# System Specification, Verification and Synthesis (SSVS) – CS 4830/7485, Fall 2019

## 11: Formal Specification:
## Temporal logic
## CTL


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# (A philosophical note)

- Your dreams, aspirations, goals in life: liveness
- Your fears: safety

# BRANCHING-TIME PROPERTIES

# Linear-Time vs. Branching-Time Properties

So far we have been talking about properties of **linear** behaviors (sequences).

But some properties are not linear, e.g.:

*"it is possible to recover from any fault"*

or

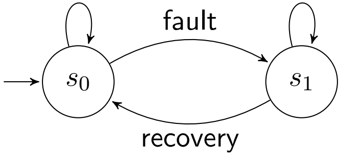
*"we can get back to the initial state from any reachable state"*

# Linear-Time vs. Branching-Time Properties

*"it is possible to recover from any fault"*

Based on *one* (linear) behavior alone,[1] we cannot conclude whether our system satisfies the property.

E.g., the following system satisfies the property, although it contains a behavior that stays forever in state $s_1$:




[1] if we had *all* linear behaviors of a system, we could in principle reconstruct its branching behavior as well – how?

# Linear-Time vs. Branching-Time Behaviors

Linear-time behavior = infinite sequence.

Branching-time behavior = infinite **tree**.

Hence the name "Computation Tree Logic" – CTL.

# Defining the semantics of CTL

We could:

1. define the semantics of CTL on trees,
2. define the "unfolding" of a transition system into a tree (or forest of trees, in case there are many initial states),
3. define what it means for a transition system to satisfy a CTL formula: its forest satisfies the formula.

Instead:

- we will simplify and define the semantics of CTL directly on the transition system (Kripke structure).

# CTL (Computation Tree Logic) – Syntax

There are two kinds of CTL formulas: state formulas and path formulas. When we just say "CTL formula" we mean CTL state formula.

- CTL **state formulas** are defined by the following grammar:

$$\begin{aligned} \phi \quad ::= \quad & p \mid q \mid ..., \text{ where } p, q, ... \in \mathsf{AP} \\ & \mid \phi_1 \wedge \phi_2 \mid \neg\phi_1 \mid \mathbf{E}\psi \mid \mathbf{A}\psi \end{aligned}$$

  where $\psi$ must be a path formula, and $\phi_1, \phi_2$ must be state formulas.

- CTL **path formulas** are defined by the following grammar:

$$\psi \quad ::= \quad \mathbf{X}\phi \quad \mid \quad \phi_1 \, \mathbf{U} \, \phi_2$$

  where $\phi, \phi_1, \phi_2$ must all be state formulas.

# CTL Syntax: Notes

- $\mathbf{E}$ ("there exists a path") and $\mathbf{A}$ ("for all paths") are called **path quantifiers**.
- As usual, we can use any Boolean operator $\vee, \rightarrow, \leftrightarrow$, etc., as abbreviation / syntactic sugar.
- Similarly, we can also use the temporal operators $\mathbf{G}$ and $\mathbf{F}$ in CTL path formulas.
  For example, $\mathbf{EF}p \equiv \mathbf{E}(true\,\mathbf{U}\,p)$, $\mathbf{AF}p \equiv$

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  $\mathbf{AG}p \equiv \neg\mathbf{EF}\neg p$, $\mathbf{EG}p \equiv \neg\mathbf{AF}\neg p$, etc.
- Alternative syntax: $\forall\Box$ instead of $\mathbf{AG}$, $\exists\Diamond$ instead of $\mathbf{EF}$, etc.

# CTL (Computation Tree Logic) – Syntax

Examples of (syntactically correct) CTL formulas:

$$\mathbf{AG}p$$

$$\mathbf{EF}q$$

$$\mathbf{AGEF}(p \rightarrow q)$$

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Syntactically incorrect CTL formulas:

$$\mathbf{G}p, \quad \mathbf{AGF}p, \quad (\mathbf{AG}p) \wedge \mathbf{F}q, \quad \mathbf{AEG}p, \quad \mathbf{A}p, \quad \mathbf{A}\neg\mathbf{F}p$$

# CTL – Semantics: Intuition

Let $s$ be a state of the Kripke structure.

Then $s$ satisfies the CTL formula $\mathbf{EG}\phi$, written

$$s \models \mathbf{EG}\phi$$

iff **there exists** an infinite path starting from $s$ and satisfying $\mathbf{G}\phi$.

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iff **every** infinite path starting from $s$ satisfies $\mathbf{G}\phi$.

# Examples

Let's construct transition systems (Kripke structures) satisfying or violating the following CTL formulas:

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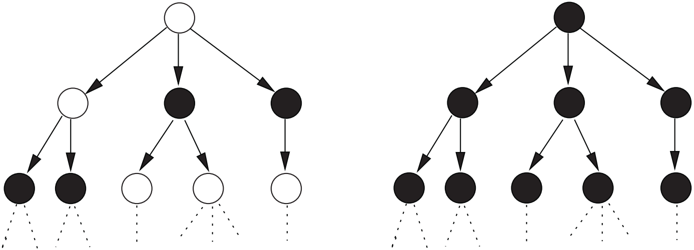
# CTL Semantics – Illustration

Figures taken from [Baier and Katoen, 2008]



$\forall\Diamond\, black$ $\qquad$ $\forall\Box\, black$

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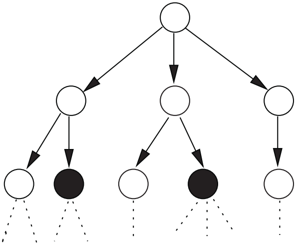
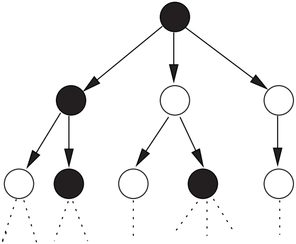


$\exists\Diamond\, black$



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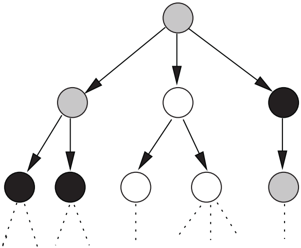
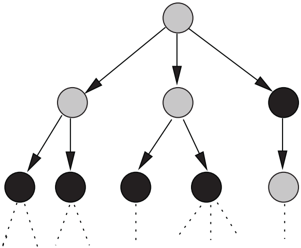


$\exists(gray \, \mathsf{U} \, black)$



$\forall(gray \, \mathsf{U} \, black)$

# CTL – Formal Semantics

The satisfaction relation $\models$ for CTL depends on the kind of CTL formula:

- CTL state formulas are evaluated on states: if $s$ is a state of the transition system, and $\phi$ is a CTL state formula, we must define what $s \models \phi$ means.
- CTL path formulas are evaluated on infinite paths (similar to LTL): if $\pi$ is an infinite path in the transition system, and $\psi$ is a CTL path formula, we must define what $\pi \models \psi$ means.

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Let $(\mathsf{AP}, S, S_0, L, R)$ be a Kripke structure and let $s \in S$.

- Recall: a path $\pi$ starting from $s$ is an infinite sequence of states and transitions: $\pi = s \to s_1 \to s_2 \to \cdots$
- $\pi(i)$ denotes the $i$-th state in the path, $s_i$, with $\pi(0) = s$.
- Let $Paths(s)$ denote the set of all paths starting from $s$.

# CTL – Formal Semantics

Let $(\mathsf{AP}, S, S_0, L, R)$ be a Kripke structure and let $s \in S$.

Satisfaction relation for CTL state formulas:

$$
\begin{array}{lll}
s \models p & \text{iff} & p \in L(s) \\
s \models \phi_1 \wedge \phi_2 & \text{iff} & s \models \phi_1 \text{ and } s \models \phi_2 \\
s \models \neg\phi & \text{iff} & s \not\models \phi \\
s \models \mathbf{E}\psi & \text{iff} & \exists \pi \in Paths(s) : \pi \models \psi \\
s \models \mathbf{A}\psi & \text{iff} & \forall \pi \in Paths(s) : \pi \models \psi
\end{array}
$$

Satisfaction relation for CTL path formulas (similar to LTL):

$$
\begin{array}{lll}
\pi \models \mathbf{X}\phi & \text{iff} & \pi(1) \models \phi \\
\pi \models \phi_1 \, \mathbf{U} \, \phi_2 & \text{iff} & \exists i \geq 0 : \pi(i) \models \phi_2 \wedge \forall 0 \leq j < i : \pi(j) \models \phi_1
\end{array}
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How would you express the last two in LTL?

We will see that when we compare LTL and CTL.

# THE MODEL-CHECKING PROBLEM FOR CTL

# The verification problem for CTL: CTL model checking

The **CTL model checking problem**: does a given transition system (Kripke structure) $M$ satisfy a given CTL (state) formula $\phi$?

Let $M = (\mathsf{AP}, S, S_0, L, R)$.
$S_0$ is a set, so $M$ generally has many initial states.

We want **every initial state** of $M$ to satisfy $\phi$:

$$\forall s \in S_0 : s \models \phi$$

We write this as:

$$M \models \phi$$

(same notation as in LTL model-checking, but here $\phi$ is a CTL formula).

# LTL vs CTL: EXPRESSIVENESS COMPARISON

# Formula equivalence

- Recall: When are two formulas $\phi_1, \phi_2$ in the same logic, say LTL, **equivalent**?

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  - When $\forall \sigma \in \Sigma^\omega : \sigma \models \phi_1 \Leftrightarrow \sigma \models \phi_2$.
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- Can we compare LTL and CTL formulas for equivalence?
  What would it even mean, since LTL is linear-time and CTL is branching-time?
  Idea: compare the transition systems that satisfy these formulas!
- Let $\phi_1$ be an LTL formula and $\phi_2$ be a CTL formula.
  We say that $\phi_1$ and $\phi_2$ are equivalent if **for any Kripke structure** $TS$**:** $TS \models \phi_1 \Leftrightarrow TS \models \phi_2$.

# Examples of equivalent formulas

| LTL formula | Equivalent CTL formula |
|---|---|
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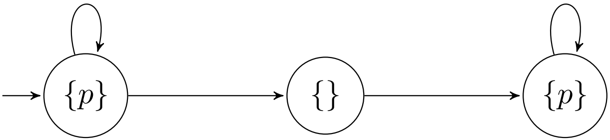
| $\mathbf{FG}p$ | $\mathbf{AFAG}p$ ??? **NO! Argh!** |

# $\mathbf{FG}p$ and $\mathbf{AFAG}p$ are **not** equivalent

Here's a transition system that distinguishes them:



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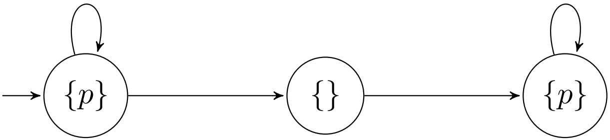


The above transition system satisfies $\mathbf{FG}p$ but violates $\mathbf{AFAG}p$.

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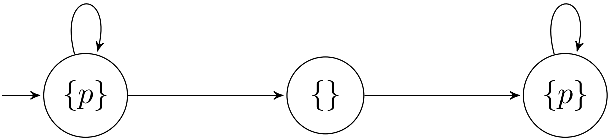


The above transition system satisfies $\mathbf{FG}p$ but violates $\mathbf{AFAG}p$.

**Homework**: Is there a transition system that satisfies $\mathbf{AFAG}p$ but violates $\mathbf{FG}p$?

# LTL and CTL are incomparable in terms of expressiveness

**Theorem**

*There is no CTL formula equivalent to the LTL formula* $\mathbf{FG}p$.

**Theorem**

*There is no LTL formula equivalent to the CTL formula* $\mathbf{AGEF}p$.

Proofs: on whiteboard.

# CTL: historical and other remarks

- Introduced by [Emerson and Clarke, 1981]
- Long intellectual "fights" over which logic is better!
  - *Sometimes is Sometimes "Not Never"– on the temporal logic of programs* [Lamport, 1980]
  - *What good is temporal logic?* [Lamport, 1983]
  - *Modalities for Model Checking: Branching Time Logic Strikes Back* [Emerson and Lei, 1985]
  - *"Sometimes" and "Not Never" revisited: On branching versus linear time temporal logic* [Emerson and Halpern, 1986]
  - *Branching versus linear logics yet again* [Carmo and Sernadas, 1990]
  - *Sometimes and not never re-revisited: on branching versus linear time* [Vardi, 1998]
  - *Branching vs. Linear Time: Final Showdown* [Vardi, 2001]
- More powerful logics:
  - CTL*: a combination of CTL and LTL, e.g., can write things like $\mathbf{AFG}p$.
  - The $\mu$-calculus [Kozen, 1983]
  - ...

# CTL and LTL in nuXmv

# CTL and LTL in nuXmv

```
-- transition system from lemma 6.19 of Baier-Katoen
MODULE TransitionSystem3
VAR   state : { s0, s1, s2 };
INIT state = s0
TRANS (state = s0 -> (next(state) = s0 | next(state) = s1))
        &
        (state = s1 -> next(state) = s2)
        &
        (state = s2 -> next(state) = s2)

MODULE main
VAR
-- this illustrates the difference between FGp and AFAGp:
  ts3: TransitionSystem3;

LTLSPEC         F G(ts3.state=s0 | ts3.state=s2)
CTLSPEC         AF AG (ts3.state=s0 | ts3.state=s2)
```

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