

# Fast and Practical DAG Decomposition with Reachability Applications

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## Abstract

We present practical linear and almost linear-time algorithms to compute a chain decomposition of a *directed acyclic graph* (DAG),  $G = (V, E)$ . The number of vertex-disjoint chains computed is very close to the minimum. The time complexity of our algorithm is  $O(|E| + c * l)$ , where  $c$  is the number of path concatenations and  $l$  is the length of a longest path of the graph. We give a comprehensive explanation on factors  $c$  and  $l$  in the following sections. Our techniques have important applications in many areas, including the design of faster practical transitive closure algorithms. We observe that  $|E_{red}| \leq width * |V|$  ( $E_{red}$ : non-transitive edges) and show how to find a substantially large subset of  $E_{tr}$  (transitive edges) using a chain decomposition in linear time, without calculating the transitive closure. Our extensive experimental results show the interplay between the width,  $E_{red}$ ,  $E_{tr}$  in various models of graphs. We show how to compute a reachability indexing scheme in  $O(k_c * |E_{red}|)$  time, where  $k_c$  is the number of chains and  $|E_{red}|$  is the number of non-transitive edges. This scheme can answer reachability queries in constant time. The space complexity of the scheme is  $O(k_c * |V|)$ . The experimental results reveal that our methods are even better in practice than the theoretical bounds imply, indicating how fast chain decomposition algorithms can be applied to the transitive closure problem.

**Keywords:** graph algorithms, hierarchy, directed acyclic graphs (DAG), path/chain decomposition, transitive closure, transitive reduction, reachability, reachability indexing scheme.

## 1 Introduction

*Directed acyclic graphs* (DAGs) are very important in many applications. The *width* of a DAG  $G = (V, E)$  is the maximum number of mutually unreachable vertices of  $G$  [11]. The width of a DAG is a crucial metric in graph theory and it is used in a wide area of applications. Any DAG can be decomposed into vertex disjoint *paths* or *chains*. In a path every vertex is connected to its successor by an edge, while in a chain any vertex is connected to its successor by a directed path, which may be an edge. The computation of the width and path/chain decomposition have many applications including bioinformatics [4, 15], evolutionary computation [19], databases [18], graph drawing [26, 27, 22], distributed systems [17, 29].

An optimum *chain decomposition* of a DAG  $G = (V, E)$  contains the minimum number of chains,  $k$ , which is equal to the width of  $G$ . Due to the multitude of applications there are several algorithms to find an optimum chain decomposition, see for example [18, 12, 10, 25, 6, 7, 21, 31]. An FPT algorithm was presented in [6] that runs in  $O(k^2 4^k |V| + k 2^k |E|)$  time. Clearly, this is practical only for very specific classes of DAGs that have very small values of  $k$ . Most DAGs have much higher width (often a function of  $n$ ), as shown in the experimental results of Section 3. Recently, better almost-linear-time algorithms were presented in [7, 21, 31] whose time complexity is  $O(k^3 |V| + |E|)$  and  $\tilde{O}(|E| + |V|^{3/2})$ , respectively. All of these algorithms solve the chain decomposition problem by reducing it to a minimum cost flow problem. Using the best minimum cost flow algorithm, which runs in almost-linear time [31], solves the problem theoretically. However, due to the heavy mechanism, this approach is challenging to implement. Additionally, for several applications it is not necessary to compute an optimum chain decomposition.

We will describe heuristic techniques that compute a chain decomposition that is close to the optimum in linear or almost linear time and are easy to implement. We present a simple greedy algorithm for the chain decomposition problem. We perform extensive experiments on several random graphs to verify its effectiveness and explore how fast chain decomposition can enhance transitive closure solutions. Our experimental results show that the algorithm produces decompositions whose sizes are close to optimal.

In [18], Jagadish categorized path and chain decomposition heuristics into two kinds, Chain-Order Heuristic and Node-Order Heuristic. The Chain-Order Heuristic constructs the paths one by one, while the Node-Order Heuristic creates the paths in parallel. Jagadish's heuristics for path decomposition run in linear time, while his chain decomposition heuristics run in  $O(n^2)$  time given a precomputed transitive closure. Few techniques have been presented to compute sub-optimal chain decomposition since then, see [9, 5], but none of them is as simple and fast as the algorithm we describe here.

In this paper, we explore new path and chain decomposition algorithms for DAGs. Our focus is on practical algorithms that run in linear or almost linear time, and produce results that are very close to the optimum. An interesting observation is that the width of some DAGs follows the curve  $width = \frac{\text{nodes}}{\text{average degree}}$ . We apply the computed decomposition in order to obtain several techniques that solve fundamental problems, such as transitive closure, faster than previous algorithms. We use the notion of chain decomposition to offer bounds to the number of non transitive edges and explore how it facilitates in various transitive closure and reduction problems. Clearly, our decomposition techniques can be used to solve various other problems in order to obtain new time and space complexity bounds.

In Section 2, we present path decomposition algorithms. In Section 3, chain decomposition and path concatenation approaches are presented. Additionally, we show experiments and evaluate the performance of our heuristics. Moreover, the value of the width is explored as the graph becomes denser. Next, we examine some applications of the heuristics. In Section 4, we show that  $|E_{red}| \leq width * |V|$ , and describe how to remove a significant subset of transitive edges,  $E'_{tr}$ , in linear time, in order to bound  $|E - E'_{tr}|$  by  $O(k * V)$  given any path/chain decomposition of size  $k$ . This can boost many known transitive closure solutions. Finally, Section 5 demonstrates how to build an indexing scheme that implicitly contains the transitive closure and we report experimental results. Our experiments shed light on the behavior of critical factors.

All experiments were conducted on a laptop PC (Intel(R) Core(TM) i5-6200U CPU, with 8 GB of main memory). Our algorithms have been developed as stand-alone java programs.

## Definitions and Abbreviations

- **DAG:** Directed acyclic graph (DAG) is a directed graph with no directed cycles.
- **Path/Chain:** In a path every vertex is connected by a direct edge to its successor, while in a chain any vertex is connected to its successor by a directed path which may be an edge. The vertices of a path/chain are in ascending topological order.
- **Paths/Chains decomposition of a DAG:** Let  $G = (V, E)$  be a DAG. A path/chain decomposition of  $G$  is a set of vertex-disjoint paths/chains. The decomposition includes all vertices of  $G$ . There is an example of a path and a chain decomposition in Figure 1.
  - $k_p$ : denotes the number of paths of a path decomposition of a DAG.
  - $k_c$ : denotes the number of chains of a chain decomposition of a DAG.
- **Width:** is the maximum number of mutually unreachable vertices of a graph [11]. The number of chains in a minimum chain decomposition of a graph is equal to its width.
- **Transitive edge:** an edge  $(v_1, v_2)$  of a DAG  $G$  is transitive if there is a path longer than one edge that connects  $v_1$  and  $v_2$ .
- **In a DAG  $G = (V, E)$ :**
  - $E_{tr}$  : is the set of all transitive edges of  $G$ .  $E_{tr} \subset E$ .
  - $E'_{tr}$  : denotes a subset of  $E_{tr}$ .
  - $E_{red}$  : denotes the set of non-transitive edges:  $E_{red} = E - E_{tr}$  ,  $E_{red} \subseteq E$ .
  - $G_{red} = (V, E_{red})$  : denotes the transitive reduction [2] of  $G = (V, E)$ . The transitive reduction is unique for DAGs. It contains the minimum number of edges needed to form the same transitive closure as  $G = (V, E)$ .

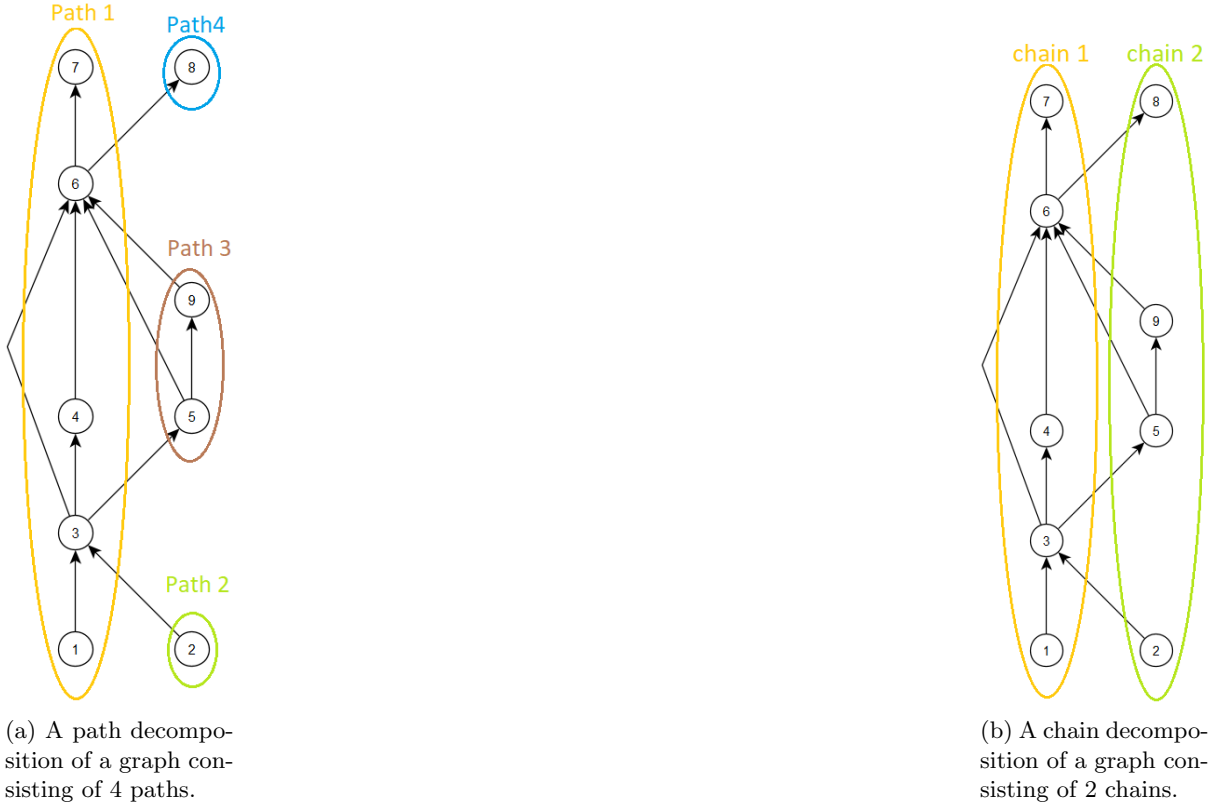


Figure 1: Path and chain decomposition of an example graph.

## 2 DAG Decomposition into Paths

Jagadish in [18] categorized path decomposition techniques into two categories: Chain Order Heuristics and Node Order Heuristics. The first constructs the paths one by one, while the second creates the paths in parallel. He also presented chain decomposition heuristics based on Chain Order and Node Order Heuristic, utilizing a list of *all successors* and not only the immediate for each vertex. His algorithms require  $O(n^2)$  time using a precomputed transitive closure. He states in [18] that in order to build a chain decomposition we must compute the transitive closure, implying that finding the minimum number of chains has the same complexity as the heuristics. We believe that, in part, this is an important reason for which most of the research on this topic is focusing on finding the optimum number of chains. In this work we argue the opposite: Namely, we show that we can build an efficient chain decomposition fast and we use it in order to solve the transitive closure problem. Theoretically, our chain decomposition heuristic has worst-case time complexity  $O(n^2)$ , without requiring any precomputation of the transitive closure. In practice however it actually behaves like a linear time algorithm as we explain later, both theoretically and experimentally. Furthermore, in Sections 4 and 5, we show how an efficient and practical chain decomposition can crucially enhance transitive closure solutions.

More precisely our algorithm decomposes the graph into  $k_c$  chains in  $O(|E| + c * l)$  time, where  $c$  is the number of path concatenations, and  $l$  is the length of a longest path of the graph. An important observation is that the factor  $c * l$  depends on the number of successful concatenations  $c$ . Hence, the more time our algorithm takes, the more concatenations it completes. If there are no concatenations then  $c * l$  equals zero. In our experiments, the factor  $(c * l)$  is less than  $|E|$  in almost all cases. We will describe our technique in detail in the next section.

In the rest of this section, we describe the linear time algorithms for path decomposition using only the immediate successors. Since paths are by definition chains of a special structure, we use the more general term chain in our algorithms. Clearly, similar algorithms can be used also for chain decomposition. We assume that topological sorting has been performed and examine the vertices in ascending order.

## Chain Order Heuristic

The chain-order heuristic starts from a vertex and keeps on extending the path to the extent possible. The path ends when no more unused immediate successors can be found. The first for loop of Algorithm 1 finds an unused vertex and creates a path. The inner while loop extends the path.

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**Algorithm 1** Path Decomposition CO

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```
1: procedure CHAINORDERHEURISTIC( $G, T$ )
   INPUT: A DAG  $G = (V, E)$ , and a topological sorting  $T(v_1, \dots, v_i, \dots, v_N)$  of  $G$ 
   OUTPUT: A path decomposition of  $G$ 
2:    $K \leftarrow \emptyset$  ▷ Set of paths
3:   Mark all nodes unused
4:   for every unused vertex  $v_i \in T$  in ascending topological order do
5:      $current \leftarrow v_i$ 
6:      $C \leftarrow \text{new Chain}()$ 
7:     Add  $current$  to  $C$ 
8:     while there is an unused immediate successor  $s$  of the current node do
9:       add  $s$  to  $C$ 
10:       $current \leftarrow s$ 
11:     end while
12:     add  $C$  to  $K$ 
13:   end for
14:   return  $K$ 
15: end procedure
```

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## Node Order Heuristic

The node-order heuristic examines each vertex (node)  $v_i$  and assigns it to an existing chain. If there is no such chain, then a new chain is created for vertex  $v_i$ . Algorithm 2 illustrates the node order heuristic.

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**Algorithm 2** Path Decomposition NO

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```
1: procedure NODEORDERHEURISTIC( $G, T$ )
   INPUT: A DAG  $G = (V, E)$ , and a topological sorting  $T(v_1, \dots, v_i, \dots, v_N)$  of  $G$ 
   OUTPUT: A path decomposition of  $G$ 
2:    $K \leftarrow \emptyset$  ▷ Set of paths
3:   for every vertex  $v_i \in T$  in ascending topological order do
4:     if  $v_i$  is an immediate successor of the last node of a chain  $C$  then
5:       add  $v_i$  to  $C$ 
6:     else
7:        $C \leftarrow \text{new Chain}()$ 
8:       add  $v_i$  to  $C$ 
9:       add  $C$  to  $K$ 
10:    end if
11:  end for
12:  return  $K$ 
13: end procedure
```

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## 3 DAG Decomposition into Chains

In this section, we present a path concatenation technique that takes as input any path decomposition of a DAG  $G = (V, E)$  and constructs a chain decomposition by performing repeated path concatenations. If it performs  $c$  path concatenations and  $l$  is the length of a longest path of the graph, then it requires  $O(|E| + c * l)$  time. In fact, the actual cost for every successful concatenation is the path between the two

concatenated paths, which is bounded by  $(c * l)$ . In order to apply our path concatenation algorithm, we first compute a path decomposition of the graph. We propose to use linear-time algorithms based on Node-Order Heuristic or Chain Order Heuristic, as described in the previous section.

### 3.1 Path/Chain Concatenation

Two chains  $c_1$  and  $c_2$  can be merged into a new chain if the last vertex of  $c_1$  can reach the first vertex of  $c_2$ , or if the last vertex of  $c_2$  can reach the first vertex of  $c_1$ . The process of merging two or more chains into a new chain is called path or chain concatenation. To reduce the number of chains of any given chain decomposition, we can find possible concatenations and merge the chains. Searching for a concatenation implies that we are searching for a path between two chains. We can start searching from the first vertex of a chain looking for the last vertex of another chain, or from the last vertex of a chain looking for the first vertex of another chain.

Given a DAG  $G = (V, E)$  and a path decomposition  $D_p$  that contains  $k_p$  paths we will build a chain decomposition of  $G$  that contains  $k_c$  chains in  $O(|E| + (k_p - k_c) * l)$  time, where  $l$  is the length of a longest path of  $G$ . This is accomplished by performing path/chain concatenations. Since each path/chain concatenation reduces the number of chains by one, the total number of such concatenations is  $(k_p - k_c)$ . Since paths are by definition chains of a special structure, and we start by concatenating paths into chains, we use the more general term chain in our algorithms.

For every path in  $D_p$  we start a reversed DFS lookup function from the first vertex of a chain, looking for the last vertex of another chain traversing the edges backward. The reversed DFS lookup function is the well-known depth-first search graph traversal for path finding. If the reversed DFS lookup function detects the last vertex of a chain, then it concatenates the chains. If we simply perform the above, as described, then the algorithm will run in  $O(k_p * |E|)$  since we will run  $k_p$  reversed DFS searches. However, every reversed DFS search can take advantage of the previous reversed DFS results. A reversed DFS for path finding returns the path between the source vertex and the target vertex, which in our case, is the path between the first vertex of a chain and the last vertex of another chain. Hence, every execution that goes through a set of vertices  $V_i$  it splits them into two vertex disjoint sets,  $R_i$  and  $P_i$ .  $P_i$  contains the vertices of the path from the source vertex to the destination vertex and  $R_i$  contains every vertex in  $V_i - P_i$ . If no path is found then  $V_i = R_i$  and  $P_i = \emptyset$ .

Notice that every vertex in the set  $R_i$  is not the last vertex of a chain. If it were then it would belong to  $P_i$  and not to  $R_i$ . Similarly, for every vertex in  $R_i$ , all its predecessors are in  $R_i$  too. Hence, if a forthcoming reversed DFS lookup function meets a vertex of  $R_i$ , there is no reason to proceed with its predecessors.

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#### Algorithm 3 Concatenation

---

```

1: procedure CONCATENATION( $G, D$ )
   INPUT: A DAG  $G = (V, E)$ , and a path decomposition  $D$  of  $G$ 
   OUTPUT: A chain decomposition of  $G$ 
2:   for each path:  $p_i \in D$  do
3:      $f_i \leftarrow$  first vertex of  $p_i$ 
4:      $(R_i, P_i) \leftarrow$  reversed_DFS_lookup( $G, f_i$ )
5:     if  $P_i \neq \emptyset$  then
6:        $l_i \leftarrow$  destination vertex of  $P_i$  ▷ Last vertex of a path
7:       Concatenate_Paths(  $l_i, f_i$  )
8:     end if
9:      $G \leftarrow G \setminus R_i$ 
10:  end for
11: end procedure

```

---

Algorithm 3 shows our path/chain concatenation technique. Observe that the reversed DFS lookup function is invoked for every starting vertex of a path. Every reversed DFS lookup function goes through the set  $R_i$  and the set  $P_i$ , examining the nodes and their incident edges.  $P_i$  is the path from the first vertex of a chain to the last vertex of another chain. The set  $R_i$  contains all of the vertices the function went through except the vertices of  $P_i$ . Hence we have the following theorem:

**Theorem 3.1.** *The time complexity of Algorithm 3 is  $O(|E| + (k_p - k_c) * l)$ , where  $l$  is the length of a longest path in  $G$ .*

*Proof.* Assume that we have  $k_p$  paths. We call  $k_p$  times the `reversed_DFS_lookup` function. Hence, we have  $(R_i, P_i)$  sets,  $0 \leq i < k_p$ . In every loop, we delete the vertices of  $R_i$ . Hence,  $R_i \cap R_j = \emptyset$ ,  $0 \leq i, j < k_p$  and  $i \neq j$ . We conclude that  $\bigcup_{i=0}^{k_p-1} R_i \subseteq V$  and  $\sum_{i=0}^{k_p-1} |R_i| \leq |V|$ .

A path/chain  $P_i$ ,  $0 \leq i < k_p$ , is not empty if and only if a concatenation has occurred. Hence,  $\sum_{i=0}^{k_p-1} |P_i| \leq c * l$  where  $c$  is the number of concatenations and  $l$  is the longest path of the graph. Since every concatenation reduces the number of paths/chains by one, we have that  $c = k_p - k_c$ .  $\square$

Please notice that according to the previous proof the actual time complexity of Algorithm 3 is  $O(|E| + \sum_{i=0}^{k_p-1} |P_i|)$  which in practice is expected to be significantly better than  $O(|E| + c * l)$ . Indeed, this is confirmed by our experimental results. We ran some extra experiments on ER graphs of size 10k, 20k, 40k, 80k, and 160k vertices and average degree 10; the run times were 9, 34, 99, 228, and 538 milliseconds, respectively, which shows again that the execution time is almost linear.

### 3.2 A Better Chain Decomposition Heuristic

Algorithm 3 describes how to produce a chain decomposition of a DAG  $G$  by applying a path/chain concatenation technique. Notice that this algorithm works for any given path/chain decomposition of  $G$ . Of course, the number of future potential concatenations depends on the starting path/chain decomposition of  $G$ , that is given as input and it will play a crucial role in obtaining a solution that is close to optimal. Hence, we will present an improved heuristic, called Algorithm 4, whose goal is to produce a path decomposition of  $G$  that will allow a large number of future concatenations. Since these variations are not computationally intensive, Algorithm 4 runs in linear time. Next, using the output path decomposition produced by Algorithm 4 we will describe Algorithm 5 that computes a chain decomposition that is very close to the optimal, according to our experimental results, and still runs in  $O(|E| + c * l)$  time.

Algorithm 4, which is a variation of the Node Order Heuristic (Algorithm 2), follows the same philosophy but with two important additions that aim to construct a path decomposition in which more concatenations will be possible in the future: (a) when we visit a vertex of out-degree 1, we immediately add its unique immediate successor to its path/chain, and (b) instead of searching for the first available immediate predecessor (that is the last vertex of a path), we choose an available vertex with the lowest out-degree. This heuristic aims to create a path decomposition that will allow more concatenations in the future. Since Algorithm 4 goes through all vertices of  $G$ , and for every vertex, it examines all the outgoing (line 8) and all the incoming edges (line 19), its time complexity is linear.

We are ready now to present our best chain decomposition algorithm, Algorithm 5, which is a combination of Algorithm 4 and the chain concatenation described in Algorithm 3. The only addition to Algorithm 4 is the block of the if-statement of line 10. In other words, if we do not find an immediate predecessor, we search all predecessors using the `reversed_DFS_lookup` function. The differentiation from our concatenation approach is that it does not use it as a post-processing step. It is applied in real time if the algorithm does not find an immediate predecessor that is the last vertex of a chain. This is done in order to avoid transitive edges that could lead to false matches.

### 3.3 DAG Decomposition: Experimental Results

In this section we present extensive experimental results on graphs, most of which were created by NetworkX [16]. We use three different random graph generator models: Erdős-Rényi [13], Barabasi-Albert [3], and Watts-Strogatz [33] models. The generated graphs are made acyclic, by orienting all edges from low to high ID number, see the documentation of networkx [16] for more information about the generators. Additionally, we use the Path-Based DAG Model that was introduced in [24] and is especially designed for DAGs with a predefined number of randomly created paths. For every model, we created 12 graphs: Six graphs of 5000 nodes and six graphs of 10000 nodes and average degree 5, 10, 20, 40, 80, and 160. We ran the heuristics on multiple copies of the graphs and examined the performance of the heuristics in terms of the number of chains in the produced chain decompositions. Additionally, we observed that the graphs generated by the same generator with the same parameters have small width deviation. For example, the percentage of deviation on ER and Path-Based model is about 5% and for the BA model is less than 10%. The width deviation of graph in the WS model is a bit higher, but this is expected since the width of these graphs is smaller.

---

**Algorithm 4** Path Decomposition (H3)

---

```
1: procedure NODE-ORDER BASED VARIATION( $G, T$ )
   INPUT: A DAG  $G = (V, E)$ , and a topological sorting  $T(v_1, \dots, v_i, \dots, v_N)$  of  $G$ 
   OUTPUT: A path decomposition of  $G$ 
2:    $K \leftarrow \emptyset$  ▷ Set of paths
3:   for every vertex  $v_i \in T$  in ascending topological order do
4:     Chain  $C$  ▷  $C$  is a pointer to a path
5:     if  $v_i$  is assigned to a chain then
6:        $C \leftarrow v_i$ 's chain
7:     else if  $v_i$  is not assigned to a chain then
8:        $l_i \leftarrow$  choose the immediate predecessor with the lowest outdegree
9:         that is the last vertex of a chain
10:      if  $l_i \neq$  null then
11:         $C \leftarrow$  path indicated by  $l_i$ 
12:        add  $v_i$  to  $C$ 
13:      else
14:         $C \leftarrow$  new Chain()
15:        add  $v_i$  to  $C$ 
16:      end if
17:      add  $C$  to  $K$ 
18:    end if
19:    if there is an immediate successor  $s_i$  of  $v_i$  with in-degree 1 then
20:      add  $s_i$  to  $C$ 
21:    end if
22:  end for
23:  return  $K$ 
24: end procedure
```

---

**Random Graph Generators:**

- **Erdős-Rényi (ER) model** [13]: The generator returns a random graph  $G_{n,p}$ , where  $n$  is the number of nodes and every edge is formed with probability  $p$ .
- **Barabási-Albert (BA) model** [3]: preferential attachment model: A graph of  $n$  nodes is grown by attaching new nodes each with  $m$  edges that are preferentially attached to existing nodes with high degree. The factors  $n$  and  $m$  are parameters to the generator.
- **Watts-Strogatz (WS) model** [33]: small-world graphs: First it creates a ring over  $n$  nodes. Then each node in the ring is joined to its  $k$  nearest neighbors. Then shortcuts are created by replacing some edges as follows: for each edge  $(u, v)$  in the underlying “ $n$ -ring with  $k$  nearest neighbors” with probability  $b$  replace it with a new edge  $(u, w)$  with uniformly random choice of an existing node  $w$ . The factors  $n, k, b$  are the parameters of the generator.
- **Path-Based DAG (PB) model** [24]: In this model, graphs are randomly generated based on a number of predefined but randomly created paths.

We compute the minimum set of chains using the method of Fulkerson [12], which in brief consists of the following steps: 1) Construct the transitive closure  $G^*(V, E')$  of  $G$ , where  $V = \{v_1, \dots, v_n\}$ . 2) Construct a bipartite graph  $B$  with partitions  $(V_1, V_2)$ , where  $V_1 = \{x_1, x_2, \dots, x_n\}$ ,  $V_2 = \{y_1, y_2, \dots, y_n\}$ . An edge  $(x_i, y_j)$  is formed whenever  $(v_i, v_j) \in E'$ . 3) Find a maximal matching  $M$  of  $B$ . The width of the graph is  $n - |M|$ . In order to construct the minimum set of chains, for any two edges  $e_1, e_2 \in M$ , if  $e_1 = (x_i, y_t)$  and  $e_2 = (x_t, y_j)$  then connect  $e_1$  to  $e_2$ .

The aim of our experiments is twofold: (a) to understand the behavior of the width of DAGs created in different models, and (b) to compare the behavior of all the heuristics used on graphs of these models. Table 1 shows the width and the number of chains created by the various heuristics for every graph of 5000 nodes. Table 2 shows the same for graphs of 10000 nodes.

- **CO:** Path decomposition using Chain Order Heuristic, (Algorithm 1)

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**Algorithm 5** Chain Decomposition (H3 conc.)

---

```
1: procedure NODEORDER BASED VARIATION WITH CONCATENATION( $G, T$ )
   INPUT: A DAG  $G = (V, E)$ , and a topological sorting  $T(v_1, \dots, v_i, \dots, v_N)$  of  $G$ 
   OUTPUT: A chain decomposition of  $G$ 
2:    $K \leftarrow \emptyset$  ▷ Set of chains
3:   for every vertex  $v_i \in T$  in ascending topological order do
4:     Chain  $C$ 
5:     if  $v_i$  is assigned to a chain then
6:        $C \leftarrow v_i$ 's chain ▷  $C$  is a pointer a to path
7:     else if  $v_i$  is not assigned to a chain then
8:        $l_i \leftarrow$  choose the immediate predecessor with the lowest outdegree
9:         that is the last vertex of a chain
10:      if  $l_i = \text{null}$  then
11:         $(R_i, P_i) \leftarrow \text{reverse\_DFS\_lookup}(G, v_i)$ 
12:        if  $P_i \neq \emptyset$  then
13:           $l_i \leftarrow$  destination vertex of  $P_i$ 
14:        end if
15:         $G \leftarrow G \setminus R_i$ 
16:      end if
17:      if  $l_i \neq \text{null}$  then
18:         $C \leftarrow$  chain indicated by  $l_i$ 
19:        add  $v_i$  to  $C$ 
20:      else
21:         $C \leftarrow \text{new Chain}()$ 
22:        add  $v_i$  to  $C$ 
23:      end if
24:      add  $C$  to  $K$ 
25:    end if
26:    if there is an immediate successor  $s_i$  of  $v_i$  with in-degree 1 then
27:      add  $s_i$  to  $C$ 
28:    end if
29:  end for
30:  return  $K$ 
31: end procedure
```

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- **CO conc.:** Chain decomposition using Chain Order Heuristic and our concatenation technique. (Algorithm 1 followed by Algorithm 3)
- **NO:** Path decomposition using Node Order Heuristic. (Algorithm 2)
- **NO conc.:** Chain decomposition using Node Order Heuristic and our concatenation technique. (Algorithm 2 followed by Algorithm 3)
- **H3:** Path decomposition using Algorithm 4, our NO Heuristic variation
- **H3 conc.:** Chain decomposition using Algorithm 5
- **Width:** The width of the graph (computed by Fulkerson's method).

Observe that in both tables our chain decomposition heuristic, H3 conc., performs better than all other heuristics since it produces fewer chains.

In order to understand the behavior of the width on DAGs of these different models we observe: (i) the BA model produces graphs with a larger width than ER, and (ii) the ER model creates graphs with a larger width than WS. For the WS model, we created two sets of graphs: The first has probability  $b$  equals 0.9 and the second 0.3. Clearly, if the probability  $b$  of rewiring an edge is 0, the width would be one, since the generator initially creates a path that goes through all vertices. As the rewiring probability  $b$  grows, the width grows. That is the reason we choose a low and a high probability. Figure 2a and 2b demonstrates the behavior of the width for each model on the graphs of 5000 and 10000 nodes. Another interesting observation is that the width of the ER model follows the curve  $width = \frac{\text{nodes}}{\text{average degree}}$ .



In order to compare the number of chains produced by all heuristics used on the graphs of these models we created several figures. Namely, we visualize how close is the result of our heuristics to the width. In Figures 4, 5, and 6, we show how close is the number of chains produced by our technique (i.e., the blue line) to the width (i.e., red line) for ER, BA, and WS models. A comparison between the ER and the Path-Based models is shown in Figure 3 for graphs of 10000 nodes and varying average degree. It is interesting to note that for sparser PB graphs the width is very close to the number of predefined (but randomly created paths) whereas the width of the ER graphs is very high. However, when the graphs become denser the width for both models seem to eventually converge.

All heuristics run very fast, in just a few milliseconds. Thus it is not interesting to elaborate on their running time. In the following sections, we present partial run-time results that are obtained for computing an indexing scheme (see Tables 3, 4, and 5). These results indicate that the running time of the heuristics is indeed very low.

**$|V| = 5000$**

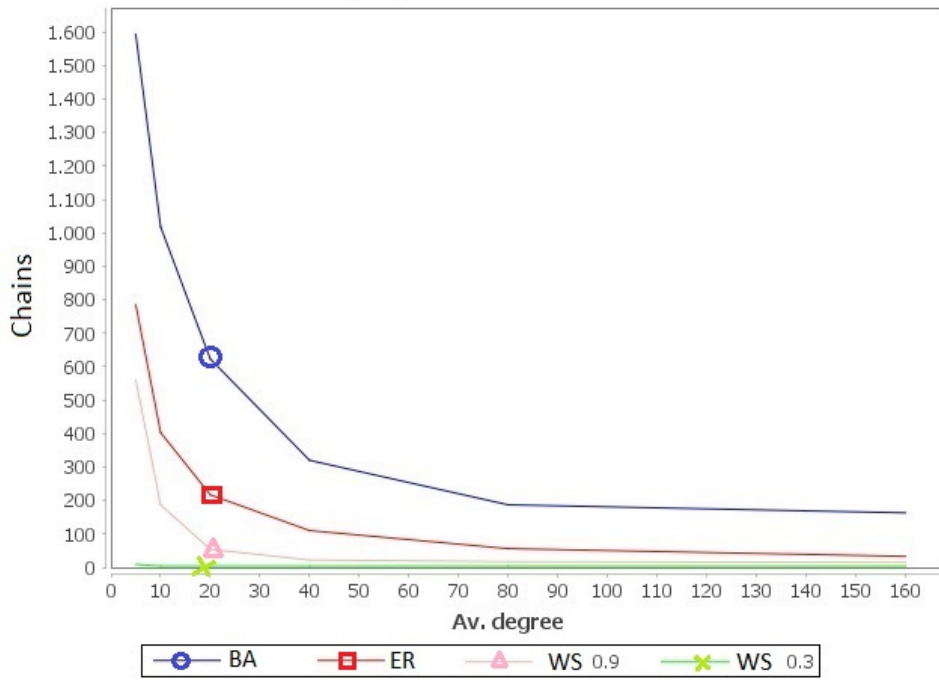
Av. Degree	5	10	20	40	80	160
	<b>BA</b>					
CO	1722	1178	801	471	296	189
CO conc.	1686	1127	747	411	252	164
NO	1792	1250	827	516	306	193
NO conc.	1743	1174	774	445	284	187
H3	1658	1102	720	424	256	165
H3 conc.	1630	1055	664	355	207	163
Width	1593	1018	623	320	187	163
	<b>ER</b>					
CO	1138	710	433	260	148	79
CO conc.	1027	593	356	217	125	69
NO	1184	744	461	263	157	83
NO conc.	1105	686	429	257	153	83
H3	1050	654	401	235	143	80
H3 conc.	923	492	252	139	70	38
Width	785	403	217	110	56	33
	<b>WS, b=0.9</b>					
CO	948	514	279	161	87	57
CO conc.	794	376	202	107	69	47
NO	995	540	272	126	60	40
NO conc.	865	441	244	119	59	40
H3	891	473	264	145	81	58
H3 conc.	687	212	60	25	20	17
Width	560	187	54	22	17	15
	<b>WS, b=0.3</b>					
CO	399	240	130	62	39	23
CO conc.	90	57	32	20	12	10
NO	275	88	23	6	7	6
NO conc.	85	40	17	6	7	6
H3	283	162	85	50	28	12
H3 conc.	9	4	4	5	4	5
Width	9	4	4	4	4	4
	<b>PB, Paths=70</b>					
CO	159	236	295	289	203	130
CO conc.	114	155	193	207	155	109
NO	210	295	328	268	197	125
NO conc.	148	215	260	242	192	124
H3	115	210	257	241	190	120
H3 conc.	86	101	107	93	73	51
Width	70	70	70	68	58	30

Table 1: Comparing the number of chains produced by the path and chain decomposition algorithms on graphs with 5000 nodes.

<b><math> V =10000</math></b>						
Av. Degree	5	10	20	40	80	160
<b>BA</b>						
CO	3501	2401	1537	985	586	357
CO conc.	3441	2301	1415	865	500	294
NO	3635	2519	1645	1033	625	387
NO conc.	3549	2413	1515	959	563	345
H3	3385	2257	1411	911	535	321
H3 conc.	3341	2159	1264	752	400	228
Width	3282	2066	1172	678	351	198
<b>ER</b>						
CO	2283	1432	871	513	294	165
CO conc.	2015	1213	730	428	251	145
NO	2369	1517	891	531	294	165
NO conc.	2172	1383	833	507	290	163
H3	2135	1325	804	482	272	166
H3 conc.	1837	1003	516	271	139	72
Width	1561	802	409	219	110	58
<b>WS, b=0.9</b>						
CO	1869	1064	566	306	170	92
CO conc.	1575	771	381	218	119	72
NO	1975	1083	528	238	101	56
NO conc.	1717	894	455	218	92	56
H3	1748	975	524	269	150	95
H3 conc.	1332	447	100	29	24	22
Width	1101	378	93	27	20	18
<b>WS, b=0.3</b>						
CO	816	434	242	133	78	37
CO conc.	184	122	57	38	24	17
NO	565	171	37	10	7	7
NO conc.	165	72	24	9	7	7
H3	534	299	180	96	34	34
H3 conc.	12	4	4	4	4	4
Width	12	4	4	4	4	4
<b>PB, Paths=100</b>						
CO	234	389	507	482	371	250
CO conc.	161	254	304	323	281	207
NO	305	504	550	512	370	238
NO conc.	205	343	440	448	343	227
H3	168	316	443	427	337	232
H3 conc.	125	141	153	142	120	89
Width	100	100	100	99	90	47

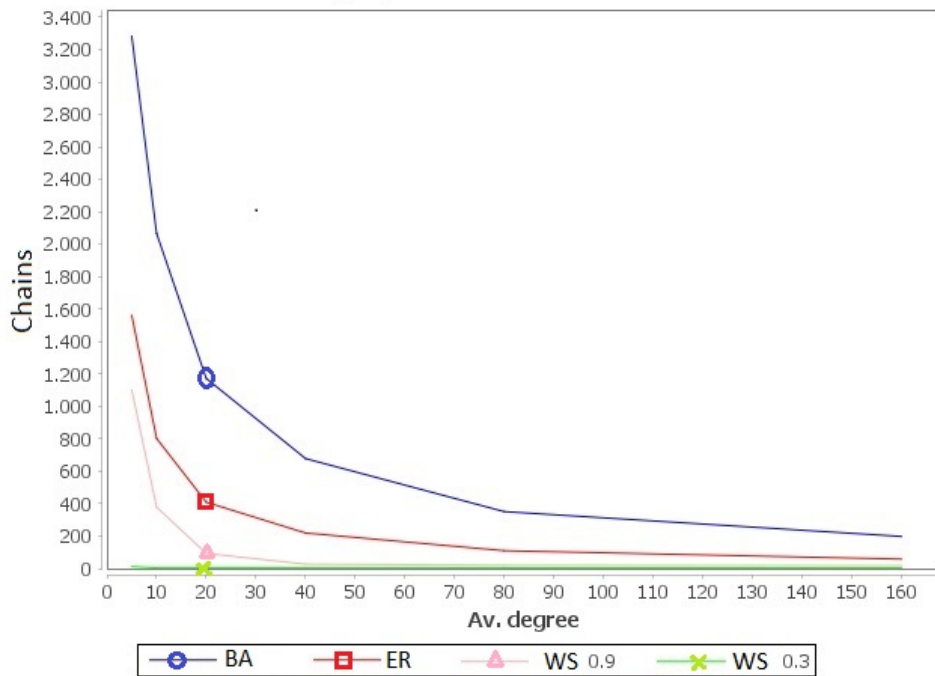
Table 2: Comparing the number of chains produced by the path and chain decomposition algorithms on graphs with 10000 nodes.

**|N| = 5000, Width**



(a) The width curve on graphs of 5000 nodes.

**|N| = 10000, Width**



(b) The width curve on graphs of 10000 nodes.

Figure 2: The width curve on graphs of 5000 and 10000 nodes using three different models.

**$|V| = 10000, PB\ Paths = 100, ER$**

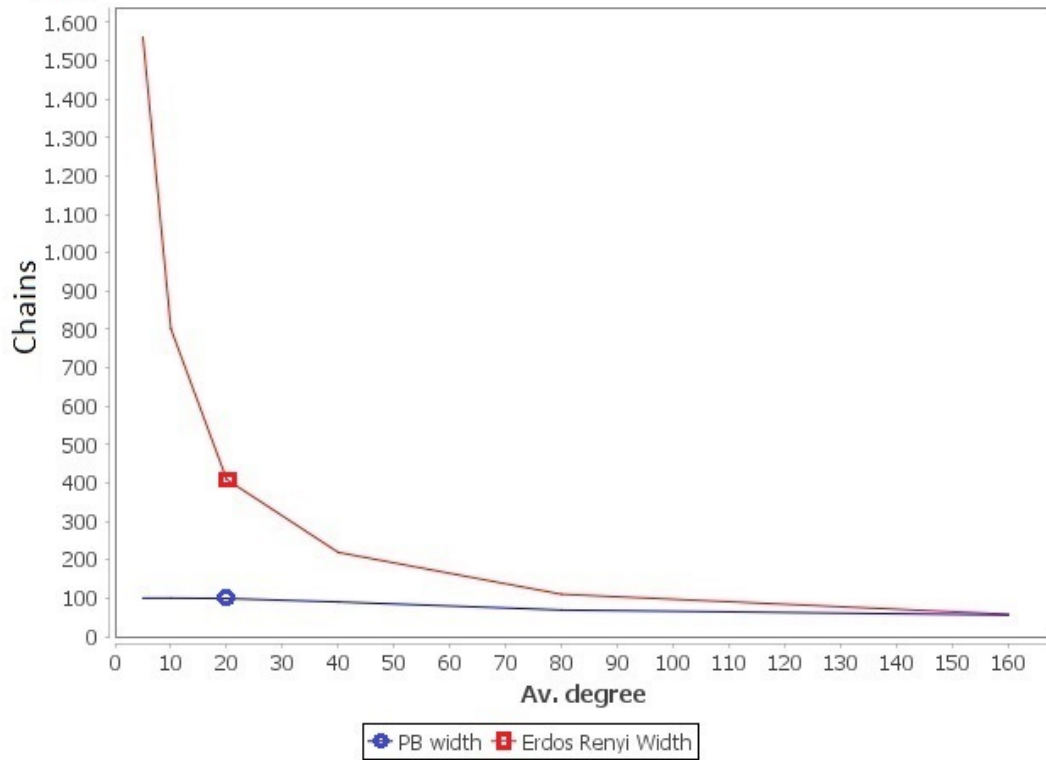


Figure 3: A comparison of width between ER model and PB model.

**$|N| = 10000, BA$**

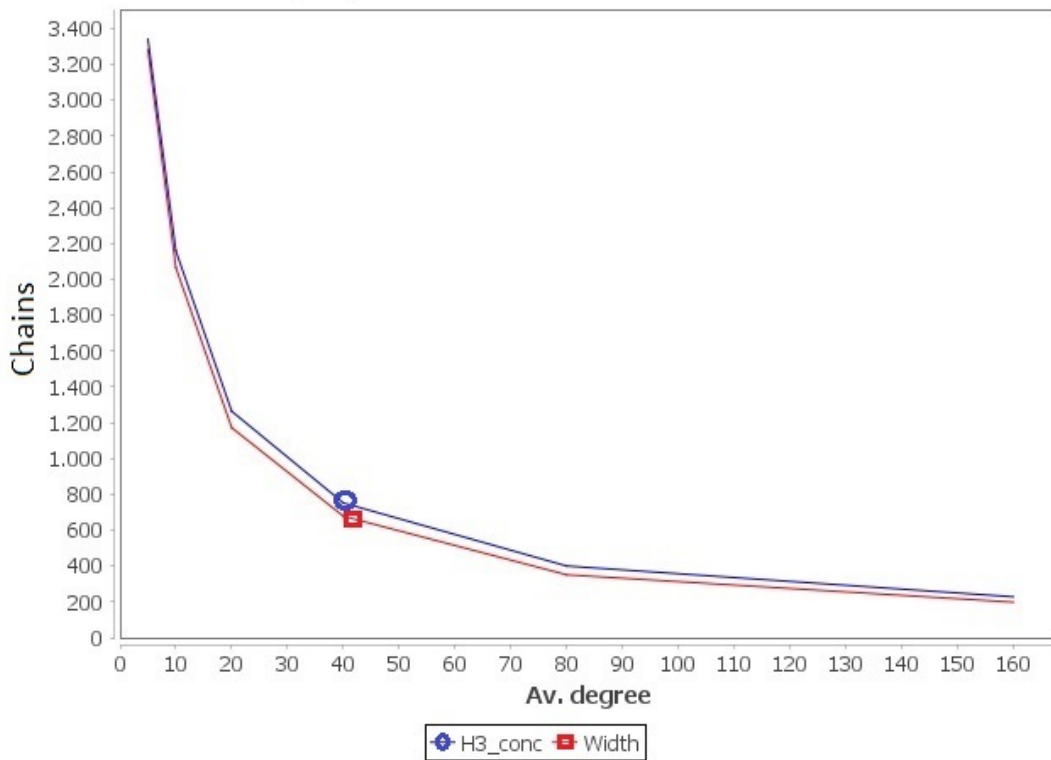


Figure 4: The efficiency of the chain decomposition algorithm in the BA model.

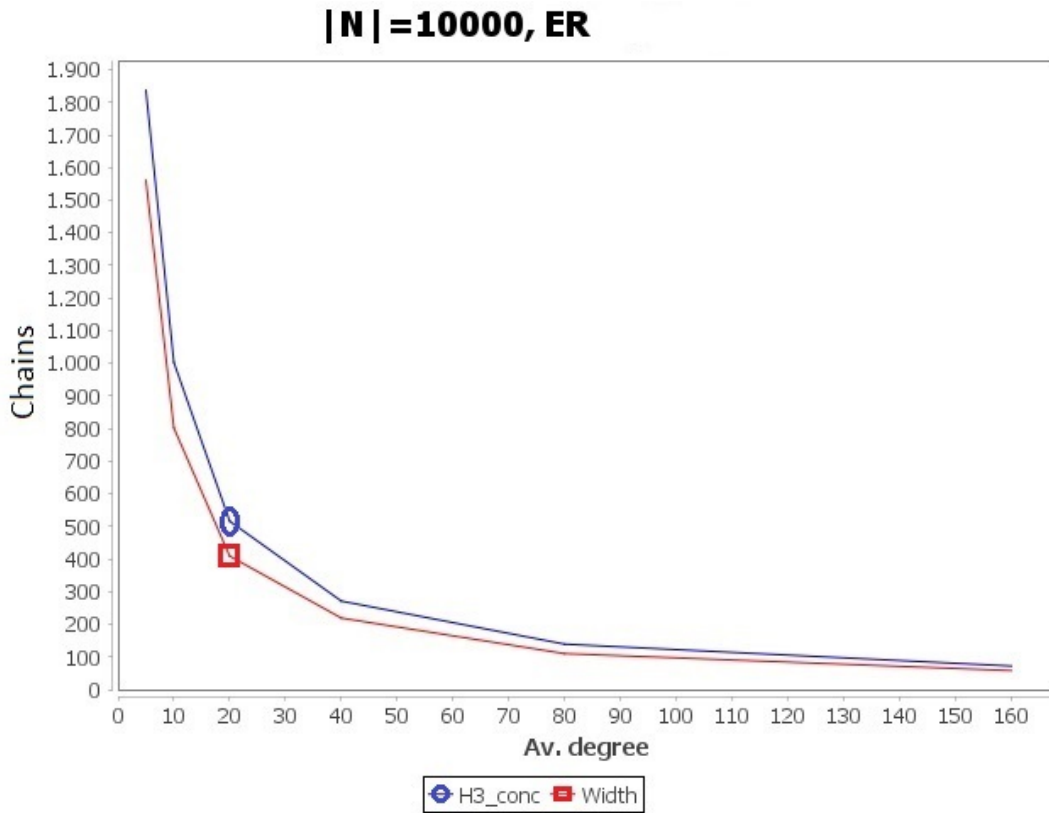


Figure 5: The efficiency of our chain decomposition algorithm in the ER model.

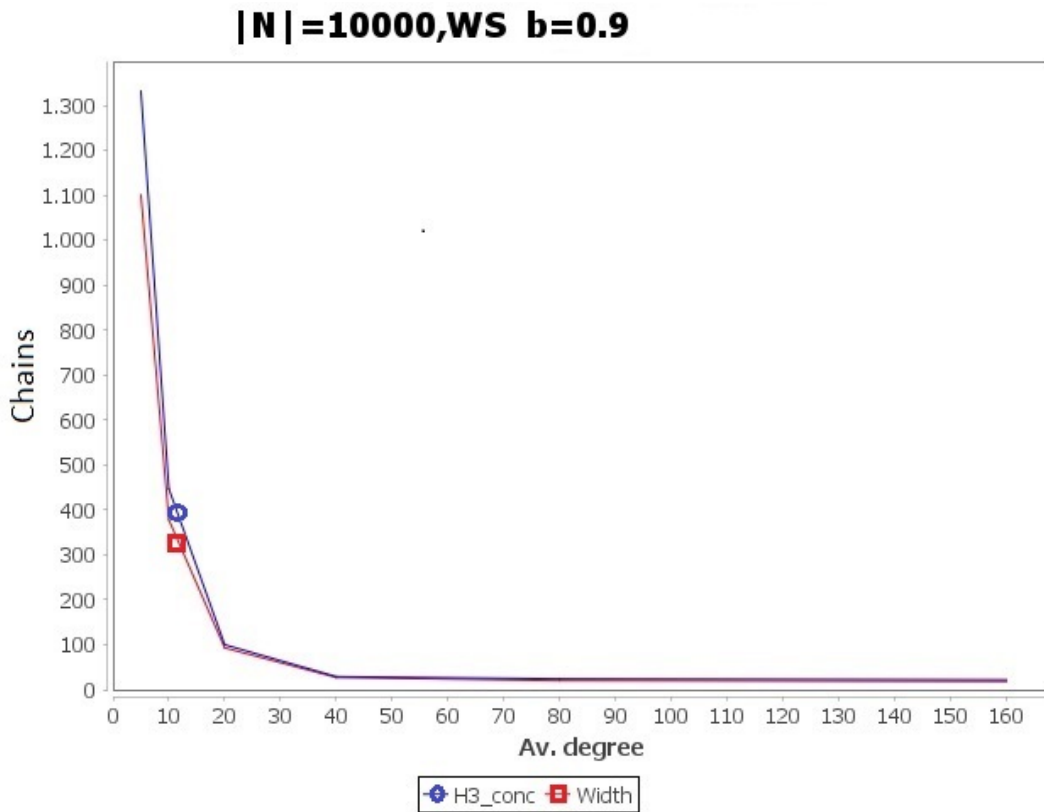


Figure 6: The efficiency of our chain decomposition algorithm in WS model.

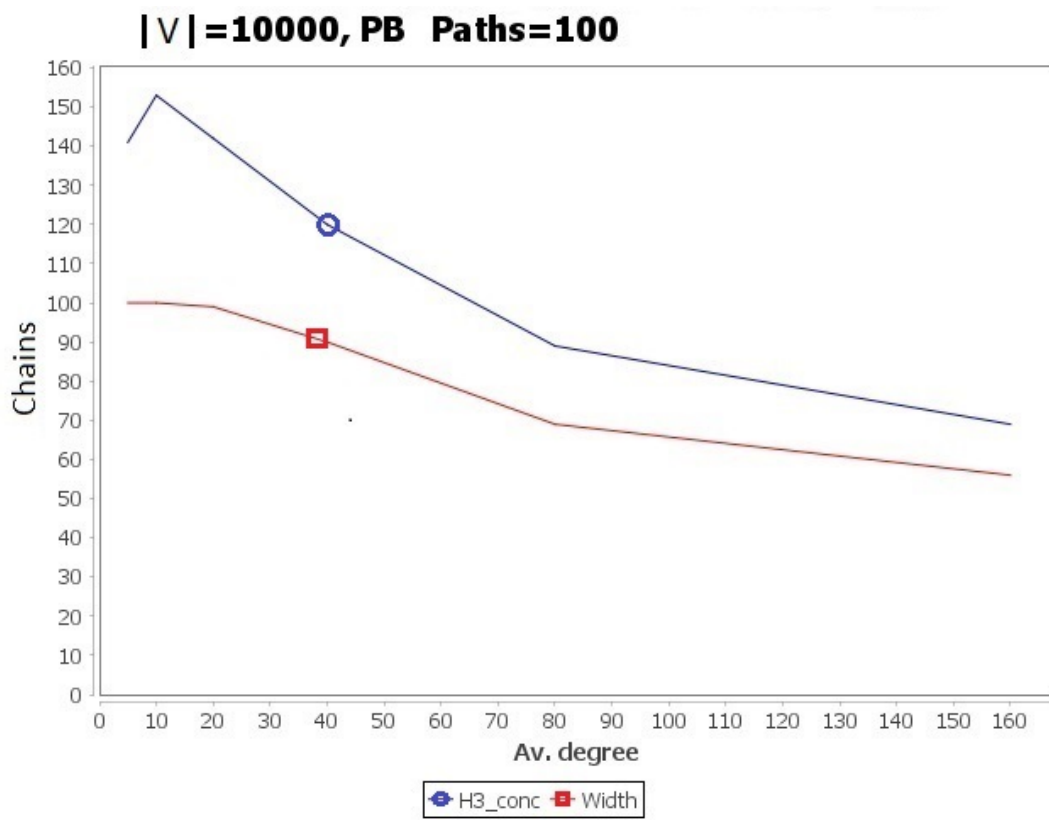


Figure 7: The efficiency of our chain decomposition algorithm in the PB model.

## 4 Hierarchies and Faster Transitivity

The importance of removing transitive edges in order to create an abstract graph utilizing paths and chains was first described in [23]. Their focus was on graph visualization techniques, while in this work we apply their abstraction to the transitive closure problem. Hence we state the following useful observations:

1. Given a chain decomposition  $D$  of a DAG  $G = (V, E)$ , each vertex  $v_i \in V$ ,  $0 \leq i < |V|$ , can have at most one outgoing non-transitive edge directed to a vertex of every other chain.
2. Given a chain decomposition  $D$  of a DAG  $G = (V, E)$ , each vertex  $v_i \in V$ ,  $0 \leq i < |V|$ , can have at most one incoming non-transitive edge directed from a vertex of every other chain.
3. Let  $G = (V, E)$  be a DAG with width  $w$ . The non-transitive edges of  $G$  are less than or equal to  $width * |V|$ , in other words  $|E_{red}| = |E| - |E_{tr}| \leq width * |V|$ .

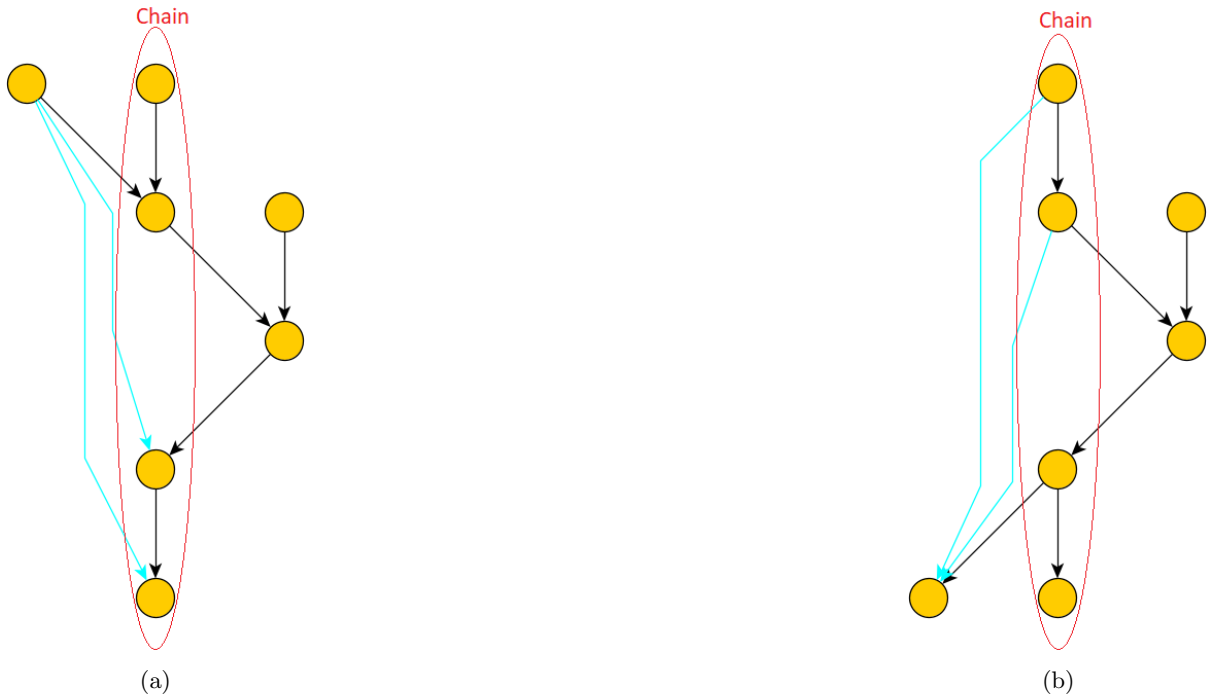


Figure 8: The light blue edges are transitive. (a) shows the outgoing transitive edges that end in the same chain. (b) shows the incoming transitive edges that start from the same chain.

An interesting application of the above observations is that we can find a significantly large subset of  $E_{tr}$  in linear time as follows: Given any chain (or path) decomposition with  $k_c$  chains, we can trace the vertices and their outgoing edges and keep the edges that point to the lowest point of each chain, rejecting the rest as transitive. We do the same for the incoming edges keeping the edges that come from the highest point (i.e., the vertex with the highest topological rank) of each chain. This way, we can find a significantly large subset  $E'_{tr} \subseteq E_{tr}$ . Hence,  $|E - E'_{tr}| \leq k_c * |V|$ . Clearly, this approach can be used as a linear-time preprocessing step in order to reduce the size of any DAG. Consequently, this will speed up every transitive closure algorithm bounding the number of edges of an input graph, and the indegree and outdegree of every vertex by  $k_c$ . For example, algorithms based on tree cover, see [1, 8, 30, 32], are practical on sparse graphs and can be enhanced further with such a preprocessing step that removes transitive edges. Additionally, this approach may have practical applications in dynamic transitive closure techniques: If we answer queries on time using graph traversal for every query, we could reduce the size of the graph with a fast (linear-time) preprocessing step that utilizes chains. Also, we could quickly decide if the edges we add and/or remove are transitive (or not). Transitive edges do not affect the transitive closure, hence no updates are required. This could be practically useful if we are adding/removing edges in a dynamic setting.



## 5 Reachability Indexing Scheme

In this section, we present an important application of a fast and practical chain decomposition technique. Namely, we solve the transitive closure problem by creating a reachability indexing scheme that is based on chain decomposition and we evaluate it by running extensive experiments. Our experiments shed light on the interplay of various important factors as the density of the graphs increases.

Jagadish described a compressed transitive closure technique in 1990 [18] applying the indexing scheme and path/chain decomposition. His method uses successor lists and focuses on the compression of the transitive closure. Thus his scheme does not answer queries in constant time. As discussed above, Jagadish’s heuristic for chain decomposition runs in  $O(n^2)$  time using a pre-computed transitive closure. Our technique runs in almost linear time without requiring a pre-computed transitive closure, and the result is close to the optimal. Simon [28], describes a technique similar to [18]. His technique is based on computing a path decomposition, thus boosting the method presented in [14]. The linear time heuristic used by Simon is very similar to the Chain Order Heuristic of [18], also described earlier. A different example is a graph structure referred to as path-tree cover introduced in [20], where the authors utilize a path decomposition algorithm to build their labeling.

In the following subsections, we describe how to compute an indexing scheme in  $O(k_c * |E_{red}|)$  time, where  $k_c$  is the number of chains (in any given chain decomposition) and  $|E_{red}|$  is the number of non-transitive edges. Following the observations of section 4, the time complexity of the scheme can be expressed as  $O(k_c * |E_{red}|) = O(k_c * width * |V|)$  since  $|E_{red}| \leq width * |V|$ . Our scheme utilizes arrays of indices, in a similar fashion as Simon’s technique [28], to answer queries in constant time. The space complexity is  $O(k_c * |V|)$ .

For our experiments we utilize our chain decomposition approach, which produces smaller decompositions than previous techniques, without any considerable run-time overhead. Thus the indexing scheme is more efficient both in terms of time and space requirements. The experimental work shows that the chains rarely have the same length. Usually, a decomposition consists of a few long chains and several short chains. Hence, for most graphs it is not even possible to have  $|E_{red}| = width * |V|$ . In fact,  $|E_{red}|$  is usually much lower than that. The experimental results presented in Tables 4 and 5 confirm this observation in practice.

Given a directed graph with cycles, we can find the strongly connected components (SCC) in linear time. Since any vertex is reachable from any other vertex in the same SCC (they form an equivalence class), all vertices in a SCC can be collapsed into a supernode. Hence, any reachability query can be reduced to a query in the resulting directed acyclic graph (DAG). This is a well-known step that has been widely used in many applications. Therefore, with loss of generality, we assume that the input graph to our method is a DAG. The following general steps describe how we compute the reachability indexing scheme:

1. Compute a Chain decomposition
2. Sort all Adjacency Lists
3. Create an Indexing Scheme

In Step 1, we use our chain decomposition technique that runs in  $O(|E| + c * l)$  time. In Step 2, we sort all the adjacency lists in  $O(|V| + |E|)$  time. Finally, we create an indexing scheme in  $O(k_c * |E_{red}|)$  time and  $O(k_c * |V|)$  space. Clearly, if the algorithm of Step 1 computes fewer chains then Step 3 becomes more efficient in terms of time and space.

### 5.1 The Indexing Scheme

Given any chain decomposition of a DAG  $G$  with size  $k_c$ , an indexing scheme will be computed for every vertex that includes a pair of integers and an array of size  $k_c$  of indexes. See for example Figure 9. The first integer of the pair indicates the node’s chain and the second its position in the chain. For example, vertex 1 of Figure 9 has a pair  $(1, 1)$ . This means that vertex 1 belongs to the 1st chain, and it is the 1st element in it. Given a chain decomposition, we can easily construct the pairs in  $O(|V|)$  time with a traversal of the chains. Every entry of the  $k_c$ -size array represents a chain. The  $i$ -th cell represents the  $i$ -th chain. The entry in the  $i$ -th cell corresponds to the lowest point of the  $i$ -th chain the vertex can reach. For example, the array of vertex 1 is  $[1, 3, 2]$ . The first cell of the array indicates that vertex 1 can reach the first vertex of the first chain (can reach itself, reflexive property). The second cell of the array indicates that vertex 1 can reach the third vertex of the second chain (There is a path from vertex

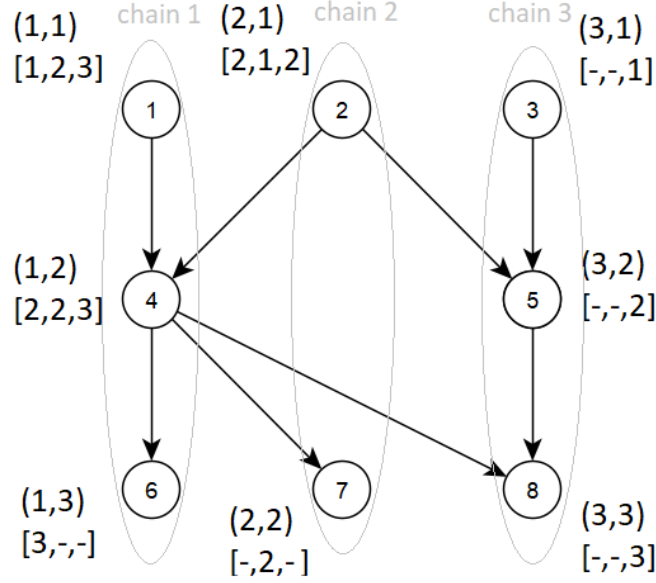


Figure 9: An example of an indexing scheme.

1 to vertex 7). Finally, the third cell of the array indicates that vertex 1 can reach the second vertex of the third chain.

Notice that we do not need the second integer of all pairs. If we know the chain a vertex belongs in, we can conclude its position using the array. We use this presentation to simplify the users' understanding.

The process of answering a reachability query is simple. Assume, there is a vertex  $s$  and a target vertex  $t$ . To find if the vertex  $t$  is reachable from the  $s$ , we get  $t$ 's chain, and we use it as an index in  $s$ 's array. Hence, we know the lowest point of  $t$ 's chain vertex  $s$  can reach.  $s$  can reach  $t$  if that point is less than or equal to  $t$ 's position, else it cannot.

## 5.2 Sorting Adjacency lists

Next we present an algorithm, Algorithm 6, that sorts all the adjacency lists of immediate successors in ascending topological order in linear time. The variable `stack` indicates the sorted adjacency list. The algorithm traverses the vertices in reverse topological order ( $v_n, \dots, v_1$ ). For every vertex  $v_i$ ,  $1 \leq i \leq n$ , it pushes  $v_i$  in the stacks of all immediate predecessors. This step may be performed even before the chain decomposition step, as a preprocessing step. We present it in this section to emphasize its crucial role in the efficient creation of the indexing scheme. If the adjacency lists are not sorted then the time complexity of the algorithm would be  $O(k_c * |E|)$  instead of  $O(k_c * |E_{red}|)$ .

---

### Algorithm 6 Sorting Adjacency lists

---

```

1: procedure SORT( $G, t$ )
   INPUT: A DAG  $G = (V, E)$ 
2:   for each vertex:  $v_i \in G$  do
3:      $v_i$ .stack  $\leftarrow$  new stack()
4:   end for
5:   for each vertex  $v_i$  in reverse topological order do
6:     for every incoming edge  $e(s_j, v_i)$  do
7:        $s_j$ .stack.push( $v_i$ )
8:     end for
9:   end for
10: end procedure

```

---

**Lemma 5.1.** *Algorithm 6 sorts the adjacency lists of immediate successors in ascending topological order.*

*Proof.* Assume that there is a stack  $(u_1, \dots, u_n)$ ,  $u_1$  is the top of the stack. Assume that there is a pair  $(u_j, u_k)$  in the stack, where  $u_j$  has a bigger topological rank than  $u_k$  and  $u_j$  precedes  $u_k$ . That means that the for-loop examined  $u_j$  before  $u_k$  since it goes through the vertices in reverse topological order. This is a contradiction. The vertex  $u_j$  cannot precede  $u_k$  if it were examined first by the for-loop.  $\square$

### 5.3 Creating the Indexing Scheme

Now we present Algorithm 7 that constructs the indexing scheme. The first for-loop initializes the array of indexes. For every vertex, it initializes the cell that corresponds to its chain. The rest of the cells are initialized to infinite. The indexing scheme initialization is illustrated in figure 10. The dashes represent the infinite. Notice that after the initialization, the indexes of all sink vertices have been calculated. Since a sink has no successors, the only vertex it can reach is itself.

---

#### Algorithm 7 Indexing Scheme

---

```

1: procedure CREATE INDEXING SCHEME( $G, T, D$ )
   INPUT: A DAG  $G = (V, E)$ , a topological sorting  $T$  of  $G$ , and the decomposition  $D$  of  $G$ .
2:   for each vertex:  $v_i \in G$  do
3:      $v_i$ .indexes  $\leftarrow$  new table[size of  $D$ ]
4:      $v_i$ .indexes.fill( $\infty$ )
5:      $ch\_no \leftarrow v_i$ 's chain index
6:      $pos \leftarrow v_i$ 's chain position
7:      $v_i$ .indexes[  $ch\_no$  ]  $\leftarrow pos$ 
8:   end for
9:   for each vertex  $v_i$  in reverse topological order do
10:    while  $v_i$ .stack  $\neq \emptyset$  do
11:       $target \leftarrow v_j$ .stack.pop()
12:       $t\_ch \leftarrow target$ 's chain index
13:       $t\_pos \leftarrow target$ 's chain position
14:      if  $t\_pos < v_i$ .indexes[ $t\_ch$ ] then  $\triangleright (v_i, target)$  is not transitive
15:         $v_i$ .updateIndexes( $target$ .indexes)
16:      end if
17:    end while
18:   end for
19: end procedure

```

---

The second for-loop builds the indexing scheme. It goes through vertices in descending topological order. For each vertex, it visits its immediate successors (outgoing edges) in ascending topological order and updates the indexes. Suppose we have the edge  $(v, s)$ , and we have calculated the indexes of vertex  $s$  ( $s$  is immediate successor of  $v$ ). The process of updating the indexes of  $v$  with its immediate successor  $s$  means that  $s$  will pass all its information to the vertex  $v$ . Hence, vertex  $v$  will be aware that it can reach  $s$  and all its successors. Assume the array of indexes of  $v$  is  $[a_1, a_2, \dots, a_{k_c}]$  and the array of  $s$  is  $[b_1, b_2, \dots, b_{k_c}]$ . To update the indexes of  $v$  using  $s$ , we merely trace the arrays and keep the smallest values. For every pair of indexes  $(a_i, b_i)$ ,  $0 \leq i < k_c$ , the new value of  $a_i$  will be  $\min\{a_i, b_i\}$ . This process needs  $k_c$  steps.

**Lemma 5.2.** *Given a vertex  $v$  and the calculated indexes of its successors, the while-loop of algorithm 7 (lines 10-17) calculates the indexes of  $v$  by updating its array with its non-transitive outgoing edges' successors.*

*Proof.* Updating the indexes of vertex  $v$  with all its immediate successors will make  $v$  aware of all its descendants. The while-loop of Algorithm 7 does not perform the update function for every direct successor. It skips all the transitive edges. Assume there is such a descendant  $t$  and the transitive edge  $(v, t)$ . Since the edge is transitive, we know by definition that there exists a path from  $v$  to  $t$  with a length of more than 1. Suppose that the path is  $(v, v_1, \dots, t)$ . Vertex  $v_1$  is a predecessor of  $t$  and immediate successor of  $v$ . Hence it has a lower topological rank than  $t$ . Since, while-loop examines the incident vertices in ascending topological order, then vertex  $t$  will be visited after vertex  $v_1$ . The opposite leads to a contradiction. Consequently, for every incident transitive edge of  $v$ , the loop firstly visits a vertex  $v_1$  which is a predecessor of  $t$ . Thus vertex  $v$  will be firstly updated by  $v_1$  and it will record the edge

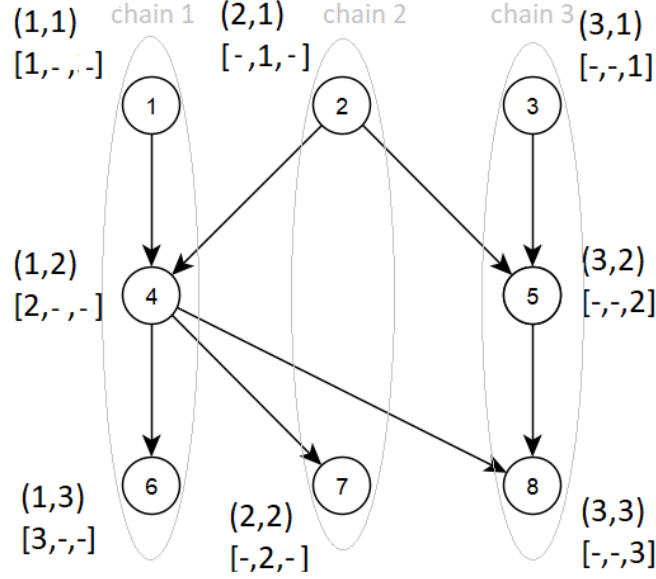


Figure 10: Initialization of indexes.

$(v, t)$  as transitive. There is no reason to update vertex  $v$  indexes with those of vertex  $t$  since the indexes of  $t$  will be greater or equal.  $\square$

**Theorem 5.3.** *Let  $G = (V, E)$  be a DAG. Algorithm 7 computes an indexing scheme for  $G$  in  $O(k_c * |E_{red}| + |E_{tr}|)$  time.*

*Proof.* In the initialization step, the indexes of all sink vertices have been computed as we described above. Taking vertices in reverse topological order, the first vertex we meet is a sink vertex. When the for-loop of line 9 visits the first non-sink vertex, the indexes of its successors are computed (all its successors are sink vertices). According to Lemma 1, we can calculate its indexes, ignoring the transitive edges. Assume the for-loop has reached vertex  $v_i$  in the  $i$ th iteration, and the indexes of its successors are calculated. Following Lemma 1, we can calculate its indexes. Hence, by induction, we can calculate the indexes of all vertices, ignoring all  $|E_{tr}|$  transitive edges in  $O(k_c * |E_{red}| + |E_{tr}|)$  time.  $\square$

As described in the introduction, a parameterized linear-time algorithm for computing the minimum number of chains was recently presented in [7]. Its time complexity is  $O(k^3|V| + |E|)$  where  $k$  is the minimum number of chains, which is equal to the width of  $G$ . Using their result Algorithm 7 computes an indexing scheme for  $G$  in parameterized linear time. This implies that the transitive closure of  $G$  can be computed in parameterized linear time. Hence we have the following:

**Corollary 5.4.** *Let  $G = (V, E)$  be a DAG. Algorithm 7 computes an indexing scheme for  $G$  in parameterized linear time. Hence the transitive closure of  $G$  can be computed in parameterized linear time.*

## 5.4 Experimental Results

We conducted an extensive experimental evaluation of our techniques and we report the results here. Tables 3a and 3b include the number of chains and real running time in ms, for the computation of an indexing scheme using two indicative decomposition techniques on ER graphs of 10000 nodes with varying average degree. In Table 3a, we have created the indexing scheme using the chain order heuristic (a path decomposition), while in Table 3b, we use our chain decomposition algorithm. Observe that even if the required time to compute the chain decomposition is higher, the total time is about the same, which shows the growing efficiency of the algorithms as the number of chains becomes smaller.

Next, we performed experiments using the same graphs of 5000 and 10000 nodes as we described in Section 3.3 produced by the three different models of the Networkx [16] and the Path-Based model [24]. We computed a chain decomposition using our best performing approach, Algorithm 5, H3\_conc, and created an indexing scheme using Algorithm 7. For simplicity, we assume that the input graph has sorted

Av. Degree	Chains	CO Time (ms)	Indexing Scheme Time(ms)	Total
5	2283	8	237	246
10	1432	11	221	231
20	871	10	170	180
40	513	12	152	164
80	294	15	162	177
160	165	21	278	299

(a) Metrics: Creating the indexing scheme in combination with the chain order heuristic.

Av. Degree	Chains	H3_conc Time (ms)	Indexing Scheme Time(ms)	Total
5	1837	9	194	203
10	1003	11	163	174
20	516	16	100	116
40	271	39	108	147
80	139	43	130	173
160	72	75	237	312

(b) Metrics: Creating the indexing scheme in combination with algorithm 5 for chain decomposition.

Table 3: Run times of the indexing scheme using path and chain decomposition.

adjacency lists, having ran the sorting of the adjacency lists, Algorithm 6, as a preprocessing step. We report our experimental results in Tables 4 and 5 for graphs with 5000 nodes and graphs with 10000 nodes, respectively. The meaning of the columns of the tables is as follows:

- **Av. Degree:** The average degree of the graph.
- **Chains:** Number of chains computed by our heuristic (H3\_conc).
- $|E_{tr}|$ : Number of transitive edges.
- $|E_{red}|$ : Number of non-transitive edges.
- $|E_{tr}|/|E|$ : The percentage of transitive edges.
- **H3\_conc Time (ms):** The running time for the chain decomposition step.
- **Indexing Scheme Time (ms):** The running time for the indexing scheme creation step.
- **Total:** The total time(ms) needed to decompose the graph and create the indexing scheme. It is the sum of the two preceding cells.
- **TC:** The time needed to perform  $n$  DFS (or BFS) starting from every vertex to mark the reachable vertices; the time complexity is  $O(|V| * |E|)$ . The results are stored in a 2-dimensional adjacency matrix.

In theory, the phase of the indexing scheme creation needs  $O(k_c * |E_{red}| + |E_{tr}|)$  steps. However, the numbers on the tables reveal some interesting findings: (a) as the average degree increases and the graph becomes denser, the cardinality of  $E_{red}$  remains almost stable; (b) the number of chains decrease; (c) we observe that the number of non-transitive edges,  $E_{red}$ , does not vary significantly as the average degree increases, i.e., the number of the transitive edges,  $|E_{tr}|$ , increases since  $(E_{tr} = E - E_{red})$ . Since the algorithm merely traces in linear time the transitive edges, the growth of  $|E_{tr}|$  affects the run time only linearly. As a result, the run time of our technique does not increase as the the size of the (edges)

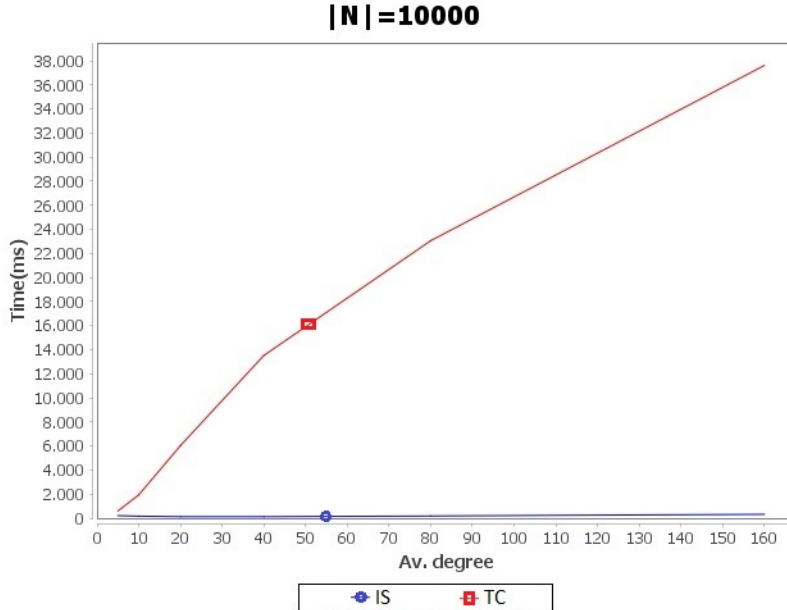


Figure 11: Run time comparison between the Indexing Scheme (blue line) and TC (red line) for ER model on graphs of 10000 nodes, see Table 5.

input graph increases. To demonstrate it clearly, we show the curves of the running time in Figure 11 for the graphs of 10000 nodes produced by the ER model. The flat (blue line) represents the run time of the indexing scheme, and the curve (red line) the run time of the DFS-based algorithm for computing the transitive closure (TC). Clearly, the time of the DFS-based algorithm increases as the average degree increases, while the time of the indexing scheme is a straight line parallel to the  $x$ -axis. All models follow this pattern, see Tables 4 and 5.

We chose to decompose the graph into chains with our most efficient algorithm since it produces the fewest chains among all heuristics. Clearly, there is a trade-off to consider when building an indexing scheme. Assume that we have a path decomposition, and then we perform chain concatenation. If there is no concatenation between two paths, the concatenation algorithm will run in linear time. On the other hand, if there are concatenations, for each of the paths, then the cost is  $O(l)$  time, but the savings in the indexing scheme creation is  $\Theta(|V|)$  in space requirements and  $\Theta(|E_{red}|)$  in time, since every concatenation reduces the needed index size for every vertex by one. Hence, in comparison with doing merely a path decomposition, applying path concatenation, we create a more compact indexing scheme faster. Another interesting and to some extent surprising observation that comes from the results of Tables 4 and 5 is that the transitive edges for almost all models of the graphs of 5000 and 10000 nodes with average degree above 20 is above 85%, i.e.,  $|E_{tr}|/|E| \geq 85\%$ , see the appropriate columns in both tables. In some cases where the graphs are a bit denser the percentage grows above 95%. This has important implications in designing practical algorithms for faster transitive closure computation.

## 6 Conclusions

We presented heuristics that find a chain decomposition in almost linear time such that the number of chains is very close to the minimum and investigate how fast and practical chain decomposition algorithms can enhance transitive closure solutions. Our extensive experiments expose the practical behavior of (1) the width, (2)  $E_{red}$ , and (3)  $E_{tr}$  as the density of graphs grows. The also show the practical efficiency of our heuristics. We show that the set  $E_{red}$  is bounded by  $width * |V|$  and show how to find a substantially large subset of  $E_{tr}$  in linear time given any path/chain decomposition. Finally, we build and evaluate an indexing scheme that allows us to answer reachability queries in constant time. The time complexity to produce the scheme is  $O(k_c * |E_{red}|)$ , and its space complexity is  $O(k_c * |V|)$ . Our experimental work reveals the practical efficiency of this approach, especially for very large graphs with relatively high average degree. Using recent results on parameterized linear time algorithms for computing the minimum number of chains, we showed that the transitive closure of a DAG can be

$|V|=5000$

Av. Degree	Chains	$ E_{tr} $	$ E_{red} $	$ E_{tr} / E $	H3_conc Time (ms)	Indexing Scheme Time(ms)	Total	TC
<b>BA</b>								
5	1630	8054	18921	0.32	3	101	104	137
10	1055	28230	21670	0.57	12	79	91	333
20	664	75801	23799	0.76	6	54	60	638
40	355	180815	22504	0.89	10	48	58	1418
80	207	382422	20854	0.95	122	118	240	3018
160	163	770771	17660	0.98	25	107	132	5464
<b>ER</b>								
5	923	3440	21466	0.14	6	67	73	172
10	492	24761	25425	0.49	10	51	61	487
20	252	75312	24646	0.75	5	26	31	1079
40	139	175809	22634	0.89	46	51	97	2896
80	70	378015	19435	0.95	16	50	66	5260
160	38	769919	16843	0.98	98	138	236	8609
<b>WS, b=0.9</b>								
5	687	7742	17258	0.30	13	71	84	393
10	212	37992	12008	0.76	11	18	29	817
20	60	89272	10728	0.89	23	22	45	1530
40	25	186486	13514	0.93	47	45	92	3704
80	20	386294	13706	0.97	115	103	218	6172
160	17	787066	12934	0.98	253	207	406	9173
<b>WS, b=0.3</b>								
5	9	18421	6579	0.74	11	8	19	910
10	4	43505	6495	0.87	8	11	19	1107
20	4	93490	6510	0.93	18	18	36	2176
40	5	193416	6584	0.97	17	18	35	4753
80	4	393348	6652	0.98	98	82	180	7949
160	5	793430	6570	0.99	250	166	416	11757
<b>PB, Paths=70</b>								
5	86	14155	10809	0.57	8	7	15	206
10	101	36801	13102	0.74	7	12	19	313
20	107	84168	15419	0.85	7	15	22	890
40	93	181388	16988	0.91	49	216	265	2584
80	73	376220	17303	0.96	128	163	291	4603
160	51	758207	16566	0.98	55	141	196	9358

Table 4: Indexing scheme analysis on graphs of 5000 nodes.

|V|=10000

Av. Degree	Chains	E <sub>tr</sub>	E <sub>red</sub>	E <sub>tr</sub>  / E	H3_conc Time (ms)	Indexing Scheme Time(ms)	Total	TC
<b>BA</b>								
5	3341	14544	35431	0.29	7	278	285	441
10	2159	53503	46397	0.54	14	231	245	1379
20	1264	147791	51809	0.74	15	218	233	3347
40	752	355854	52465	0.85	28	188	216	7700
80	400	764926	48350	0.94	271	322	593	14632
160	228	1560464	42967	0.97	81	264	345	24601
<b>ER</b>								
5	1837	5595	44401	0.11	12	200	212	600
10	1003	44813	55366	0.45	9	161	170	1935
20	516	144276	55310	0.72	16	110	126	6031
40	271	347323	52620	0.87	25	101	126	13522
80	139	749781	46666	0.94	40	145	185	23052
160	72	1548153	39710	0.97	73	249	322	37613
<b>WS, b=0.9</b>								
5	1332	13353	36647	0.27	12	175	187	1213
10	447	74782	25218	0.75	9	53	62	3829
20	100	178930	21070	0.89	13	32	45	9279
40	29	373054	26946	0.93	24	60	84	13144
80	24	771374	28626	0.96	266	247	513	25585
160	22	1571957	28043	0.98	80	232	312	36507
<b>WS, b=0.3</b>								
5	12	36816	13184	0.73	27	19	46	3468
10	4	86804	13196	0.86	18	45	63	5063
20	4	186756	13244	0.93	10	42	52	12156
40	4	386751	13249	0.97	19	48	67	21055
80	4	786840	13160	0.98	237	187	424	31016
160	4	1586896	13104	0.99	62	167	229	40704
<b>PB, Paths=100</b>								
5	125	8182	16810	0.33	12	16	28	240
10	141	74182	25722	0.74	11	30	41	937
20	153	168839	30728	0.85	13	43	56	5015
40	142	363753	34606	0.91	27	78	105	13797
80	120	756578	36918	0.96	56	142	198	27904
160	89	1538101	36496	0.98	77	265	342	41235

Table 5: Indexing scheme analysis on graphs of 10000 nodes.



computed in parameterized linear time.

The potential applications of these techniques in a dynamic setting where edges and nodes are inserted and deleted from a (very large) graph are significant in practice. Although our techniques were not developed for the dynamic case, the picture that emerges is very interesting. According to our experimental results, see Tables 4 and 5, the overwhelming majority of edges in a DAG are transitive. The insertion or deletion of a transitive edge clearly requires a constant time update since it does not affect transitivity, and can be detected in constant time. On the other hand, the insertion or removal of a non-transitive edge may require a minor or major recomputation in order to reestablish a good chain decomposition. However, even if the insertion/deletion of new nodes/edges causes significant changes in the reachability index (transitive closure) one can simply recompute a chain decomposition in linear or almost linear time, and then recompute the reachability scheme in  $O(k_c * |E_{red}|)$  time, and  $O(k_c * |V|)$  space, which is still very efficient in practice. We plan to work on such dynamic indexes in the future.

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