# Transfinite Ordinals and Their Notations: For The Uninitiated

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## Intoduction

This is supposed to be a primer to give you a feel for the subject fast. You're not expected to understand immediately why everything that is asserted is true, but there's enough there so you could fill in the gaps if you wanted to.

The theory of transfinite ordinals is a part of set theory. While the concept is tied up with the completed infinite and high cardinalities, we'll emphasize more constructive aspects of the theory. There have been applications of constructive treatments of ordinals to recursive function theory, and proof theory.

Pretty much everything we need to know about ordinals follows from the following three properties of the class of all ordinals O. Just how the existence of such a class can be shown we'll regard as a matter of mathematical foundations.

## Basic Properties of the System of Ordinal Numbers.

1) O has a well ordering, which will be denoted by $<$.

2) Any well ordered set whatsoever is order isomorphic to (a unique) initial segment of O. The ordinal determining the segment is called the ordinal of the set. This makes the system of ordinals in some sense the "mother of all well ordered sets"

3) Any set of members of O has a strict upper bound (and therefore by well ordering, a least upper bound) in O.

Warning: The collection of all ordinals itself is not a "set". You'll get hit with a famous paradox if you treat it like one. Sub-collections of initial segments of the ordinals are sets, and these are all we use.

As a consequence of the well ordering, we see easily that:

i) One can do arguments by induction over the ordinals, or over any initial segment of them, if we use the $<$ form of the induction principle. Recall that for natural numbers we also have induction in the $n$ to $n+1$

form, but that fails as soon as one has an object greater than every natural number. Induction in the $<$ form frequently goes under the fancy name "transfinite induction"

ii) We let 0 denote the smallest ordinal. Every ordinal has an immediate successor, so by taking repeated successors of zero, we generate all the natural numbers as an initial segment of the system of ordinals. Not all non-zero ordinals have an immediate predecessor. Those without an immediate predecessor are called *limit ordinals.* The smallest number to follow all the natural numbers is denoted by $\omega$ and it is a limit ordinal. Each limit ordinal is the least upper bound ("supremum") of the set of smaller ordinals.

iii) On account of the well ordering we can define by recursion operations of addition, multiplication, and exponentiation. The only added feature over recursion on natural numbers is that we need to specify what to do at limit ordinals.

$x + 0 = x$

$x + y' = (x + y)'$

$x + \bar{y} = \sup_{y<\bar{y}} x + y$ when $\bar{y}$ is a limit ordinal.

$x \times 0 = 0$

$x \times y' = x \times y + x$

$x \times \bar{y} = \sup_{y<\bar{y}} x \times y$ when $\bar{y}$ is limit ordinal.

$x^0 = 1$

$x^{y'} = x^y \times x$

$x^{\bar{y}} = \sup_{y<\bar{y}} x^y$ when $\bar{y}$ is a limit ordinal.

It turns out that:

a) Addition and Multiplication are associative. Neither is commutative. Multiplication distributes from the left over addition; that is

$x \times (y + z) = x \times y + x \times z.$

Right distributivity fails, but we do have the inequality

$(x + y) \times z \leq x \times z + y \times z$

b) Addition, multiplication and exponentiation are (with trivial exceptions):

i) Continuous strictly increasing function of the right argument.

ii) Weakly monotone increasing functions of the left argument.

### Cantor Normal Form

The theorem of ordinary number theory that justifies writing any non-zero number to a number base, like base 10 or base 2, applies to transfinite ordinals as well.

> **Theorem**. Any non-zero ordinal can be written uniquely as a polynomial to any base greater than 1 with descending exponents and coefficients less than the base. The coefficients are written to the right of the base. Such a representation is called a Cantor normal form.

It's common to use as the base the number $\omega$, in which case the coefficients are natural numbers, thus a typical normal form looks like:

$\omega^{\alpha_1} \times n_1 + \omega^{\alpha_2} \times n_2 + ...... + \omega^{\alpha_k} \times n_k$

where

$\alpha_1 > \alpha_2 > ..... > \alpha_k$

1) The rule for comparing two normal forms to see which represents the bigger ordinal is as follows: One first looks to see which has the highest leading exponent, if they are the same then you look to see which has the highest leading coefficient, etc. Such a method could go into an infinite regress on account of the fact (shown in the next section) that some ordinals can equal their own leading exponent.

2) Numbers of the form $\omega^x$ determine initial segments closed under addition, and numbers of the form $\omega^{\omega^x}$ dertermine initial segments of the ordinals closed under multiplication.

**Definition of the ordinal** $\epsilon_0$

1) The limit of the sequence

$\omega \ , \omega^{\omega}, \omega^{\omega^{\omega}} ..........$

was named $\epsilon_0$ by Cantor. We can see that it's a solution of the the equation $\omega^x = x$ by the following simple argument:

Consider the recursively defined sequence

i) $a_o = 0$

ii) $a_{n+1} = \omega^{a_n}$

then

$\omega^{lim\ a_n} = lim\ \omega^{a_n} = lim\ a_{n+1} = lim\ a_n.$

Another way to say that a number is a solution of $\omega^x = x$ is to say that it's a fixed point of the function $\omega^x$.

2) $\epsilon_0$ is the smallest ordinal bigger than $\omega$ that determines an initial segment closed under ordinal addition, multiplication, and exponentiation. From this it follows that you can't denote it just using symbols 0, $\omega$, plus, times, and exponentiation. One can show using Cantor Normal Form that anything smaller can be so represented.

3) Actually a similar argument can be used to show additional fixed points and they run clear through the ordinals. They can be arranged in a transfinite sequence as

$\epsilon_0, \epsilon_1, \epsilon_2 .......\epsilon_\omega ......$

**Computation With Ordinals On Real Computers**

We describe below a system of formal terms $T$ as follows:

$i)$ The symbol $a$ is an element of $T$

$ii)$ If $x$ and $y$ are elements of $T$, then so is the formal term $f(x, y)$.

Our intention here is that $a$ denote the ordinal number 0. The meaning of the function symbol $f$ here we shall leave mysterious.

Recursive Definition of an ordering on $T$:

$i)$ $a < f(x, y)$ for all terms $x$ and $y$.

$ii)$ $f(x, y) < f(u, v)$ iff one of the following holds.

$a)$ $x = u$ and $y < v$

$b)$ $x < u$ and $y < f(u, v)$

$c)$ $u < x$ and ( $f(x, y) = v$ or $f(x, y) < v$ )

This ordering is easily shown to be linear. It turns out to be a well ordering whose ordinal is $\epsilon_0$. It is known that it can't be proven to be a well ordering in first order number theory, yet any initial segment can be. (Gentzen 1941)

Each term can be regarded as a notation for an ordinal less than $\epsilon_0$. To demonstrate that we are dealing with objects that can be manipulated by computer, we give below a PROLOG program for telling of two terms $u$ and $v$ whether $u < v$.

$less(a, f(U, V)).$

$less(f(X, Y), f(U, V) : -X = U, less(Y, V).$

$less(f(X, Y), f(U, V)) : -less(X, U), less(Y, f(U, V)).$

$less(f(X, Y), f(U, V)) : -less(U, X), f(X, Y) = V.$

$less(f(X, Y), f(U, V)) : -less(U, X), less(f(X, Y), V).$

Now if $u$ and $v$ are are formal terms of $T$, the query:

less($u$,$v$)?

will cause he machine to return the answer "yes" if the ordinal denoted by $u$ is less than the ordinal denoted by $v$, and "no" otherwise.

**Connections to the Multi-set Orderings of Term Rewriting Theory**.

$i)$ The set of natural numbers ordered by the multi-set ordering has ordinal $\omega^\omega$.

In fact, in such a case the multi-set $\{m_1, m_2, ....m_k\}$ with $m_1 \geq m_2 \geq .... \geq m_k$ corresponds to the ordinal

$\omega^{m_1} + \omega^{m_2} + ...... + \omega^{m_k}$.

$ii)$ If we permit multi-sets of multi-sets of multi-sets etc. of natural numbers, the ordinal is $\epsilon_0$

In fact, in such a case the multi-set $\{M_1, M_2, ...., M_k\}$, where $M_1 \geq M_2, ...., \geq M_k$ are multi-sets, corresponds to the ordinal

$\omega^{ordM_1} + \omega^{ordM_2} + ...... + \omega^{ordM_k}$.

**Problem of Skolem.**

Nothing to do with logic really, but just about every one who's made a contribution works in logic..

Consider the set of formal terms S defined inductively below:

i) The symbols 1 and $X$ are in S.

ii) If $u$ and $v$ are in S, then so are $(u + v)$, $(u \times v)$, and $u^v$ .

Each term determines in a natural way a function of one variable on positive natural numbers.

Consider the ordering of the functions.

$f < g$ iff $f(x) < g(x)$ for sufficiently large $x$.

Questions:

1) Is it a linear ordering? (Yes, Richardson)

2) Is it a well ordering? (Yes, Ehrenfeucht using Kruskal's Theorem)

3) If yes to the above, what's the ordinal? (Unknown)

Skolem showed $\epsilon_0$ is a lower bound. Levitz showed that the first critical epsilon number is an upper bound. The first critical epsilon number is defined as follows. Arrange the solutions of $\omega^x = x$ in order and call them $\epsilon_0$, $\epsilon_1$, $\epsilon_2$, .... etc. Then the first critical epsilon number is the smallest member of the sequence equal to own subscript.

Levitz, Van den Dries, and Dahn have partial results supporting the conjecture that the actual ordinal is $\epsilon_0$

**The Least Uncountable Ordinal $\Omega$.**

It is a theorem of general set theory that there are uncountable sets, and a further theorem that any set can be well ordered. Let $\Omega$ denote the smallest ordinal that is the ordinal of an uncountable set. The initial segment determined by $\Omega$ has the following properties.

*I*) If $\alpha < \Omega$ then are only countably many ordinals less than $\alpha$.

*II*) Every countable set of ordinals less than $\Omega$ has a strict upper bound less than $\Omega$.

It turns out that any well ordered set with these properties is order isomorphic to the initial segment determined by $\Omega$. More precisely, if W is a well ordered set each of whose members has only countably many smaller members in W, and each countable subset of W has a strict upper bound in W, then W is order isomorphix to the initial segment determined $\Omega$.

### Normal Functions and Ordinal Notations

This is not intended to give a rigorous definition of a system of ordinal notations, but rather to help develop some insight and intuition for a commonly used one.

That cute little argument used to show that $\omega^x$ has fixed points can be abstracted to show that if a sequence of successive iterations of a continuous function approaches a limit, then that limit is a fixed point. What's more, if the function is strictly increasing and $f(0) > 0$, the iterations are increasing and, therefore, approach a limit. This brings us to the notion of a *normal function.*

> **Definition**. A normal function $f$ from $\Omega$ into $\Omega$ is a strictly increasing continuous function which has the additional property that $f(0) > 0$.

*i*) Normal functions have fixed points and the set of such is order isomorphic to the set of ordinals less than $\Omega$.

*ii*) The function which enumerates the fixed points of a normal function is itself a normal function. This suggests a hierarchy of normal functions $\theta_\alpha$ for $\alpha < \Omega$ given by:

*a*) $\theta_0(x) = \omega^x$.

*b*) For $\alpha < \Omega$, the function $\theta_\alpha$ enumerates, in order, the simultaneous fixed points of all $\theta_\beta$ for all $\beta < \alpha$.

One might attempt to define a function $\theta_\Omega$ in this way, however the attempt to do so fails as it turns out that the set of simultaneous fixed points of all $\theta_\beta$ for all $\beta < \Omega$ is empty.

Bachmann's idea (essentially) was to get around this collapse through the following considerations. The function $\theta_x(0)$ turns out to be a normal function, so he let $\theta_\Omega$ be defined as the function which enumerates, in order, the set of fixed points of $\theta_x(0)$. Now we can define $\theta_{\Omega+1}$ to be the function which enumerates in order the fixed points of $\theta_\Omega$. We can continue this way but face a similar collapse in trying to define $\theta_{\Omega+\Omega}$. To keep things going, let $\theta_{\Omega+\Omega}$ be defined as the function which enumerates, in order, the fixed points of $\theta_{\Omega+x}(0)$ .

With no more than what was said above, you can, without a general rule governing this phenomenon, work your way to things like $\theta_{\Omega^2}$ and and $\theta_{\Omega^\Omega}$ and even $\theta_{\Omega^{\Omega^{\cdot^{\cdot}}}}$ .

Feferman indicated how one can give a general rule for describing the hierarchy. His ideas have been worked out and refined variously by Weyrauch, Aczel, Bridge, and Bucholz. Not only is the least uncountable ordinal $\Omega$ used in subscripts for this hierarchy, but ordinals from even higher cardinality classes are used. For example, ordinals like $\Omega_2$, $\Omega_3$, ....... $\Omega_\omega$....... appear in subscripts.

Despite the presence of uncountable ordinals in the description of the hierarchy, the ordinals we present below are countable and, in fact, recursive. To show this, one has to show how all ordinals in initial segments they determine can be represented by means of smaller ordinals using functions of the hierarchy, and work out recursion relations that show how to compare two representations, assuming knowledge of how to compare their sub-terms. [If one starts the hierarchy using the function $\omega^x$ as we did here, then one also needs the addition function on ordinals to get representations, since $\omega^x$ is additively prime.]

One could now, if one so desires, build up the representations from scratch as purely formal terms and recursively define the ordering on them, thus throwing away the whole "scaffolding" of ordinal functions and uncountable ordinals used in the construction.

### An Ordinal Menagerie

Below we list some ordinals in order of size. There is an important body of literature in proof theory relating some of these to the "strength" of various formal systems. To keep things simple, we won't get into it here.

$\theta_0(0) =$ the ordinal number 1.

$\theta_1(0) = \epsilon_0$.

$\theta_2(0) =$ the least critical epsilon number; that is, the smallest solution of $\epsilon_x = x$

$\theta_\omega(0) =$ the closure ordinal for primitive recursive functions on the ordinals.

$\theta_\Omega(0) =$ the Feferman-Schütte ordinal $\Gamma_0$. Maximal ordinal for recursive path orderings (Dershowitz)

$\theta_{\Omega^2}(0) =$ ordinal of the Ackermann notation system.

$\theta_{\Omega^\omega}(0) =$ the maximal ordinal for simplification orderings. (Dershowitz-Okada) An ordering of the natural numbers having this ordinal is presented in the German edition of Schütte's book *Beweistheorie*.

$\theta_{\Omega^{\omega+1}}(0) =$ Closure ordinal for finite, structured, labeled, trees, under homeomorphic embedding (as defined by Kruskal) with labels coming from a well quasi-ordered set. This is the closure ordinal in the sense that, if the maximal ordinal of well ordered extensions of the set of labels is smaller than this ordinal, then so will be the maximal ordinal for well ordered extensions of the set of structured, labeled, trees. (D.Schmidt)

$\theta_{\Omega^\Omega}(0) =$ Veblen's ordinal.

$\theta_{\Omega^{\Omega^{\cdot^{\cdot^{\cdot}}}}}(0) = \theta_{\epsilon_{\Omega+1}}(0) =$ Howard's Ordinal.

$\theta_{\Omega_\omega}(0) =$ the limiting ordinal of Takeuti's ordinal notations of finite order (Levitz).

## References

A nice introduction to transfinite numbers generally is P. Halmos, *Naive Set Theory*, Princeton, N.J., Van Nostrand 1960.

An excellent treatment of ordinal notations is in K. Schütte, *Proof Theory*, Springer-Verlag, Berlin/Heidelberg/New York 1977.