} q^{ji} = \prod_{j=1}^{\infty} (1-q^j)^{-1}.$$

No closed-form expression for the partition function is known, but it has both asymptotic expansions that accurately approximate it and recurrence relations by which it can be calculated exactly. It grows as an exponential function of the square root of its argument.[3] The multiplicative inverse of its generating function is the Euler function; by Euler's pentagonal number theorem this function is an alternating sum of pentagonal number powers of its argument.



Using Euler's method to find $p(40)$: A ruler with plus and minus signs (grey box) is slid downwards, the relevant terms added or subtracted. The positions of the signs are given by differences of alternating natural (blue) and odd (orange) numbers. In the SVG file, (https://upload.wikimedia.org/wikipedia/commons/0/05/Euler_partition_function.svg) hover over the image to move the ruler.

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots$$

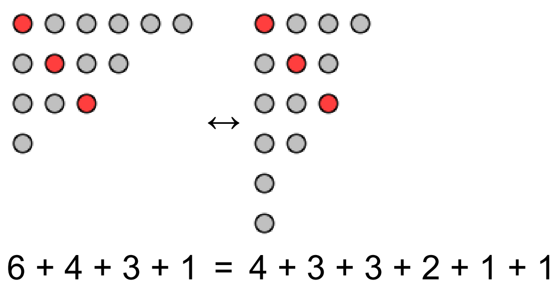
Srinivasa Ramanujan first discovered that the partition function has nontrivial patterns in modular arithmetic, now known as Ramanujan's congruences. For instance, whenever the decimal representation of n ends in the digit 4 or 9, the number of partitions of n will be divisible by 5.[4]

Restricted partitions

In both combinatorics and number theory, families of partitions subject to various restrictions are often studied.[5] This section surveys a few such restrictions.

Conjugate and self-conjugate partitions

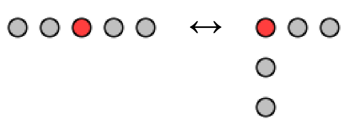
If we flip the diagram of the partition 6 + 4 + 3 + 1 along its main diagonal, we obtain another partition of 14:



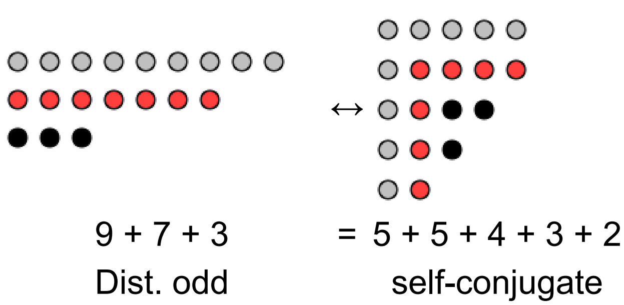
By turning the rows into columns, we obtain the partition 4 + 3 + 3 + 2 + 1 + 1 of the number 14. Such partitions are said to be *conjugate* of one another.[6] In the case of the number 4, partitions 4 and 1 + 1 + 1 + 1 are conjugate pairs, and partitions 3 + 1 and 2 + 1 + 1 are conjugate of each other. Of particular interest is the partition 2 + 2, which has itself as conjugate. Such a partition is said to be *self-conjugate*.[7]

Claim: The number of self-conjugate partitions is the same as the number of partitions with distinct odd parts.

Proof (outline): The crucial observation is that every odd part can be "*folded*" in the middle to form a self-conjugate diagram:



One can then obtain a bijection between the set of partitions with distinct odd parts and the set of self-conjugate partitions, as illustrated by the following example:



Odd parts and distinct parts

Among the 22 partitions of the number 8, there are 6 that contain only *odd parts*:

- 7 + 1
- 5 + 3
- 5 + 1 + 1 + 1
- 3 + 3 + 1 + 1
- 3 + 1 + 1 + 1 + 1 + 1
- 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1

Alternatively, we could count partitions in which no number occurs more than once. Such a partition is called a *partition with distinct parts*. If we count the partitions of 8 with distinct parts, we also obtain 6:

- 8
- 7 + 1
- 6 + 2
- 5 + 3
- 5 + 2 + 1
- 4 + 3 + 1

This is a general property. For each positive number, the number of partitions with odd parts equals the number of partitions with distinct parts, denoted by $q(n)$.[8][9] This result was proved by Leonhard Euler in 1748[10] and later was generalized as Glaisher's theorem.

For every type of restricted partition there is a corresponding function for the number of partitions satisfying the given restriction. An important example is $q(n)$. The first few values of $q(n)$ are (starting with $q(0)=1$):

1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 10, ... (sequence A000009 in the OEIS).

The generating function for $q(n)$ (partitions into distinct parts) is given by[11]

$$\sum_{n=0}^{\infty} q(n)x^n = \prod_{k=1}^{\infty}(1+x^k) = \prod_{k=1}^{\infty} \frac{1}{1-x^{2k-1}}.$$

The pentagonal number theorem gives a recurrence for q:[12]

$$q(k) = a_k + q(k-1) + q(k-2) - q(k-5) - q(k-7) + q(k-12) + q(k-15) - q(k-22) - \dots$$

where a_k is $(-1)^m$ if $k = 3m^2 - m$ for some integer m and is 0 otherwise.

Restricted part size or number of parts

By taking conjugates, the number $p_k(n)$ of partitions of n into exactly k parts is equal to the number of partitions of n in which the largest part has size k. The function $p_k(n)$ satisfies the recurrence

$$p_k(n) = p_k(n-k) + p_{k-1}(n-1)$$

with initial values $p_0(0) = 1$ and $p_k(n) = 0$ if $n \leq 0$ or $k \leq 0$ and n and k are not both zero.[13]

One recovers the function $p(n)$ by

$$p(n) = \sum_{k=0}^{n} p_k(n).$$

One possible generating function for such partitions, taking k fixed and n variable, is

$$\sum_{n \geq 0} p_k(n) x^n = x^k \cdot \prod_{i=1}^{k} \frac{1}{1 - x^i}.$$

More generally, if T is a set of positive integers then the number of partitions of n, all of whose parts belong to T, has generating function

$$\prod_{t \in T} (1 - x^t)^{-1}.$$

This can be used to solve change-making problems (where the set T specifies the available coins). As two particular cases, one has that the number of partitions of n in which all parts are 1 or 2 (or, equivalently, the number of partitions of n into 1 or 2 parts) is

$$\left\lfloor \frac{n}{2} + 1 \right\rfloor,$$

and the number of partitions of n in which all parts are 1, 2 or 3 (or, equivalently, the number of partitions of n into at most three parts) is the nearest integer to $(n + 3)^2 / 12$.[14]

Asymptotics

The asymptotic growth rate for $p(n)$ is given by

$$\log p(n) \sim C\sqrt{n} \text{ as } n \to \infty$$

where $C = \pi\sqrt{\frac{2}{3}}$..[15] The more precise asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \text{ as } n \to \infty$$

was first obtained by G. H. Hardy and Ramanujan in 1918 and independently by J. V. Uspensky in 1920. A complete asymptotic expansion was given in 1937 by Hans Rademacher.

If A is a set of natural numbers, we let $p_A(n)$ denote the number of partitions of n into elements of A. If A possesses positive natural density α then

$$\log p_A(n) \sim C\sqrt{\alpha n}$$

and conversely if this asymptotic property holds for $p_A(n)$ then A has natural density α.[16] This result was stated, with a sketch of proof, by Erdős in 1942.[17][18]

If A is a finite set, this analysis does not apply (the density of a finite set is zero). If A has k elements whose greatest common divisor is 1, then[19]

$$p_A(n) = \left(\prod_{a\in A} a^{-1}\right) \cdot \frac{n^{k-1}}{(k-1)!} + O(n^{k-2}).$$

Partitions in a rectangle and Gaussian binomial coefficients

One may also simultaneously limit the number and size of the parts. Let $p(N, M; n)$ denote the number of partitions of n with at most M parts, each of size at most N. Equivalently, these are the partitions whose Young diagram fits inside an $M \times N$ rectangle. There is a recurrence relation

$$p(N, M; n) = p(N, M-1; n) + p(N-1, M; n-M)$$

obtained by observing that $p(N, M; n) - p(N, M-1; n)$ counts the partitions of n into exactly M parts of size at most N, and subtracting 1 from each part of such a partition yields a partition of $n - M$ into at most M parts.[20]

The Gaussian binomial coefficient is defined as:

$$\binom{k+\ell}{\ell}_q = \binom{k+\ell}{k}_q = \frac{\prod_{j=1}^{k+\ell}(1-q^j)}{\prod_{j=1}^{k}(1-q^j)\prod_{j=1}^{\ell}(1-q^j)}.$$

The Gaussian binomial coefficient is related to the generating function of $p(N, M; n)$ by the equality

$$\sum_{n=0}^{MN} p(N, M; n)q^n = \binom{M+N}{M}_q.$$

Rank and Durfee square

The *rank* of a partition is the largest number k such that the partition contains at least k parts of size at least k. For example, the partition 4 + 3 + 3 + 2 + 1 + 1 has rank 3 because it contains 3 parts that are ≥ 3, but does not contain 4 parts that are ≥ 4. In the Ferrers diagram or Young diagram of a partition of rank r, the $r \times r$ square of entries in the upper-left is known as the Durfee square: