that is, the set of vertices adjacent to some member of X. Prove ***Hall's theorem***: there exists a perfect matching in G if and only if $|A| \leq |N(A)|$ for every subset $A \subseteq L$.

26.3-5 ⋆

We say that a bipartite graph $G = (V, E)$, where $V = L \cup R$, is ***d-regular*** if every vertex $v \in V$ has degree exactly d. Every d-regular bipartite graph has $|L| = |R|$. Prove that every d-regular bipartite graph has a matching of cardinality $|L|$ by arguing that a minimum cut of the corresponding flow network has capacity $|L|$.

Push-relabel algorithms

In this section, we present the "push-relabel" approach to computing maximum flows. To date, many of the asymptotically fastest maximum-flow algorithms are push-relabel algorithms, and the fastest actual implementations of maximum-flow algorithms are based on the push-relabel method. Push-relabel methods also efficiently solve other flow problems, such as the minimum-cost flow problem. This section introduces Goldberg's "generic" maximum-flow algorithm, which has a simple implementation that runs in $O(V^2E)$ time, thereby improving upon the $O(VE^2)$ bound of the Edmonds-Karp algorithm. Section 26.5 refines the generic algorithm to obtain another push-relabel algorithm that runs in $O(V^3)$ time.

Push-relabel algorithms work in a more localized manner than the Ford-Fulkerson method. Rather than examine the entire residual network to find an augmenting path, push-relabel algorithms work on one vertex at a time, looking only at the vertex's neighbors in the residual network. Furthermore, unlike the Ford-Fulkerson method, push-relabel algorithms do not maintain the flow-conservation property throughout their execution. They do, however, maintain a ***preflow***, which is a function $f : V \times V \to \mathbb{R}$ that satisfies the capacity constraint and the following relaxation of flow conservation:

$$\sum_{v \in V} f(v, u) - \sum_{v \in V} f(u, v) \geq 0$$

for all vertices $u \in V - \{s\}$. That is, the flow into a vertex may exceed the flow out. We call the quantity

$$e(u) = \sum_{v \in V} f(v, u) - \sum_{v \in V} f(u, v) \tag{26.14}$$

the ***excess flow*** into vertex u. The excess at a vertex is the amount by which the flow in exceeds the flow out. We say that a vertex $u \in V - \{s, t\}$ is ***overflowing*** if $e(u) > 0$.

We shall begin this section by describing the intuition behind the push-relabel method. We shall then investigate the two operations employed by the method: "pushing" preflow and "relabeling" a vertex. Finally, we shall present a generic push-relabel algorithm and analyze its correctness and running time.

Intuition

You can understand the intuition behind the push-relabel method in terms of fluid flows: we consider a flow network $G = (V, E)$ to be a system of interconnected pipes of given capacities. Applying this analogy to the Ford-Fulkerson method, we might say that each augmenting path in the network gives rise to an additional stream of fluid, with no branch points, flowing from the source to the sink. The Ford-Fulkerson method iteratively adds more streams of flow until no more can be added.

The generic push-relabel algorithm has a rather different intuition. As before, directed edges correspond to pipes. Vertices, which are pipe junctions, have two interesting properties. First, to accommodate excess flow, each vertex has an outflow pipe leading to an arbitrarily large reservoir that can accumulate fluid. Second, each vertex, its reservoir, and all its pipe connections sit on a platform whose height increases as the algorithm progresses.

Vertex heights determine how flow is pushed: we push flow only downhill, that is, from a higher vertex to a lower vertex. The flow from a lower vertex to a higher vertex may be positive, but operations that push flow push it only downhill. We fix the height of the source at $|V|$ and the height of the sink at 0. All other vertex heights start at 0 and increase with time. The algorithm first sends as much flow as possible downhill from the source toward the sink. The amount it sends is exactly enough to fill each outgoing pipe from the source to capacity; that is, it sends the capacity of the cut $(s, V - \{s\})$. When flow first enters an intermediate vertex, it collects in the vertex's reservoir. From there, we eventually push it downhill.

We may eventually find that the only pipes that leave a vertex u and are not already saturated with flow connect to vertices that are on the same level as u or are uphill from u. In this case, to rid an overflowing vertex u of its excess flow, we must increase its height—an operation called "relabeling" vertex u. We increase its height to one unit more than the height of the lowest of its neighbors to which it has an unsaturated pipe. After a vertex is relabeled, therefore, it has at least one outgoing pipe through which we can push more flow.

Eventually, all the flow that can possibly get through to the sink has arrived there. No more can arrive, because the pipes obey the capacity constraints; the amount of flow across any cut is still limited by the capacity of the cut. To make the preflow a "legal" flow, the algorithm then sends the excess collected in the reservoirs of overflowing vertices back to the source by continuing to relabel vertices to above

the fixed height $|V|$ of the source. As we shall see, once we have emptied all the reservoirs, the preflow is not only a "legal" flow, it is also a maximum flow.

The basic operations

From the preceding discussion, we see that a push-relabel algorithm performs two basic operations: pushing flow excess from a vertex to one of its neighbors and relabeling a vertex. The situations in which these operations apply depend on the heights of vertices, which we now define precisely.

Let $G = (V, E)$ be a flow network with source s and sink t, and let f be a preflow in G. A function $h : V \to \mathbb{N}$ is a ***height function***[3] if $h(s) = |V|$, $h(t) = 0$, and

$$h(u) \leq h(v) + 1$$

for every residual edge $(u, v) \in E_f$. We immediately obtain the following lemma.

Lemma 26.12
Let $G = (V, E)$ be a flow network, let f be a preflow in G, and let h be a height function on V. For any two vertices $u, v \in V$, if $h(u) > h(v) + 1$, then (u, v) is not an edge in the residual network. ■

The push operation
The basic operation PUSH(u, v) applies if u is an overflowing vertex, $c_f(u, v) > 0$, and $h(u) = h(v) + 1$. The pseudocode below updates the preflow f and the excess flows for u and v. It assumes that we can compute residual capacity $c_f(u, v)$ in constant time given c and f. We maintain the excess flow stored at a vertex u as the attribute $u.e$ and the height of u as the attribute $u.h$. The expression $\Delta_f(u, v)$ is a temporary variable that stores the amount of flow that we can push from u to v.

---

[3] In the literature, a height function is typically called a "distance function," and the height of a vertex is called a "distance label." We use the term "height" because it is more suggestive of the intuition behind the algorithm. We retain the use of the term "relabel" to refer to the operation that increases the height of a vertex. The height of a vertex is related to its distance from the sink t, as would be found in a breadth-first search of the transpose G^{T}.

PUSH(u, v)

1 **// Applies when**: u is overflowing, $c_f(u, v) > 0$, and $u.h = v.h + 1$.
2 **// Action:** Push $\Delta_f(u, v) = \min(u.e, c_f(u, v))$ units of flow from u to v.
3 $\Delta_f(u, v) = \min(u.e, c_f(u, v))$
4 **if** $(u, v) \in E$
5 $\quad (u, v).f = (u, v).f + \Delta_f(u, v)$
6 **else** $(v, u).f = (v, u).f - \Delta_f(u, v)$
7 $u.e = u.e - \Delta_f(u, v)$
8 $v.e = v.e + \Delta_f(u, v)$

The code for PUSH operates as follows. Because vertex u has a positive excess $u.e$ and the residual capacity of (u, v) is positive, we can increase the flow from u to v by $\Delta_f(u, v) = \min(u.e, c_f(u, v))$ without causing $u.e$ to become negative or the capacity $c(u, v)$ to be exceeded. Line 3 computes the value $\Delta_f(u, v)$, and lines 4–6 update f. Line 5 increases the flow on edge (u, v), because we are pushing flow over a residual edge that is also an original edge. Line 6 decreases the flow on edge (v, u), because the residual edge is actually the reverse of an edge in the original network. Finally, lines 7–8 update the excess flows into vertices u and v. Thus, if f is a preflow before PUSH is called, it remains a preflow afterward.

Observe that nothing in the code for PUSH depends on the heights of u and v, yet we prohibit it from being invoked unless $u.h = v.h + 1$. Thus, we push excess flow downhill only by a height differential of 1. By Lemma 26.12, no residual edges exist between two vertices whose heights differ by more than 1, and thus, as long as the attribute h is indeed a height function, we would gain nothing by allowing flow to be pushed downhill by a height differential of more than 1.

We call the operation PUSH(u, v) a ***push*** from u to v. If a push operation applies to some edge (u, v) leaving a vertex u, we also say that the push operation applies to u. It is a ***saturating push*** if edge (u, v) in the residual network becomes ***saturated*** ($c_f(u, v) = 0$ afterward); otherwise, it is a ***nonsaturating push***. If an edge becomes saturated, it disappears from the residual network. A simple lemma characterizes one result of a nonsaturating push.

Lemma 26.13
After a nonsaturating push from u to v, the vertex u is no longer overflowing.

Proof Since the push was nonsaturating, the amount of flow $\Delta_f(u, v)$ actually pushed must equal $u.e$ prior to the push. Since $u.e$ is reduced by this amount, it becomes 0 after the push. ■

The relabel operation

The basic operation RELABEL(u) applies if u is overflowing and if $u.h \leq v.h$ for all edges $(u, v) \in E_f$. In other words, we can relabel an overflowing vertex u if for every vertex v for which there is residual capacity from u to v, flow cannot be pushed from u to v because v is not downhill from u. (Recall that by definition, neither the source s nor the sink t can be overflowing, and so s and t are ineligible for relabeling.)

RELABEL(u)

1 // **Applies when:** u is overflowing and for all $v \in V$ such that $(u, v) \in E_f$, we have $u.h \leq v.h$.
2 // **Action:** Increase the height of u.
3 $u.h = 1 + \min\{v.h : (u, v) \in E_f\}$

When we call the operation RELABEL(u), we say that vertex u is ***relabeled***. Note that when u is relabeled, E_f must contain at least one edge that leaves u, so that the minimization in the code is over a nonempty set. This property follows from the assumption that u is overflowing, which in turn tells us that

$$u.e = \sum_{v \in V} f(v, u) - \sum_{v \in V} f(u, v) > 0 .$$

Since all flows are nonnegative, we must therefore have at least one vertex v such that $(v, u).f > 0$. But then, $c_f(u, v) > 0$, which implies that $(u, v) \in E_f$. The operation RELABEL(u) thus gives u the greatest height allowed by the constraints on height functions.

The generic algorithm

The generic push-relabel algorithm uses the following subroutine to create an initial preflow in the flow network.

INITIALIZE-PREFLOW(G, s)

1 **for** each vertex $v \in G.V$
2 $\quad v.h = 0$
3 $\quad v.e = 0$
4 **for** each edge $(u, v) \in G.E$
5 $\quad (u, v).f = 0$
6 $s.h = |G.V|$
7 **for** each vertex $v \in s.Adj$
8 $\quad (s, v).f = c(s, v)$
9 $\quad v.e = c(s, v)$
10 $\quad s.e = s.e - c(s, v)$

INITIALIZE-PREFLOW creates an initial preflow f defined by

$$(u, v).f = \begin{cases} c(u, v) & \text{if } u = s \text{ ,} \\ 0 & \text{otherwise .} \end{cases} \tag{26.15}$$

That is, we fill to capacity each edge leaving the source s, and all other edges carry no flow. For each vertex v adjacent to the source, we initially have $v.e = c(s, v)$, and we initialize $s.e$ to the negative of the sum of these capacities. The generic algorithm also begins with an initial height function h, given by

$$u.h = \begin{cases} |V| & \text{if } u = s \text{ ,} \\ 0 & \text{otherwise .} \end{cases} \tag{26.16}$$

Equation (26.16) defines a height function because the only edges (u, v) for which $u.h > v.h + 1$ are those for which $u = s$, and those edges are saturated, which means that they are not in the residual network.

Initialization, followed by a sequence of push and relabel operations, executed in no particular order, yields the GENERIC-PUSH-RELABEL algorithm:

GENERIC-PUSH-RELABEL(G)

1 INITIALIZE-PREFLOW(G, s)
2 **while** there exists an applicable push or relabel operation
3 select an applicable push or relabel operation and perform it

The following lemma tells us that as long as an overflowing vertex exists, at least one of the two basic operations applies.

Lemma 26.14 (An overflowing vertex can be either pushed or relabeled)
Let $G = (V, E)$ be a flow network with source s and sink t, let f be a preflow, and let h be any height function for f. If u is any overflowing vertex, then either a push or relabel operation applies to it.

Proof For any residual edge (u, v), we have $h(u) \leq h(v) + 1$ because h is a height function. If a push operation does not apply to an overflowing vertex u, then for all residual edges (u, v), we must have $h(u) < h(v) + 1$, which implies $h(u) \leq h(v)$. Thus, a relabel operation applies to u. ■

Correctness of the push-relabel method

To show that the generic push-relabel algorithm solves the maximum-flow problem, we shall first prove that if it terminates, the preflow f is a maximum flow. We shall later prove that it terminates. We start with some observations about the height function h.

Lemma 26.15 (Vertex heights never decrease)
During the execution of the GENERIC-PUSH-RELABEL procedure on a flow network $G = (V, E)$, for each vertex $u \in V$, the height $u.h$ never decreases. Moreover, whenever a relabel operation is applied to a vertex u, its height $u.h$ increases by at least 1.

Proof Because vertex heights change only during relabel operations, it suffices to prove the second statement of the lemma. If vertex u is about to be relabeled, then for all vertices v such that $(u, v) \in E_f$, we have $u.h \leq v.h$. Thus, $u.h < 1 + \min\{v.h : (u, v) \in E_f\}$, and so the operation must increase $u.h$. ■

Lemma 26.16
Let $G = (V, E)$ be a flow network with source s and sink t. Then the execution of GENERIC-PUSH-RELABEL on G maintains the attribute h as a height function.

Proof The proof is by induction on the number of basic operations performed. Initially, h is a height function, as we have already observed.

We claim that if h is a height function, then an operation RELABEL(u) leaves h a height function. If we look at a residual edge $(u, v) \in E_f$ that leaves u, then the operation RELABEL(u) ensures that $u.h \leq v.h + 1$ afterward. Now consider a residual edge (w, u) that enters u. By Lemma 26.15, $w.h \leq u.h + 1$ before the operation RELABEL(u) implies $w.h < u.h + 1$ afterward. Thus, the operation RELABEL(u) leaves h a height function.

Now, consider an operation PUSH(u, v). This operation may add the edge (v, u) to E_f, and it may remove (u, v) from E_f. In the former case, we have $v.h = u.h - 1 < u.h + 1$, and so h remains a height function. In the latter case, removing (u, v) from the residual network removes the corresponding constraint, and h again remains a height function. ■

The following lemma gives an important property of height functions.

Lemma 26.17
Let $G = (V, E)$ be a flow network with source s and sink t, let f be a preflow in G, and let h be a height function on V. Then there is no path from the source s to the sink t in the residual network G_f.

Proof Assume for the sake of contradiction that G_f contains a path p from s to t, where $p = \langle v_0, v_1, \ldots, v_k \rangle$, $v_0 = s$, and $v_k = t$. Without loss of generality, p is a simple path, and so $k < |V|$. For $i = 0, 1, \ldots, k - 1$, edge $(v_i, v_{i+1}) \in E_f$. Because h is a height function, $h(v_i) \leq h(v_{i+1}) + 1$ for $i = 0, 1, \ldots, k - 1$. Combining these inequalities over path p yields $h(s) \leq h(t) + k$. But because $h(t) = 0$,

we have $h(s) \leq k < |V|$, which contradicts the requirement that $h(s) = |V|$ in a height function. ■

We are now ready to show that if the generic push-relabel algorithm terminates, the preflow it computes is a maximum flow.

Theorem 26.18 (Correctness of the generic push-relabel algorithm)
If the algorithm GENERIC-PUSH-RELABEL terminates when run on a flow network $G = (V, E)$ with source s and sink t, then the preflow f it computes is a maximum flow for G.

Proof We use the following loop invariant:

> Each time the **while** loop test in line 2 in GENERIC-PUSH-RELABEL is executed, f is a preflow.

Initialization: INITIALIZE-PREFLOW makes f a preflow.

Maintenance: The only operations within the **while** loop of lines 2–3 are push and relabel. Relabel operations affect only height attributes and not the flow values; hence they do not affect whether f is a preflow. As argued on page 739, if f is a preflow prior to a push operation, it remains a preflow afterward.

Termination: At termination, each vertex in $V - \{s, t\}$ must have an excess of 0, because by Lemma 26.14 and the invariant that f is always a preflow, there are no overflowing vertices. Therefore, f is a flow. Lemma 26.16 shows that h is a height function at termination, and thus Lemma 26.17 tells us that there is no path from s to t in the residual network G_f. By the max-flow min-cut theorem (Theorem 26.6), therefore, f is a maximum flow. ■

Analysis of the push-relabel method

To show that the generic push-relabel algorithm indeed terminates, we shall bound the number of operations it performs. We bound separately each of the three types of operations: relabels, saturating pushes, and nonsaturating pushes. With knowledge of these bounds, it is a straightforward problem to construct an algorithm that runs in $O(V^2E)$ time. Before beginning the analysis, however, we prove an important lemma. Recall that we allow edges into the source in the residual network.

Lemma 26.19
Let $G = (V, E)$ be a flow network with source s and sink t, and let f be a preflow in G. Then, for any overflowing vertex x, there is a simple path from x to s in the residual network G_f.

Proof For an overflowing vertex x, let $U = \{v : \text{there exists a simple path from } x \text{ to } v \text{ in } G_f\}$, and suppose for the sake of contradiction that $s \notin U$. Let $\overline{U} = V - U$.

We take the definition of excess from equation (26.14), sum over all vertices in U, and note that $V = U \cup \overline{U}$, to obtain

$$\begin{aligned}
&\sum_{u \in U} e(u) \\
&\quad = \sum_{u \in U} \left(\sum_{v \in V} f(v,u) - \sum_{v \in V} f(u,v) \right) \\
&\quad = \sum_{u \in U} \left(\left(\sum_{v \in U} f(v,u) + \sum_{v \in \overline{U}} f(v,u) \right) - \left(\sum_{v \in U} f(u,v) + \sum_{v \in \overline{U}} f(u,v) \right) \right) \\
&\quad = \sum_{u \in U} \sum_{v \in U} f(v,u) + \sum_{u \in U} \sum_{v \in \overline{U}} f(v,u) - \sum_{u \in U} \sum_{v \in U} f(u,v) - \sum_{u \in U} \sum_{v \in \overline{U}} f(u,v) \\
&\quad = \sum_{u \in U} \sum_{v \in \overline{U}} f(v,u) - \sum_{u \in U} \sum_{v \in \overline{U}} f(u,v) \, .
\end{aligned}$$

We know that the quantity $\sum_{u \in U} e(u)$ must be positive because $e(x) > 0$, $x \in U$, all vertices other than s have nonnegative excess, and, by assumption, $s \notin U$. Thus, we have

$$\sum_{u \in U} \sum_{v \in \overline{U}} f(v,u) - \sum_{u \in U} \sum_{v \in \overline{U}} f(u,v) > 0 \, . \tag{26.17}$$

All edge flows are nonnegative, and so for equation (26.17) to hold, we must have $\sum_{u \in U} \sum_{v \in \overline{U}} f(v,u) > 0$. Hence, there must exist at least one pair of vertices $u' \in U$ and $v' \in \overline{U}$ with $f(v',u') > 0$. But, if $f(v',u') > 0$, there must be a residual edge (u',v'), which means that there is a simple path from x to v' (the path $x \rightsquigarrow u' \rightarrow v'$), thus contradicting the definition of U. ■

The next lemma bounds the heights of vertices, and its corollary bounds the number of relabel operations that are performed in total.

Lemma 26.20

Let $G = (V,E)$ be a flow network with source s and sink t. At any time during the execution of GENERIC-PUSH-RELABEL on G, we have $u.h \leq 2|V| - 1$ for all vertices $u \in V$.

Proof The heights of the source s and the sink t never change because these vertices are by definition not overflowing. Thus, we always have $s.h = |V|$ and $t.h = 0$, both of which are no greater than $2|V| - 1$.

Now consider any vertex $u \in V - \{s,t\}$. Initially, $u.h = 0 \leq 2|V| - 1$. We shall show that after each relabeling operation, we still have $u.h \leq 2|V| - 1$. When u is

relabeled, it is overflowing, and Lemma 26.19 tells us that there is a simple path p from u to s in G_f. Let $p = \langle v_0, v_1, \ldots, v_k \rangle$, where $v_0 = u$, $v_k = s$, and $k \leq |V|-1$ because p is simple. For $i = 0, 1, \ldots, k-1$, we have $(v_i, v_{i+1}) \in E_f$, and therefore, by Lemma 26.16, $v_i.h \leq v_{i+1}.h + 1$. Expanding these inequalities over path p yields $u.h = v_0.h \leq v_k.h + k \leq s.h + (|V| - 1) = 2|V| - 1$. ■

Corollary 26.21 (Bound on relabel operations)
Let $G = (V, E)$ be a flow network with source s and sink t. Then, during the execution of GENERIC-PUSH-RELABEL on G, the number of relabel operations is at most $2|V| - 1$ per vertex and at most $(2|V| - 1)(|V| - 2) < 2|V|^2$ overall.

Proof Only the $|V|-2$ vertices in $V - \{s, t\}$ may be relabeled. Let $u \in V - \{s, t\}$. The operation RELABEL(u) increases $u.h$. The value of $u.h$ is initially 0 and by Lemma 26.20, it grows to at most $2|V| - 1$. Thus, each vertex $u \in V - \{s, t\}$ is relabeled at most $2|V| - 1$ times, and the total number of relabel operations performed is at most $(2|V| - 1)(|V| - 2) < 2|V|^2$. ■

Lemma 26.20 also helps us to bound the number of saturating pushes.

Lemma 26.22 (Bound on saturating pushes)
During the execution of GENERIC-PUSH-RELABEL on any flow network $G = (V, E)$, the number of saturating pushes is less than $2|V||E|$.

Proof For any pair of vertices $u, v \in V$, we will count the saturating pushes from u to v and from v to u together, calling them the saturating pushes between u and v. If there are any such pushes, at least one of (u, v) and (v, u) is actually an edge in E. Now, suppose that a saturating push from u to v has occurred. At that time, $v.h = u.h - 1$. In order for another push from u to v to occur later, the algorithm must first push flow from v to u, which cannot happen until $v.h = u.h + 1$. Since $u.h$ never decreases, in order for $v.h = u.h + 1$, the value of $v.h$ must increase by at least 2. Likewise, $u.h$ must increase by at least 2 between saturating pushes from v to u. Heights start at 0 and, by Lemma 26.20, never exceed $2|V|-1$, which implies that the number of times any vertex can have its height increase by 2 is less than $|V|$. Since at least one of $u.h$ and $v.h$ must increase by 2 between any two saturating pushes between u and v, there are fewer than $2|V|$ saturating pushes between u and v. Multiplying by the number of edges gives a bound of less than $2|V||E|$ on the total number of saturating pushes. ■

The following lemma bounds the number of nonsaturating pushes in the generic push-relabel algorithm.

Lemma 26.23 (Bound on nonsaturating pushes)
During the execution of GENERIC-PUSH-RELABEL on any flow network $G = (V, E)$, the number of nonsaturating pushes is less than $4|V|^2(|V| + |E|)$.

Proof Define a potential function $\Phi = \sum_{v:e(v)>0} v.h$. Initially, $\Phi = 0$, and the value of Φ may change after each relabeling, saturating push, and nonsaturating push. We will bound the amount that saturating pushes and relabelings can contribute to the increase of Φ. Then we will show that each nonsaturating push must decrease Φ by at least 1, and will use these bounds to derive an upper bound on the number of nonsaturating pushes.

Let us examine the two ways in which Φ might increase. First, relabeling a vertex u increases Φ by less than $2|V|$, since the set over which the sum is taken is the same and the relabeling cannot increase u's height by more than its maximum possible height, which, by Lemma 26.20, is at most $2|V| - 1$. Second, a saturating push from a vertex u to a vertex v increases Φ by less than $2|V|$, since no heights change and only vertex v, whose height is at most $2|V| - 1$, can possibly become overflowing.

Now we show that a nonsaturating push from u to v decreases Φ by at least 1. Why? Before the nonsaturating push, u was overflowing, and v may or may not have been overflowing. By Lemma 26.13, u is no longer overflowing after the push. In addition, unless v is the source, it may or may not be overflowing after the push. Therefore, the potential function Φ has decreased by exactly $u.h$, and it has increased by either 0 or $v.h$. Since $u.h - v.h = 1$, the net effect is that the potential function has decreased by at least 1.

Thus, during the course of the algorithm, the total amount of increase in Φ is due to relabelings and saturated pushes, and Corollary 26.21 and Lemma 26.22 constrain the increase to be less than $(2|V|)(2|V|^2) + (2|V|)(2|V||E|) = 4|V|^2(|V| + |E|)$. Since $\Phi \geq 0$, the total amount of decrease, and therefore the total number of nonsaturating pushes, is less than $4|V|^2(|V| + |E|)$. ∎

Having bounded the number of relabelings, saturating pushes, and nonsaturating push, we have set the stage for the following analysis of the GENERIC-PUSH-RELABEL procedure, and hence of any algorithm based on the push-relabel method.

Theorem 26.24
During the execution of GENERIC-PUSH-RELABEL on any flow network $G = (V, E)$, the number of basic operations is $O(V^2E)$.

Proof Immediate from Corollary 26.21 and Lemmas 26.22 and 26.23. ∎

Thus, the algorithm terminates after $O(V^2E)$ operations. All that remains is to give an efficient method for implementing each operation and for choosing an appropriate operation to execute.

Corollary 26.25
There is an implementation of the generic push-relabel algorithm that runs in $O(V^2E)$ time on any flow network $G = (V, E)$.

Proof Exercise 26.4-2 asks you to show how to implement the generic algorithm with an overhead of $O(V)$ per relabel operation and $O(1)$ per push. It also asks you to design a data structure that allows you to pick an applicable operation in $O(1)$ time. The corollary then follows. ■

Exercises

26.4-1
Prove that, after the procedure INITIALIZE-PREFLOW(G, s) terminates, we have $s.e \leq -|f^*|$, where f^* is a maximum flow for G.

26.4-2
Show how to implement the generic push-relabel algorithm using $O(V)$ time per relabel operation, $O(1)$ time per push, and $O(1)$ time to select an applicable operation, for a total time of $O(V^2E)$.

26.4-3
Prove that the generic push-relabel algorithm spends a total of only $O(VE)$ time in performing all the $O(V^2)$ relabel operations.

26.4-4
Suppose that we have found a maximum flow in a flow network $G = (V, E)$ using a push-relabel algorithm. Give a fast algorithm to find a minimum cut in G.

26.4-5
Give an efficient push-relabel algorithm to find a maximum matching in a bipartite graph. Analyze your algorithm.

26.4-6
Suppose that all edge capacities in a flow network $G = (V, E)$ are in the set $\{1, 2, \ldots, k\}$. Analyze the running time of the generic push-relabel algorithm in terms of $|V|$, $|E|$, and k. (*Hint:* How many times can each edge support a nonsaturating push before it becomes saturated?)

26.4-7
Show that we could change line 6 of INITIALIZE-PREFLOW to

6 $s.h = |G.V| - 2$

without affecting the correctness or asymptotic performance of the generic push-relabel algorithm.

26.4-8
Let $\delta_f(u, v)$ be the distance (number of edges) from u to v in the residual network G_f. Show that the GENERIC-PUSH-RELABEL procedure maintains the properties that $u.h < |V|$ implies $u.h \leq \delta_f(u, t)$ and that $u.h \geq |V|$ implies $u.h - |V| \leq \delta_f(u, s)$.

26.4-9 ⋆
As in the previous exercise, let $\delta_f(u, v)$ be the distance from u to v in the residual network G_f. Show how to modify the generic push-relabel algorithm to maintain the property that $u.h < |V|$ implies $u.h = \delta_f(u, t)$ and that $u.h \geq |V|$ implies $u.h - |V| = \delta_f(u, s)$. The total time that your implementation dedicates to maintaining this property should be $O(VE)$.

26.4-10
Show that the number of nonsaturating pushes executed by the GENERIC-PUSH-RELABEL procedure on a flow network $G = (V, E)$ is at most $4|V|^2|E|$ for $|V| \geq 4$.

The relabel-to-front algorithm

The push-relabel method allows us to apply the basic operations in any order at all. By choosing the order carefully and managing the network data structure efficiently, however, we can solve the maximum-flow problem faster than the $O(V^2E)$ bound given by Corollary 26.25. We shall now examine the relabel-to-front algorithm, a push-relabel algorithm whose running time is $O(V^3)$, which is asymptotically at least as good as $O(V^2E)$, and even better for dense networks.

The relabel-to-front algorithm maintains a list of the vertices in the network. Beginning at the front, the algorithm scans the list, repeatedly selecting an overflowing vertex u and then "discharging" it, that is, performing push and relabel operations until u no longer has a positive excess. Whenever we relabel a vertex, we move it to the front of the list (hence the name "relabel-to-front") and the algorithm begins its scan anew.

The correctness and analysis of the relabel-to-front algorithm depend on the notion of "admissible" edges: those edges in the residual network through which flow can be pushed. After proving some properties about the network of admissible edges, we shall investigate the discharge operation and then present and analyze the relabel-to-front algorithm itself.

Admissible edges and networks

If $G = (V, E)$ is a flow network with source s and sink t, f is a preflow in G, and h is a height function, then we say that (u, v) is an ***admissible edge*** if $c_f(u, v) > 0$ and $h(u) = h(v) + 1$. Otherwise, (u, v) is ***inadmissible***. The ***admissible network*** is $G_{f,h} = (V, E_{f,h})$, where $E_{f,h}$ is the set of admissible edges.

The admissible network consists of those edges through which we can push flow. The following lemma shows that this network is a directed acyclic graph (dag).

Lemma 26.26 (The admissible network is acyclic)
If $G = (V, E)$ is a flow network, f is a preflow in G, and h is a height function on G, then the admissible network $G_{f,h} = (V, E_{f,h})$ is acyclic.

Proof The proof is by contradiction. Suppose that $G_{f,h}$ contains a cycle $p = \langle v_0, v_1, \ldots, v_k \rangle$, where $v_0 = v_k$ and $k > 0$. Since each edge in p is admissible, we have $h(v_{i-1}) = h(v_i) + 1$ for $i = 1, 2, \ldots, k$. Summing around the cycle gives

$$\begin{aligned} \sum_{i=1}^{k} h(v_{i-1}) &= \sum_{i=1}^{k} (h(v_i) + 1) \\ &= \sum_{i=1}^{k} h(v_i) + k \ . \end{aligned}$$

Because each vertex in cycle p appears once in each of the summations, we derive the contradiction that $0 = k$. ■

The next two lemmas show how push and relabel operations change the admissible network.

Lemma 26.27
Let $G = (V, E)$ be a flow network, let f be a preflow in G, and suppose that the attribute h is a height function. If a vertex u is overflowing and (u, v) is an admissible edge, then PUSH(u, v) applies. The operation does not create any new admissible edges, but it may cause (u, v) to become inadmissible.

Proof By the definition of an admissible edge, we can push flow from u to v. Since u is overflowing, the operation PUSH(u, v) applies. The only new residual edge that pushing flow from u to v can create is (v, u). Since $v.h = u.h - 1$, edge (v, u) cannot become admissible. If the operation is a saturating push, then $c_f(u, v) = 0$ afterward and (u, v) becomes inadmissible. ■

Lemma 26.28
Let $G = (V, E)$ be a flow network, let f be a preflow in G, and suppose that the attribute h is a height function. If a vertex u is overflowing and there are no admissible edges leaving u, then RELABEL(u) applies. After the relabel operation, there is at least one admissible edge leaving u, but there are no admissible edges entering u.

Proof If u is overflowing, then by Lemma 26.14, either a push or a relabel operation applies to it. If there are no admissible edges leaving u, then no flow can be pushed from u and so RELABEL(u) applies. After the relabel operation, $u.h = 1 + \min\{v.h : (u, v) \in E_f\}$. Thus, if v is a vertex that realizes the minimum in this set, the edge (u, v) becomes admissible. Hence, after the relabel, there is at least one admissible edge leaving u.

To show that no admissible edges enter u after a relabel operation, suppose that there is a vertex v such that (v, u) is admissible. Then, $v.h = u.h + 1$ after the relabel, and so $v.h > u.h + 1$ just before the relabel. But by Lemma 26.12, no residual edges exist between vertices whose heights differ by more than 1. Moreover, relabeling a vertex does not change the residual network. Thus, (v, u) is not in the residual network, and hence it cannot be in the admissible network. ■

Neighbor lists

Edges in the relabel-to-front algorithm are organized into "neighbor lists." Given a flow network $G = (V, E)$, the ***neighbor list*** $u.N$ for a vertex $u \in V$ is a singly linked list of the neighbors of u in G. Thus, vertex v appears in the list $u.N$ if $(u, v) \in E$ or $(v, u) \in E$. The neighbor list $u.N$ contains exactly those vertices v for which there may be a residual edge (u, v). The attribute $u.N.head$ points to the first vertex in $u.N$, and $v.next\text{-}neighbor$ points to the vertex following v in a neighbor list; this pointer is NIL if v is the last vertex in the neighbor list.

The relabel-to-front algorithm cycles through each neighbor list in an arbitrary order that is fixed throughout the execution of the algorithm. For each vertex u, the attribute $u.current$ points to the vertex currently under consideration in $u.N$. Initially, $u.current$ is set to $u.N.head$.

Discharging an overflowing vertex

An overflowing vertex u is ***discharged*** by pushing all of its excess flow through admissible edges to neighboring vertices, relabeling u as necessary to cause edges leaving u to become admissible. The pseudocode goes as follows.

```
DISCHARGE(u)
1  while u.e > 0
2      v = u.current
3      if v == NIL
4          RELABEL(u)
5          u.current = u.N.head
6      elseif c_f(u, v) > 0 and u.h == v.h + 1
7          PUSH(u, v)
8      else u.current = v.next-neighbor
```

Figure 26.9 steps through several iterations of the **while** loop of lines 1–8, which executes as long as vertex u has positive excess. Each iteration performs exactly one of three actions, depending on the current vertex v in the neighbor list $u.N$.

1. If v is NIL, then we have run off the end of $u.N$. Line 4 relabels vertex u, and then line 5 resets the current neighbor of u to be the first one in $u.N$. (Lemma 26.29 below states that the relabel operation applies in this situation.)
2. If v is non-NIL and (u, v) is an admissible edge (determined by the test in line 6), then line 7 pushes some (or possibly all) of u's excess to vertex v.
3. If v is non-NIL but (u, v) is inadmissible, then line 8 advances $u.current$ one position further in the neighbor list $u.N$.

Observe that if DISCHARGE is called on an overflowing vertex u, then the last action performed by DISCHARGE must be a push from u. Why? The procedure terminates only when $u.e$ becomes zero, and neither the relabel operation nor advancing the pointer $u.current$ affects the value of $u.e$.

We must be sure that when PUSH or RELABEL is called by DISCHARGE, the operation applies. The next lemma proves this fact.

Lemma 26.29

If DISCHARGE calls PUSH(u, v) in line 7, then a push operation applies to (u, v). If DISCHARGE calls RELABEL(u) in line 4, then a relabel operation applies to u.

Proof The tests in lines 1 and 6 ensure that a push operation occurs only if the operation applies, which proves the first statement in the lemma.

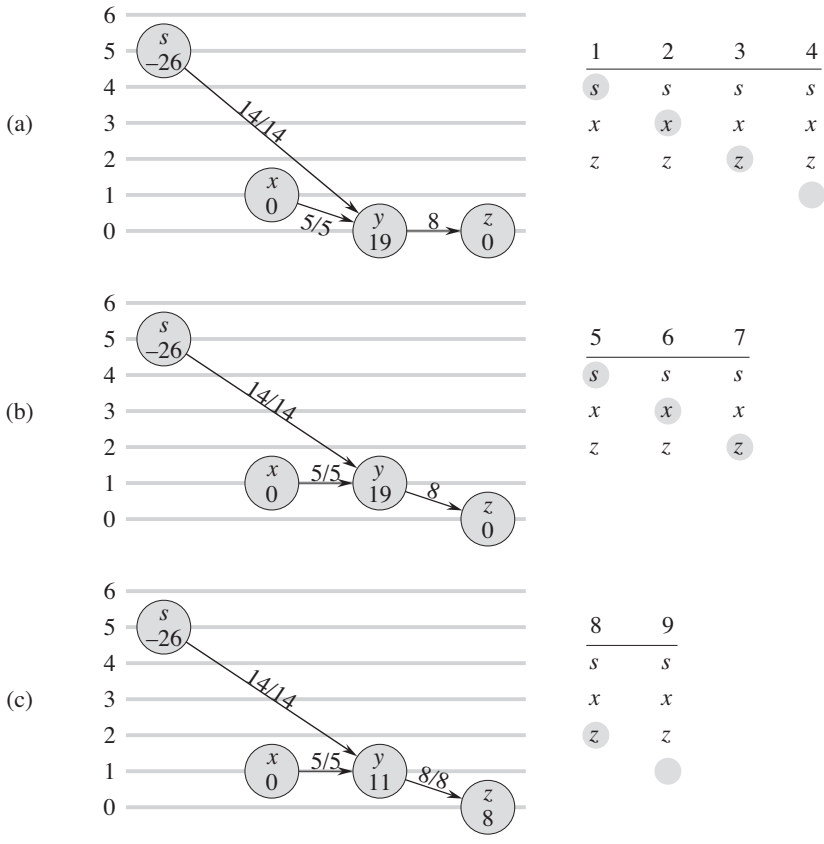
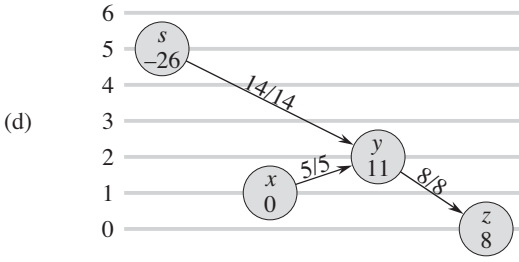
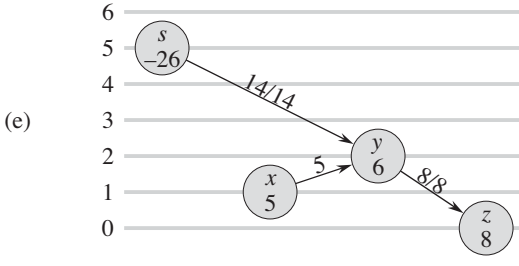


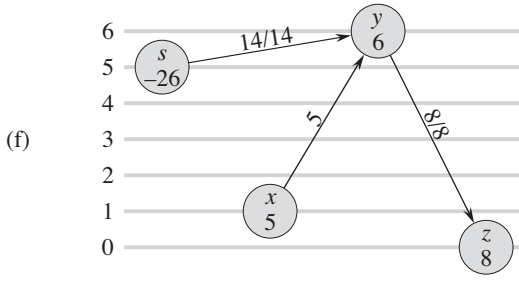
Figure 26.9 Discharging a vertex y. It takes 15 iterations of the **while** loop of DISCHARGE to push all the excess flow from y. Only the neighbors of y and edges of the flow network that enter or leave y are shown. In each part of the figure, the number inside each vertex is its excess at the beginning of the first iteration shown in the part, and each vertex is shown at its height throughout the part. The neighbor list $y.N$ at the beginning of each iteration appears on the right, with the iteration number on top. The shaded neighbor is $y.current$. **(a)** Initially, there are 19 units of excess to push from y, and $y.current = s$. Iterations 1, 2, and 3 just advance $y.current$, since there are no admissible edges leaving y. In iteration 4, $y.current = \text{NIL}$ (shown by the shading being below the neighbor list), and so y is relabeled and $y.current$ is reset to the head of the neighbor list. **(b)** After relabeling, vertex y has height 1. In iterations 5 and 6, edges (y, s) and (y, x) are found to be inadmissible, but iteration 7 pushes 8 units of excess flow from y to z. Because of the push, $y.current$ does not advance in this iteration. **(c)** Because the push in iteration 7 saturated edge (y, z), it is found inadmissible in iteration 8. In iteration 9, $y.current = \text{NIL}$, and so vertex y is again relabeled and $y.current$ is reset.



10	11
s	*s*
x	*x*
z	*z*



12	13	14
s	*s*	*s*
x	*x*	*x*
z	*z*	*z*



15
s
x
z

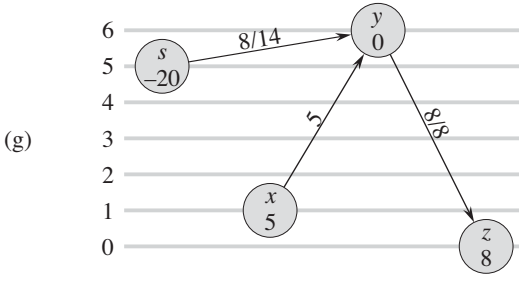


Figure 26.9, continued **(d)** In iteration 10, (y, s) is inadmissible, but iteration 11 pushes 5 units of excess flow from y to x. **(e)** Because $y.current$ did not advance in iteration 11, iteration 12 finds (y, x) to be inadmissible. Iteration 13 finds (y, z) inadmissible, and iteration 14 relabels vertex y and resets $y.current$. **(f)** Iteration 15 pushes 6 units of excess flow from y to s. **(g)** Vertex y now has no excess flow, and DISCHARGE terminates. In this example, DISCHARGE both starts and finishes with the current pointer at the head of the neighbor list, but in general this need not be the case.

To prove the second statement, according to the test in line 1 and Lemma 26.28, we need only show that all edges leaving u are inadmissible. If a call to DISCHARGE(u) starts with the pointer $u.current$ at the head of u's neighbor list and finishes with it off the end of the list, then all of u's outgoing edges are inadmissible and a relabel operation applies. It is possible, however, that during a call to DISCHARGE(u), the pointer $u.current$ traverses only part of the list before the procedure returns. Calls to DISCHARGE on other vertices may then occur, but $u.current$ will continue moving through the list during the next call to DISCHARGE(u). We now consider what happens during a complete pass through the list, which begins at the head of $u.N$ and finishes with $u.current = \text{NIL}$. Once $u.current$ reaches the end of the list, the procedure relabels u and begins a new pass. For the $u.current$ pointer to advance past a vertex $v \in u.N$ during a pass, the edge (u, v) must be deemed inadmissible by the test in line 6. Thus, by the time the pass completes, every edge leaving u has been determined to be inadmissible at some time during the pass. The key observation is that at the end of the pass, every edge leaving u is still inadmissible. Why? By Lemma 26.27, pushes cannot create any admissible edges, regardless of which vertex the flow is pushed from. Thus, any admissible edge must be created by a relabel operation. But the vertex u is not relabeled during the pass, and by Lemma 26.28, any other vertex v that is relabeled during the pass (resulting from a call of DISCHARGE(v)) has no entering admissible edges after relabeling. Thus, at the end of the pass, all edges leaving u remain inadmissible, which completes the proof. ■

The relabel-to-front algorithm

In the relabel-to-front algorithm, we maintain a linked list L consisting of all vertices in $V - \{s, t\}$. A key property is that the vertices in L are topologically sorted according to the admissible network, as we shall see in the loop invariant that follows. (Recall from Lemma 26.26 that the admissible network is a dag.)

The pseudocode for the relabel-to-front algorithm assumes that the neighbor lists $u.N$ have already been created for each vertex u. It also assumes that $u.next$ points to the vertex that follows u in list L and that, as usual, $u.next = \text{NIL}$ if u is the last vertex in the list.

```
RELABEL-TO-FRONT(G, s, t)
 1  INITIALIZE-PREFLOW(G, s)
 2  L = G.V − {s, t}, in any order
 3  for each vertex u ∈ G.V − {s, t}
 4       u.current = u.N.head
 5  u = L.head
 6  while u ≠ NIL
 7       old-height = u.h
 8       DISCHARGE(u)
 9       if u.h > old-height
10            move u to the front of list L
11       u = u.next
```

The relabel-to-front algorithm works as follows. Line 1 initializes the preflow and heights to the same values as in the generic push-relabel algorithm. Line 2 initializes the list L to contain all potentially overflowing vertices, in any order. Lines 3–4 initialize the *current* pointer of each vertex u to the first vertex in u's neighbor list.

As Figure 26.10 illustrates, the **while** loop of lines 6–11 runs through the list L, discharging vertices. Line 5 makes it start with the first vertex in the list. Each time through the loop, line 8 discharges a vertex u. If u was relabeled by the DISCHARGE procedure, line 10 moves it to the front of list L. We can determine whether u was relabeled by comparing its height before the discharge operation, saved into the variable *old-height* in line 7, with its height afterward, in line 9. Line 11 makes the next iteration of the **while** loop use the vertex following u in list L. If line 10 moved u to the front of the list, the vertex used in the next iteration is the one following u in its new position in the list.

To show that RELABEL-TO-FRONT computes a maximum flow, we shall show that it is an implementation of the generic push-relabel algorithm. First, observe that it performs push and relabel operations only when they apply, since Lemma 26.29 guarantees that DISCHARGE performs them only when they apply. It remains to show that when RELABEL-TO-FRONT terminates, no basic operations apply. The remainder of the correctness argument relies on the following loop invariant:

> At each test in line 6 of RELABEL-TO-FRONT, list L is a topological sort of the vertices in the admissible network $G_{f,h} = (V, E_{f,h})$, and no vertex before u in the list has excess flow.

Initialization: Immediately after INITIALIZE-PREFLOW has been run, $s.h = |V|$ and $v.h = 0$ for all $v \in V - \{s\}$. Since $|V| \geq 2$ (because V contains at

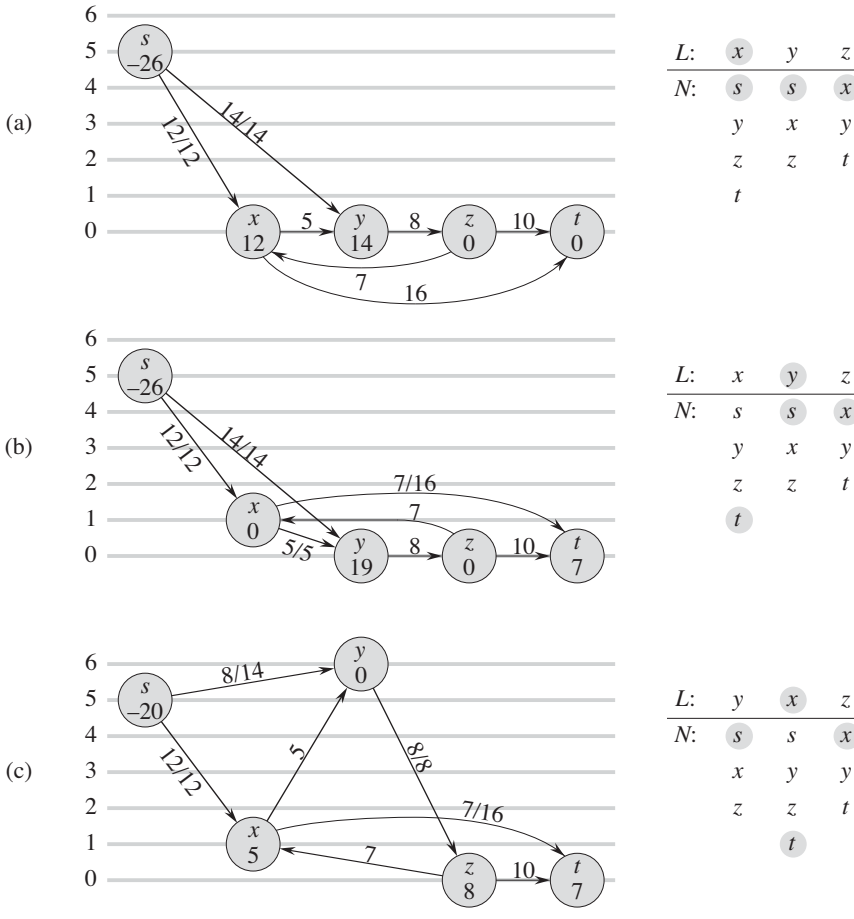
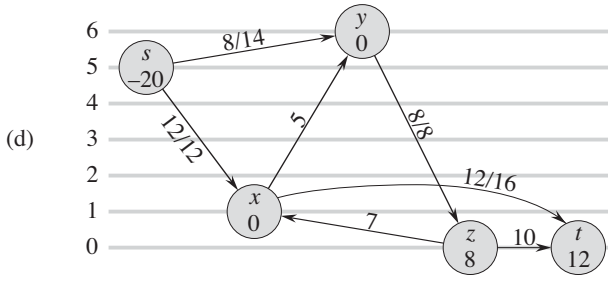


Figure 26.10 The action of RELABEL-TO-FRONT. **(a)** A flow network just before the first iteration of the **while** loop. Initially, 26 units of flow leave source s. On the right is shown the initial list $L = \langle x, y, z \rangle$, where initially $u = x$. Under each vertex in list L is its neighbor list, with the current neighbor shaded. Vertex x is discharged. It is relabeled to height 1, 5 units of excess flow are pushed to y, and the 7 remaining units of excess are pushed to the sink t. Because x is relabeled, it moves to the head of L, which in this case does not change the structure of L. **(b)** After x, the next vertex in L that is discharged is y. Figure 26.9 shows the detailed action of discharging y in this situation. Because y is relabeled, it is moved to the head of L. **(c)** Vertex x now follows y in L, and so it is again discharged, pushing all 5 units of excess flow to t. Because vertex x is not relabeled in this discharge operation, it remains in place in list L.



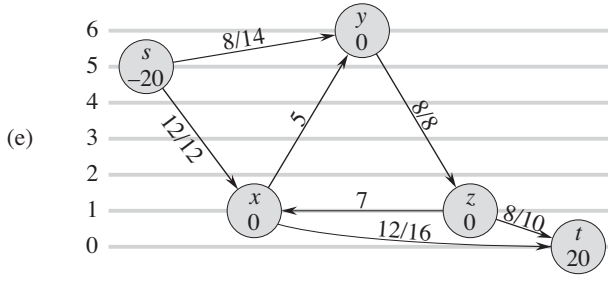


Figure 26.10, continued **(d)** Since vertex z follows vertex x in L, it is discharged. It is relabeled to height 1 and all 8 units of excess flow are pushed to t. Because z is relabeled, it moves to the front of L. **(e)** Vertex y now follows vertex z in L and is therefore discharged. But because y has no excess, DISCHARGE immediately returns, and y remains in place in L. Vertex x is then discharged. Because it, too, has no excess, DISCHARGE again returns, and x remains in place in L. RELABEL-TO-FRONT has reached the end of list L and terminates. There are no overflowing vertices, and the preflow is a maximum flow.

least s and t), no edge can be admissible. Thus, $E_{f,h} = \emptyset$, and any ordering of $V - \{s, t\}$ is a topological sort of $G_{f,h}$.

Because u is initially the head of the list L, there are no vertices before it and so there are none before it with excess flow.

Maintenance: To see that each iteration of the **while** loop maintains the topological sort, we start by observing that the admissible network is changed only by push and relabel operations. By Lemma 26.27, push operations do not cause edges to become admissible. Thus, only relabel operations can create admissible edges. After a vertex u is relabeled, however, Lemma 26.28 states that there are no admissible edges entering u but there may be admissible edges leaving u. Thus, by moving u to the front of L, the algorithm ensures that any admissible edges leaving u satisfy the topological sort ordering.

To see that no vertex preceding u in L has excess flow, we denote the vertex that will be u in the next iteration by u'. The vertices that will precede u' in the next iteration include the current u (due to line 11) and either no other vertices (if u is relabeled) or the same vertices as before (if u is not relabeled). When u is discharged, it has no excess flow afterward. Thus, if u is relabeled during the discharge, no vertices preceding u' have excess flow. If u is not relabeled during the discharge, no vertices before it on the list acquired excess flow during this discharge, because L remained topologically sorted at all times during the discharge (as just pointed out, admissible edges are created only by relabeling, not pushing), and so each push operation causes excess flow to move only to vertices further down the list (or to s or t). Again, no vertices preceding u' have excess flow.

Termination: When the loop terminates, u is just past the end of L, and so the loop invariant ensures that the excess of every vertex is 0. Thus, no basic operations apply.

Analysis

We shall now show that RELABEL-TO-FRONT runs in $O(V^3)$ time on any flow network $G = (V, E)$. Since the algorithm is an implementation of the generic push-relabel algorithm, we shall take advantage of Corollary 26.21, which provides an $O(V)$ bound on the number of relabel operations executed per vertex and an $O(V^2)$ bound on the total number of relabel operations overall. In addition, Exercise 26.4-3 provides an $O(VE)$ bound on the total time spent performing relabel operations, and Lemma 26.22 provides an $O(VE)$ bound on the total number of saturating push operations.

Theorem 26.30
The running time of RELABEL-TO-FRONT on any flow network $G = (V, E)$ is $O(V^3)$.

Proof Let us consider a "phase" of the relabel-to-front algorithm to be the time between two consecutive relabel operations. There are $O(V^2)$ phases, since there are $O(V^2)$ relabel operations. Each phase consists of at most $|V|$ calls to DISCHARGE, which we can see as follows. If DISCHARGE does not perform a relabel operation, then the next call to DISCHARGE is further down the list L, and the length of L is less than $|V|$. If DISCHARGE does perform a relabel, the next call to DISCHARGE belongs to a different phase. Since each phase contains at most $|V|$ calls to DISCHARGE and there are $O(V^2)$ phases, the number of times DISCHARGE is called in line 8 of RELABEL-TO-FRONT is $O(V^3)$. Thus, the total

work performed by the **while** loop in RELABEL-TO-FRONT, excluding the work performed within DISCHARGE, is at most $O(V^3)$.

We must now bound the work performed within DISCHARGE during the execution of the algorithm. Each iteration of the **while** loop within DISCHARGE performs one of three actions. We shall analyze the total amount of work involved in performing each of these actions.

We start with relabel operations (lines 4–5). Exercise 26.4-3 provides an $O(VE)$ time bound on all the $O(V^2)$ relabels that are performed.

Now, suppose that the action updates the $u.current$ pointer in line 8. This action occurs $O(\text{degree}(u))$ times each time a vertex u is relabeled, and $O(V \cdot \text{degree}(u))$ times overall for the vertex. For all vertices, therefore, the total amount of work done in advancing pointers in neighbor lists is $O(VE)$ by the handshaking lemma (Exercise B.4-1).

The third type of action performed by DISCHARGE is a push operation (line 7). We already know that the total number of saturating push operations is $O(VE)$. Observe that if a nonsaturating push is executed, DISCHARGE immediately returns, since the push reduces the excess to 0. Thus, there can be at most one nonsaturating push per call to DISCHARGE. As we have observed, DISCHARGE is called $O(V^3)$ times, and thus the total time spent performing nonsaturating pushes is $O(V^3)$.

The running time of RELABEL-TO-FRONT is therefore $O(V^3 + VE)$, which is $O(V^3)$. ■

Exercises

26.5-1

Illustrate the execution of RELABEL-TO-FRONT in the manner of Figure 26.10 for the flow network in Figure 26.1(a). Assume that the initial ordering of vertices in L is $\langle v_1, v_2, v_3, v_4 \rangle$ and that the neighbor lists are

$$\begin{aligned} v_1.N &= \langle s, v_2, v_3 \rangle , \\ v_2.N &= \langle s, v_1, v_3, v_4 \rangle , \\ v_3.N &= \langle v_1, v_2, v_4, t \rangle , \\ v_4.N &= \langle v_2, v_3, t \rangle . \end{aligned}$$

26.5-2 ⋆

We would like to implement a push-relabel algorithm in which we maintain a first-in, first-out queue of overflowing vertices. The algorithm repeatedly discharges the vertex at the head of the queue, and any vertices that were not overflowing before the discharge but are overflowing afterward are placed at the end of the queue. After the vertex at the head of the queue is discharged, it is removed. When the