1 Lattice Theory

1.1 Basic Lattices

Recall that a lattice $\langle L, \leq \rangle$ consists of a set L and a partial order $\leq$ on L such that any pair of elements has a greatest lower bound, the meet ($\wedge$), and a least upper bound, the join ($\vee$). From a more algebraic, but equivalent, viewpoint, a lattice is a triple $\langle L, \wedge, \vee \rangle$ where $\wedge$ and $\vee$ satisfy the associative, commutative, idempotency, and absorption laws.

$$\begin{array}{rrcl} \text{(associative law)} & (a \vee b) \vee c & = & a \vee (b \vee c) \\ \text{(commutative law)} & a \vee b & = & b \vee a \\ \text{(idempotency law)} & a \vee a & = & a \\ \text{(absorption law)} & a \vee (a \wedge b) & = & a \end{array}$$

Each law also has a dual, which is obtained by interchanging $\vee$ and $\wedge$. We can then define $a \leq b \;\equiv\; (a \wedge b) = a$ and it follows that $a \leq b \;\equiv\; (a \vee b) = b$. We will stick to the algebraic view. Note that in light of associativity and commutativity, we do not need parentheses for sequences of joins or meets.

Here is a simple lemma about lattices.

Lemma 1 (1) $a \leq b \;\Rightarrow\; a \vee c \;\leq\; b \vee c$ (2) $a \leq b \;\Rightarrow\; a \wedge c \;\leq\; b \wedge c$

Proof

$a \vee c$

$\leq$ { Absorption ($x \leq x \vee y$), Associativity, Commutativity }

$a \vee c \vee b$

$=$ { $a \leq b$, hence $a \vee b \;=\; b$ }

$b \vee c$

The proof of (2) is similar. □

1.2 Tarski-Knaster Fixpoint Theorem

A lattice $\langle L, \vee, \wedge \rangle$ is *complete* iff $\vee S$, $\wedge S$ is defined for all $S \subseteq L$.

$f : L \rightarrow L$ is *monotonic* (*order preserving*) iff:

$$\langle \forall x, y \in L :: x \leq y \rightarrow f.x \leq f.y \rangle$$

x is a *fixpoint* of f iff $f.x = x$.

f^n is defined as: f^0 is the identity function and $f^{n+1} = f \circ f^n$.

Note: f is monotonic implies that f^n is monotonic: The identity function is monotonic and composing two monotonic functions gives a monotonic function.

Theorem 1 *(Tarski-Knaster) Let f be a monotonic function on $\langle L, \vee, \wedge, \leq \rangle$, a complete lattice. Let $S = \{b | f.b \leq b\}, \alpha = \wedge S$. Then α is the least fixpoint of f.*

Proof (1): note that $f.\alpha \leq \alpha$. $\forall x \in S, \alpha \leq x$, thus $f.\alpha \leq f.x \leq x$. So, $f.\alpha \leq \wedge S$, *i.e.*, $f.\alpha \leq \alpha$.

(2) note that $\alpha \leq f.\alpha$. By monotonicity applied to (1), we get $f(f.\alpha) \leq f.\alpha$, so $f.\alpha \in S$, so $f.\alpha \leq \alpha$.

Obviously α is a fixpoint and below all fixpoints (they are all in S), thus the least fixpoint. $\square$

One can similarly define the greatest fixpoint. With some set theory, one can prove that the least fixpoint can be obtained by starting with $\perp$ and applying f until a fixpoint is reached. This will require at most κ steps, where $\kappa = |L|^+$, the next cardinal after $|L|$. (Recall that cardinals are just ordinals, but cardinal arithmetic is not ordinal arithmetic, *e.g.*, w^+ is the ordinal Ω, not $\omega + 1$.)